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CHARACTERIZATIONS OF H^1 BY SINGULAR INTEGRAL TRANSFORMS ON MARTINGALES AND $\mathbf{R}^n$

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1. Introduction.

This paper consists of two related but logically independent parts, Sections 2–4 where we study martingales, and Section 5 where we prove an analogous result for $\mathbf{R}^n$.

Singular integral transforms have been defined on local fields by Phillips and Taibleson [10], [11], [14]. These transforms have many properties in common with singular integrals on $\mathbf{R}^n$ [2], [12], and in particular Chao and Taibleson [3], [14], have shown that they can be used to define conjugate systems and H^p-spaces. However, as was noted by Gundy and Varopoulos [7] and Carleson who suggested this study to me, their theorems and proofs are independent of the algebraic structure and seem to be most naturally formulated for martingales.

The main result (Theorem 4) is a necessary and sufficient condition for a set of such transforms to characterize H^1 as the set of integrable functions having these transforms integrable.

In the case when the transforms are convolutions, especially on local fields, the condition is that the corresponding Fourier multipliers separate every character from its inverse. This is shown in Section 5 also to be a necessary condition for multipliers to characterize $H^1(\mathbf{R}^n)$. These theorems contain the counter examples with even multipliers by Gandulfo, Garcia-Cuerva and Taibleson [5].

2. Basic definitions.

Here are collected some facts about martingales that can be found e.g. in [6] or [9].

We assume that $(\Omega, \mathscr{F}, \mu)$ is a probability space and that for every $n \geqq 1$ there is a partition $\{E_{i_1 \ldots i_n}\}^d_{i_1 \ldots i_n = 1}$ of Ω into measurable subsets such that $\mu(E_{i_1 \ldots i_n}) = d^{-n}$ and $\bigcup_{i_n=1}^d E_{i_1 \ldots i_n} = E_{i_1 \ldots i_{n-1}}$ where d is a fixed integer.

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EXAMPLES. a) Let X be a set with d points, each having probability $1/d$. Take $(\Omega, \mathscr{F}, \mu)$ as X^∞ and $E_{i_1 \dots i_n}$ as the subset with the first n coordinates prescribed.

b) $(\Omega, \mathscr{F}, \mu)$ is $[0,1)$ with Lebesgue measure and $E_{i_1 \dots i_n}$ is the interval of points whose decimal expansion in the scale of d begins with $(i_1-1) \dots (i_n-1)$.

Let $\mathscr{F}_n$ be the sub-σ-field of $\mathscr{F}$ generated by $\{E_{i_1 \dots i_n}\}$. Thus $\mathscr{F}_n$-measurable functions are constant on each $E_{i_1 \dots i_n}$. If f is integrable, its conditional expectations are given by

$$E(f \mid \mathscr{F}_n) = d^n \int_{E_{i_1 \dots i_n}} f \, d\mu, \quad \text{on } E_{i_1 \dots i_n},$$

and in particular

$$E(f \mid \mathscr{F}_0) \equiv E(f) = \int_\Omega f \, d\mu .$$

A martingale is a sequence of integrable functions $\{f_n\}_0^\infty$ such that $E(f_{n+1} \mid \mathscr{F}_n) = f_n$. We define $\Delta f_n = f_n - f_{n-1}$.

An integrable function f defines a martingale by $f_n = E(f \mid \mathscr{F}_n)$ which can be identified with f. In order to obtain uniqueness we assume that $\mathscr{F}$ is generated by $\bigcup \mathscr{F}_n$. f^* is defined as $\sup_n |f_n|$.

Some well-known lemmas follow.

LEMMA 1. *If $f \in L^p$, $\infty \geqq p \geqq 1$, then $\|f_n\|_{L^p} \leqq \|f\|_{L^p}$ and $f_n \to f$ a.e. and in L^p.*

LEMMA 2. *If $\|f_n\|_{L^p} \leqq C$, $p > 1$, then $f_n = E(f \mid \mathscr{F}_n)$ for some $f \in L^p$ and $\|f\|_{L^p} \leqq \sup \|f_n\|_{L^p}$.*

LEMMA 3. *If $\sup_n |f_n| \in L^1$, then $f_n = E(f \mid \mathscr{F}_n)$ for some $f \in L^1$.*

LEMMA 4. *If $\{f_n\}$ is a positive submartingale, i.e., f_n is non-negative and $\mathscr{F}_n$-measurable and $E(f_{n+1} \mid \mathscr{F}_n) \geqq f_n$, then*

$$\|\sup f_n\|_{L^p} \leqq \frac{p}{p-1} \sup \|f_n\|_{L^p} .$$

Two Banach spaces are defined by

$$H^1 = \{f \, ; \, f^* \in L^1\} \qquad \mathrm{BMO} = \{f \, ; \, E(|f - f_n|^2 \mid \mathscr{F}_n) \leqq C, \forall n\} .$$

(Equivalent definitions exist.)

The Fefferman duality holds in the following form.

LEMMA 5. BMO *is isomorphic to the dual space of* H^1, *with the duality given by* $(f, g) = \lim_{n\to\infty} E(f_n g_n)$.

We have the inclusions $L^1 \supset H^1 \supset L^p \supset \mathrm{BMO} \supset L^\infty$, $1 < p < \infty$.

LEMMA 6. *If* $x_1 \dots x_d$ *are real numbers such that* $\sum x_i = 0$ *and* $\min x_i = -1$, *then there is a function* $f \in L^1$, *but* $f \notin H^1$ *such that*

$$\Delta f_n(E_{i_1 \dots i_{n-1}, j}) = \lambda_{i_1 \dots i_{n-1}} \cdot x_j .$$

PROOF. Define g_n by

$$g_n(E_{i_1 \dots i_n}) = \prod_1^n (1 + x_{i_k}), \quad . g_0 = 1 .$$

Thus

$$g_n(E_{i_1 \dots i_n}) = (1 + x_{i_n}) g_{n-1}(E_{i_1 \dots i_{n-1}})$$

and $\{g_n\}$ is a positive martingale. $\|g_n\|_{L^1} = E(g_n) = E(g_0) = 1$.

We assume that $x_1 = -1$. g_n is thus 0 if any i_k, $k \leqq n$, is equal to 1. Set

$$F_n = \bigcup_{i_1 \dots i_{n-1} \neq 1} E_{i_1 \dots i_{n-1}, 1} .$$

F_n are disjoint and

$$\int_{F_n} g_{n-1}\, d\mu = \frac{1}{d} \int_{\bigcup_{i_1 \dots i_n \neq 1} E_{i_1 \dots i_{n-1}}} g_{n-1}\, d\mu = \frac{1}{d} .$$

Set

$$f = \sum_1^\infty \frac{g_k}{k^2} \in L^1, \quad f_n = \sum_1^{n-1} \frac{g_k}{k^2} + \sum_n^\infty \frac{g_n}{k^2} \geqq \frac{g_n}{n+1} .$$

Consequently

$$\int_{F_n} f^* \geqq \int_{F_n} f_{n-1} \geqq \int_{F_n} \frac{g_{n-1}}{n} = \frac{1}{dn} \quad \text{and} \quad \int_\Omega f^* = \sum \int_{F_n} f^* = \infty .$$

3. The transform.

Let A be a linear operator in the space $V = \{x \in \mathbf{C}^d ; \sum x_i = 0\}$. Given a martingale $\{f_n\}$, we can regard $\{\Delta f_n(E_{i_1 \dots i_{n-1}, j})\}_{j=1}^d$ as an element in V for every $i_1 \dots i_{n-1}$. Define Δg_n as $A(\Delta f_n)$ on every set $E_{i_1 \dots i_{n-1}}$, that is,

$$\Delta g_n(E_{i_1 \dots i_{n-1}, i_n}) = \sum_{j=1}^{d} a_{i_n j} \Delta f_n(E_{i_1 \dots i_{n-1}, j})$$

where (a_{ij}) is the matrix representation of an arbitrary extension of A to C^d. Define $g_n = \sum_1^n \Delta g_n$. g_n is $\mathscr{F}_n$-measurable.

Since $\sum_{i_n=1}^{d} \Delta g_n(E_{i_1 \dots i_{n-1}, i_n}) = 0$, $\{g_n\}_{n=0}^{\infty}$ is a martingale denoted by $T\{f_n\}$. This gives an algebra homomorphism from the linear operators in V into the linear operators on martingales.

The most important case, including the previous results referred to above, is when A is the convolution with a fixed function on a finite group G, i.e. when $a_{i,j} = \alpha_{i \circ j^{-1}}$. Then T is a convolution with a homogenous function on G^∞.

Remark. One can also define more general transforms by taking different operators $A_{i_1 \dots i_{n-1}}$ on different subsets $E_{i_1 \dots i_{n-1}}$. They will have similar properties; in particular the results of this section are valid, assumed that the operators are uniformly bounded. The proofs below are of a well-known nature and are included for completeness. Cf. also Burkholder [1].

We are interested especially in the case of martingales of integrable functions, i.e. when $f_n = E(f \mid \mathscr{F}_n)$ and $g_n = E(g \mid \mathscr{F}_n)$, where $f, g \in L^1$ and we will then define Tf as g in accordance with the identification of an integrable function with the corresponding martingale.

It follows immediately from the definitions that Tf exists when f is measurable with respect to some $\mathscr{F}_n$. In particular, Tf_n exists and $T\{f_n\} = \{Tf_n\}$ for any martingale.

Thus if Tf exists, then $Tf = \lim Tf_n$ a.e.

We assume for the sake of simplicity that the (euclidean) norm of A is less than or equal to 1.

Lemma 7. *$E(|\Delta Tf_n|^2 \mid \mathscr{F}_m) \leqq E(|\Delta f_n|^2 \mid \mathscr{F}_m)$ if $n > m$.*

Proof.

$$E(|\Delta Tf_n|^2 \mid \mathscr{F}_{n-1})(E_{i_1 \dots i_{n-1}}) = \frac{1}{d} \sum_{k=1}^{d} \left| \sum_{j=1}^{d} a_{kj} \Delta f_n(E_{i_1 \dots i_{n-1}, j}) \right|^2$$

$$\leqq \frac{1}{d} \sum_{j=1}^{d} |\Delta f_n(E_{i_1 \dots i_{n-1}, j})|^2 = E(|\Delta f_n|^2 \mid \mathscr{F}_{n-1})(E_{i_1 \dots i_{n-1}})$$

and the result follows immediately.

Lemma 8. *If $f \in L^2$, then Tf exists and $\|Tf\|_{L^2} \leqq \|f\|_{L^2}$.*

PROOF.

$$E(|Tf_n|^2) = \sum_1^n E(|\Delta Tf_n|^2) \leqq \sum_1^n E(|\Delta f_n|^2)$$
$$= E(|f_n - f_0|^2) \leqq E(|f_n|^2) \leqq E(|f|^2) .$$

Thus $\|Tf_n\|_{L^2} \leqq \|f\|_{L^2}$. Lemma 2 shows that Tf exists and $\|Tf\|_{L^2} \leqq \|f\|_{L^2}$.

LEMMA 9. *If $f \in L^1$, then*

$$\mu\left\{x \; ; \; \sup_n |Tf_n|(x) > \lambda\right\} \leqq 5d\frac{\|f\|_{L^1}}{\lambda}$$

and $\lim Tf_n$ *exists a.e.*

PROOF. Let F_n be the set where $E(|f| \mid \mathscr{F}_n) > \lambda$, $E(|f| \mid \mathscr{F}_m) \leqq \lambda$, $m < n$. Thus

$$\bigcup F_n = \{x \; ; \; |f|^*(x) > \lambda\} \qquad \text{and} \qquad \mu(F_n) \leqq \frac{1}{\lambda} \int_{F_n} |f| \, d\mu .$$

Hence

$$\sum \mu(F_n) \leqq \frac{1}{\lambda} \|f\|_{L^1} .$$

Define $F = \bigcup F_n$ and

$$h(x) = \begin{cases} f_n(x), & c \in F_n , \\ f(x), & x \notin F . \end{cases}$$

$|h(x)| \leqq d\lambda$ a.e., since $x \in F_n$ implies

$$\lambda \geqq |f|_{n-1}(x) \geqq \frac{1}{d}|f|_n(x) \geqq \frac{1}{d}|f_n(x)| .$$

Consequently,

$$\int |h|^2 \, d\mu \leqq d\lambda \int |h| \, d\mu = d\lambda\left(\sum \int_{F_n} |f_n| \, d\mu + \int_{CF} |f| \, d\mu\right)$$
$$\leqq d\lambda\left(\sum \int_{F_n} |f| \, d\mu + \int_{CF} |f| \, d\mu\right) = d\lambda \|f\|_{L^1} .$$

Thus $h \in L^2$ and according to Lemmas 1, 4 and 8, $Th = \lim Th_n$ exists a.e. and

$$\left\|\sup_n |Th_n|\right\|_{L^2} \leqq 2\|Th\|_{L^2} \leqq 2(d\lambda \|f\|_{L^1})^{\frac{1}{2}} .$$

This implies

$$\mu\left\{x\ ;\ \sup_n |Th_n|(x) > \lambda\right\} \leqq \frac{1}{\lambda^2} 4d\lambda \|f\|_{L^1} .$$

Since $h_n = f_n$ except on $\bigcup F_n$, $Tf_n = Th_n$ except on $\bigcup F_n^*$, where F_n^* is the union of intervals $E_{i_1 \ldots i_{n-1}}$ containing some interval $E_{i_1 \ldots i_{n-1}, i_n}$ contained in F_n. $\mu(F_n^*) \leqq d\mu(F_n)$, thus

$$\mu(\bigcup F_n^*) \leqq d \sum \mu(F_n) \leqq \frac{d}{\lambda} \|f\|_{L^1} ,$$

which proves the first assertion. The second is obtained when $\lambda \to \infty$.

THEOREM 1. *T is a bounded operator on each L^p, $1 < p < \infty$.*

PROOF. The preceding lemmas and the Marcinkiewicz interpolation theorem show that

$$\left\|\sup_n |Tf_n|\right\|_{L^p} \leqq C_p \|f\|_{L^p}$$

if $1 < p \leqq 2$ and Lemma 2 shows that Tf exists in this case. The result for $2 < p$ follows by duality since the adjoint operator is the transform obtained from A^*.

The operator is not (except in trivial cases) bounded on L^1 or L^∞ as will be seen from Theorem 4 and Corollary 2. Exactly as in the classical case we have the following substitute.

THEOREM 2. *T is a bounded operator on* BMO *and H^1.*

PROOF. $f \in$ BMO implies $f \in L^2$ and $Tf \in L^2$.

$$E(|Tf - Tf_n|^2 \mid \mathscr{F}_n) = \sum_{n+1}^{\infty} E(|\Delta Tf_k|^2 \mid \mathscr{F}_n) \leqq \sum_{n+1}^{\infty} E(|\Delta f_k|^2 \mid \mathscr{F}_n)$$

$$= E(|f - f_n|^2 \mid \mathscr{F}_n) \leqq \|f\|_{\mathrm{BMO}}^2 .$$

Thus $\|Tf\|_{\mathrm{BMO}} \leqq \|f\|_{\mathrm{BMO}}$. If $h \in H^1$, duality gives

$$\|Th_n\|_{H^1} \leqq C\|h_n\|_{H^1} \leqq C\|h\|_{H^1} .$$

Hence

$$\left\|\sup_{k \leqq n} |Th_k|\right\|_{L^1} \leqq C\|h\|_{H^1} \quad \text{and} \quad \|\sup |Th_k|\|_{L^1} \leqq C\|h\|_{H^1}$$

by monotone convergence. Lemma 3 shows that Th exists and thus belongs to H^1.

4. A characterization of H^1.

Assume that $A_1 \dots A_m$ are one or more linear operators in V and let $T_1 \dots T_m$ be the corresponding martingale transforms. Theorem 2 shows that $f \in H^1$ implies that $T_1 f \dots T_m f$ belongs to H^1 and thus L^1. To prove a converse of this we first prove the following technical lemma.

LEMMA 10. *Let $A_1 \dots A_m$ be linear operators in V not having a common eigenvector in $\mathbf{R}^d \cap V$. Then there is $p_0 < 1$ such that*

(i) $a = (a_i)_0^m \in \mathbf{C}^{m+1}$,

(ii) $x_0 \in V$,

(iii) $x_i = A_i x_0, \quad i = 1, \dots, m,$

and $p > p_0$ implies

$$(*) \qquad \|a\|^p \leqq \frac{1}{d} \sum_{k=1}^{d} \|(a_i + x_{ik})_0^m\|^p .$$

PROOF. Since $x_i \in V$, we have $a_i = d^{-1} \sum_k (a_i + x_{ik})$ and thus

$$\|a\| \leqq \frac{1}{d} \sum_k \|(a_i + x_{ik})_0^m\| \leqq \left(\frac{1}{d} \sum_{k=1}^{d} \|(a_i + x_{ik})_0^m\|^p \right)^{1/p}$$

which proves (*) for $p \geqq 1$. Assume now that $0 < p < 1$. First we assume that $\|x_0\|/\|a\|$ is small and use the binomial expansion.

$$\begin{aligned}
\sum_k \|(a_i + x_{ik})_0^m\|^p &= \sum_k \left(\sum_i |a_i + x_{ik}|^2 \right)^{p/2} \\
&= \sum_k \left(\sum_i |a_i|^2 + \sum_i 2 \operatorname{Re} \bar{a}_i x_{ik} + \sum_i |x_{ik}|^2 \right)^{p/2} \\
&= \|a\|^p \sum_k \left(1 + \frac{2 \operatorname{Re} \sum_i \bar{a}_i x_{ik} + \sum_i |x_{ik}|^2}{\|a\|^2} \right)^{p/2} \\
&= \|a\|^p \sum_k \left(1 + \frac{p}{2} \frac{2 \operatorname{Re} \sum_i \bar{a}_i x_{ik}}{\|a\|^2} + \frac{p}{2} \frac{\sum_i |x_{ik}|^2}{\|a\|^2} \right. \\
&\qquad \left. + \frac{1}{2} \frac{p}{2} \left(\frac{p}{2} - 1 \right) \left(\frac{2 \operatorname{Re} \sum_i \bar{a}_i x_{ik}}{\|a\|^2} \right)^2 + O\left(\frac{\|x_0\|^3}{\|a\|^3} \right) \right) .
\end{aligned}$$

The second term will disappear since $\sum_k x_{ik}=0$. To estimate the fourth term, we set α as the maximum of the continuous function $\sum_k (\operatorname{Re}\sum_i \bar{a}_i x_{ik})^2$ on the compact set

$$K_1 = \left\{a \in \mathbf{C}^{m+1},\ x_i \in V;\ \|a\|=1,\ \sum_i \|x_i\|^2=1,\ x_i=A_i x_0\right\}.$$

Schwarz' inequality gives

$$\sum_k \left(\operatorname{Re}\sum_i \bar{a}_i x_{ik}\right)^2 \leqq \sum_k \left|\sum_i \bar{a}_i x_{ik}\right|^2 = \sum_k \sum_{i,j} \bar{a}_i x_{ik} a_j \bar{x}_{jk}$$
$$\leqq \left(\sum_{ijk} |a_j x_{ik}|^2\right)^{\frac{1}{2}} = \sum_i |a_j|^2 \sum_i \|x_i\|^2 = 1$$

on K_1. Equality would imply that $\sum \bar{a}_i x_{ik} \in \mathbf{R}$ and $a_i x_{jk}=\lambda a_j x_{ik}$. Symmetry gives $\lambda=1$, thus we have $a_0 x_i = a_i x_0$. Now $x_0 \neq 0$ on K_1, thus $a_0=0$ implies $a_i=0\ \forall i$ which is impossible. Consequently $A_i x_0 = x_i = (a_i/a_0)x_0$ and x_0 is a common eigenvector to A_i. So is

$$\frac{x_0}{a_0} = \sum_i \bar{a}_i a_i \frac{x_0}{a_0} = \sum_i \bar{a}_i x_i \in \mathbf{R}^d,$$

which contradicts the assumptions.

We conclude that $\alpha<1$. Homogenity shows that in general

$$\sum_k \left(\operatorname{Re}\sum_i \bar{a}_i x_{ik}\right)^2 \leqq \alpha\|a\|^2 \sum_i \|x_i\|^2 .$$

This gives

$$\|a\|^{-p} \sum_k \|(a_i+x_{ik})_0^m\|^p \geqq d+\frac{p}{2}\frac{\sum_i \|x_i\|^2}{\|a\|^2}$$
$$+\frac{p}{2}(p-2)\alpha\frac{\sum_i \|x_i\|^2}{\|a\|^2}+O\left(\frac{\|x_0\|^3}{\|a\|^3}\right) \geqq d$$

if $p>\alpha$ and $\|x_0\|/\|a\|<\varepsilon$ where ε is some positive number. Thus (*) is proved in this case.

To complete the proof we use another compactness argument.

Set

$$K_2 = \left\{a \in \mathbf{C}^{m+1},\ x_i \in V;\ x_i=A_i x_0,\ \frac{1}{d}\sum_k \|(a_i+x_{ik})_0^m\|=1,\ \|x_0\| \geqq \varepsilon\|a\|\right\}.$$

As remarked in the beginning of the proof, $a=d^{-1}\sum_k (a_i+x_{ik})_0^m$. Thus $\|a\| \leqq 1$ on K_2 and $\|a\|=1$ only if $a_i+x_{ik}=\lambda_k a_i$ with $\lambda_k \geqq 0$. This gives $x_{ik}=(\lambda_k-1)a_i$ and since $\|x_0\| \geqq \varepsilon$,

$$(\lambda_k - 1)_1^d = \frac{x_0}{a_0}$$

is a common real eigenvector to A_i which again contradicts the assumptions.

Consequently $\|a\| \leqq \beta < 1$ on K_2 which implies thst

$$\|a\| \leqq \frac{\beta}{d} \sum_{k=1}^{d} \|(a_i + x_{ik})_0^m\| \quad \text{if } \|x_0\| > \varepsilon \|a\| .$$

Thus

$$\|a\|^p \leqq \left(\frac{\beta}{d}\right)^p \left(\sum_{k=1}^{d} \|(a_i + x_{ik})_0^m\|\right)^p \leqq \frac{1}{d} \sum_{k=1}^{d} \|(a_i + x_{ik})_0^m\|^p$$

if $1 \geqq p \geqq \log d / \log (d/\beta)$ and $\|x\| \geqq \varepsilon \|a\|$, and the lemma is proved.

Returning to martingales, we obtain

THEOREM 3. *Assume that $A_1 \ldots A_m$ do not have a common real eigenvector and that $\{f_n\}$ is a martingale such that $\|f_n\|_{L^1}$ and $\|T_i f_n\|_{L^1}$ are bounded. Then $\{f_n\}$ and $\{T_i f_n\}$ are martingales of functions in H^1.*

PROOF. Set $T_0 f_n = f_n$ and $g_n = \|(T_i f_n)_0^m\|$. Lemma 10 with

$$a_i = T_i f_n(E_{i_1 \ldots i_n}) \quad \text{and} \quad x_{ik} = \Delta T_i f_{n+1}(E_{i_1 \ldots i_n, k})$$

shows that $g_n^p \leqq E(g_{n+1}^p \mid \mathscr{F}_n)$ for some $p < 1$, thus g_n^p is a positive submartingale.

$$\|g_n^p\|_{L^{1/p}} = \|g_n\|_{L^1}^p \leqq \left(\sum \|T_i f_n\|_{L^1}\right)^p \leqq C ,$$

thus we can use Lemma 4 to conclude $\sup g_n^p \in L^{1/p}$, hence $\sup g_n \in L^1$. Lemma 3 completes the argument since $|T_i f_n| \leqq g_n$.

A finite measure ν on $(\Omega \mathscr{F})$ defines a martingale $\{f_n\}$ by

$$f_n(E_{i_1 \ldots i_n}) = d^n \nu(E_{i_1 \ldots i_n}) .$$

Clearly $\|f_n\|_{L^1} \leqq \|\nu\|$ so we have the following martingale version of the F. and M. Riesz theorem.

COROLLARY 1. *Assume that $A_1 \ldots A_m$ do not have a common real eigenvector. If ν and $T_i \nu$ are measures, then ν is absolutely continuous.*

The condition of $\{A_i\}$ is necessary for Theorem 3 to hold as follows from the following specialization to integrable functions.

THEOREM 4. *$H^1=\{f\in L^1\ ;\ T_i f\in L^1\}$ if and only if $A_1 \dots A_m$ do not have a common real eigenvector.*

PROOF. Theorem 2 shows that $H^1\subset\{f\in L^1\ ;\ T_i f\in L^1\}$. If $A_1 \dots A_m$ do not have a common real eigenvector, the reverse inclusion is provided by Theorem 3. Conversely, assume that $x=(x_k)_1^d$ is a common real eigenvector. We can further assume that $\min x_k=-1$. The function f constructed in Lemma 6 is an eigenfunction of $T_1 \dots T_m$. Thus $T_i f=\lambda_i f\in L^1$ but $f\notin H^1$.

We do also obtain a characterization of BMO.

COROLLARY 2. *$\mathrm{BMO}=L^\infty+\sum_1^m T_i L^\infty$ if and only if $A_1^* \dots A_m^*$ do not have a common real eigenvector.*

PROOF. A continuous linear functional on $\{f\in L^1\ ;\ T_i^* f\in L^1\}$ can by the Hahn–Banach theorem be extended to a continuous linear functional on $\oplus_0^n L^1$ and can thus be represented as

$$f\to\sum_0^m (T_i^* f, g_i) = \sum_0^m (f, T_i g_i) \quad \text{where } g_i\in L^\infty .$$

Conversely, $\sum_0^m T_i g_i$, $g_i\in L^\infty$, gives a continuous linear functional on $\{f\in L^1\ ;\ T_i^* f\in L^1\}$. Therefore

$$\mathrm{BMO} = \left\{g_0+\sum_1^m T_i g_i\ ;\ g_i\in L^\infty\right\}$$

if and only if $H^1=\{f\in L^1\ ;\ T_i^* f\in L^1\}$.

If $d=2$, which corresponds to the Walsh–Paley group, then V is one-dimensional. Thus $Ax=\lambda x$ and $Tf=\lambda(f-f_0)$ and these results do not apply.

If $d\geqq 3$, there exists an operator A not having real eigenvectors and consequently a transform T characterizing H^1.

Restricting our attention to when A is a convolution, say with α, A do not have a real eigenvector if and only if $\hat\alpha(\chi)\neq\hat\alpha(-\chi)$, $\chi\neq 0$. Consequently, when d is odd H^1 can be characterized by one such transform, but when d is even G has a real character which always will be an eigenvector and H^1 cannot be characterized by any finite number of them.

The difference between odd and even d is also seen if we want A to be a real operator. This again possible if d is odd, but if d is even, A will always have a real eigenvector in V and two real operators are needed to characterized H^1.

This difference is also seen in the related results by Gundy and Varopoulos [7].

We may apply Theorem 4 to K^n, where K is a local field, if we regard only the set $\{x\ ;\ |x| \leqq q^k\}$ and let $k \to \infty$. This gives the singular integrals in [14, p. 235] with Ω ramified of degree one, that is, $\Omega(x+y) = \Omega(x)$, $|y| < |x|$. G now is the nth power of the residual field and the characters on G correspond to characters on K^n.

Consequently we obtain

COROLLARY 3. *$H^1(K^n) = \{f \in L^1\ ;\ m_i\hat{f} \in \hat{L}^1\}$, where m_i are homogeneous of degree zero and ramified of degree one, if and only if there is no $\chi \neq 0$ such that $m_i(\chi) = m_i(-\chi)$ for every i.*

5. Multipliers on R^n.

It may be conjectured that the condition in Corollary 3 applies on R^n also. We will prove that it is necessary.

THEOREM 5. *If $H^1(\mathsf{R}^n) = \{f \in L^1\ ;\ m_i\hat{f} \in \hat{L}^1\}$ where m_i, $i = 1,\ldots,k$, are homogeneous of degree zero, then there is no $x \neq 0$ such that $m_i(x) = m_i(-x)$ for every i.*

PROOF. Assume not, e.g. that

$$m_i(1,0,\ldots,0) = m_i(-1,0,\ldots,0) = \lambda_i .$$

Duality [4] shows that if $g \in \mathrm{BMO}$, there exist $g_i \in L^\infty$, such that $f \in H^1_{00}$ and $\hat{f}_i = m_i\hat{f}$ implies

$$\int gf = \sum \int g_i f_i .$$

Set $g(x_1,\ldots,x_n) = h(x_1)$ where h is an arbitrary unbounded function in BMO (R). We may assume that $g_i(x)$ also depend only on x_1, otherwise we convolve with

$$\delta_{x_1} N^{1-n}\psi\left(\frac{x_2}{N}\right)\psi\left(\frac{x_3}{N}\right)\ldots\psi\left(\frac{x_n}{N}\right)dx_2dx_3\ldots dx_n \qquad \left(\psi \in C_0^\infty \text{ and } \int\psi = 1\right)$$

and take weak *-limits. Consequently $g_i(x) = h_i(x_1)$, $h_i \in L^\infty(\mathsf{R})$.

Define $P\colon L^1(\mathsf{R}^n) \to L^1(\mathsf{R})$ by

$$Pf(x_1) = \int_{\mathsf{R}^{n-1}} f(x_1,x_2,\ldots,x_n)\,dx_2\ldots dx_n .$$

Thus $\widehat{Pf}(t) = \hat{f}(t,0\ldots 0)$, and if $\hat{f}_i = m_i\hat{f}$,

$$\widehat{Pf_i}(t) = m_i(t,0\ldots 0)\,\hat{f}(t,0\ldots 0) = \lambda_i\widehat{Pf}(t) .$$

That is, $Pf_i = \lambda_i Pf$.

This gives, if $f \in H^1_{00}(\mathbf{R}^n)$,

$$\int hPf = \int gf = \sum \int g_i f_i = \sum \int h_i Pf_i = \int \sum \lambda_i h_i Pf.$$

Since Pf may be any function in $H^1_{00}(\mathbf{R})$ this gives $h = c + \sum \lambda_i h_i \in L^\infty$, a contradiction.

REMARK. Without the assumption that m_i are homogeneous, this proof shows that the restriction of them to every line through the origin must characterize $H^1(\mathbf{R})$.

This shows immediately that no subset of the Riesz transforms is sufficient to characterize H^1. In fact their number is minimal among all real multipliers characterizing H^1.

COROLLARY 4. *With the same condition as in Theorem 5, if m_i are real, then $k \geqq n$.*

PROOF. $M = (m_1, \ldots, m_k)$ is a continuous function from $\mathbf{R}^n \setminus \{0\}$ to $\mathbf{R}^k$ such that $M(x) \neq M(-x)$. Consequently $(M(x) - M(-x))/\|M(x) - M(-x)\|$ is an odd continuous function from S^{n-1} to S^{k-1}, which is impossible if $k < n$ [8, p. 138].

If we allow the multipliers to be complex at least $n/2$ ($(n+1)/2$ if n is odd) are required by the same argument. This bound is sharp since $(x_1 + ix_2)/|x|$, $(x_3 + ix_4)/|x| \ldots$ characterize H^1. This follows since these multipliers define a generalized Cauchy–Riemann system of linear partial differential equations [13, p. 231].

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