MATH. SCAND. 41 (1977), 153–158

TRANSFORMS VANISHING AT INFINITY IN A CERTAIN DIRECTION AND SEMI-IDEMPOTENT MEASURES

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Let G be a compact abelian group with character group Γ. Throughout the paper, we shall assume there is a non-trivial group homomorphism

$$\varphi : \Gamma \to \mathsf{R}$$

where R is the additive group of reals. If φ is an isomorphism, then the semigroup $\mathscr{P}$ is the set $\{\gamma \in \Gamma : \varphi(\gamma) \geqq 0\}$. For convenience, assume Γ is countable.

Let $M(G)$ be the usual convolution algebra of finite complex-valued Borel measures on G. The Fourier–Stieltjes transform of the measure $\mu \in M(G)$ is the function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int_G \gamma(-x)\,d\mu(x) .$$

We will also use $\hat{}$ to denote the Gelfand transform. Let $M_0(G)$ be the ideal of measures $\mu \in M(G)$ such that $\hat{\mu} \in C_0(\Gamma)$.

We designate by $M_\varphi(G)$ the set of those $\mu \in M(G)$ such that $\hat{\mu}$ vanishes at infinity in the direction of φ. By this is meant $\{\gamma_n\}_1^\infty \subset \Gamma$ with $\varphi(\gamma_n) \to \infty \Rightarrow \hat{\mu}(\gamma_n) \to 0$. Here, $\varphi(\gamma_n) \to \infty$ in the usual topology of R.

It is easy to check that $M_\varphi(G)$ is a closed ideal of $M(G)$ such that if $\tau \in M_\varphi(G)$ and $\xi \ll \tau$ then $\xi \in M_\varphi(G)$. Thus

$$M(G) = M_\varphi(G) \oplus M_\varphi^\perp(G)$$

where

$$M_\varphi^\perp(G) = \{\varrho \in M(G) : \varrho \perp \tau \text{ for each } \tau \in M_\varphi(G)\} .$$

Let δ_0 be the identity measure in $M(G)$ and for any set of non-zero integers $\{N_1,\ldots,N_m\}$ put $\delta_i = N_i\delta_0$, $i=1,2,\ldots,m$. We state our first result.

THEOREM 1. *Let $\mu \in M(G)$ with $\mu * \prod_{i=1}^m (\mu-\delta_i) \in M_\varphi(G)$. Then $\mu=\mu_0+\mu_\perp$ where $\mu_0 \in M_\varphi(G)$, $\mu_\perp \in M_\varphi^\perp(G)$ and $\hat{\mu}_\perp(\Gamma) \subset \mathsf{Z}$.*

Received July 27, 1976; in revised form September 19, 1976.

PROOF. We adapt the method of [5]: Suppose $\mu \in M(G)$ and

(1) $$\mu * \prod_{i=1}^{m} (\mu-\delta_i) \in M_\varphi(G) .$$

Since $M_\varphi(G)$ is an ideal (1) gives:

(2) $$\mu_\perp * \prod_{i=1}^{m} (\mu_\perp-\delta_i) \in M_\varphi(G) .$$

Let S be the structure semi-group for $M(G)$ and consider the image of $M(G)$ in $M(S)$ in the usual way; see [8]. For $\xi \in M(G)$ the image of ξ is denoted by $(\xi)_s$. Then (2) becomes:

(3) $$(\mu_\perp)_s * \prod_{i=1}^{m} (\mu_\perp-\delta_i)_s \in M_\varphi(S)$$

where $M_\varphi(S)$ is the image of $M_\varphi(G)$.

We shall assume

(4) $$(\mu_\perp)_s \neq 0 .$$

Let $\hat{S}$ denote the semi-characters of S and $\bar{\Gamma}$ the closure of Γ in $\hat{S}$. Recall that $\hat{S}$ is the maximal ideal space of $M(G)$. Now (4) implies the existence of an infinite set $\{\gamma_n\}_1^\infty \subset \Gamma$ and an $\varepsilon>0$ such that $|\hat{\mu}_\perp(\gamma_n)| \geqq \varepsilon$ and $\varphi(\gamma_n) \to \infty$.

Thus $\{\gamma_n\}_1^\infty$ has a cluster point $\beta_0 \in \bar{\Gamma}\backslash\Gamma$. Since conjugation is continuous and multiplication of semi-characters separately continuous we may infer that $|\beta_0|^2 \in \bar{\Gamma}\backslash\Gamma$.

Put $T(\beta_0)=\{s \in S : \beta_0(s)=0\}$. Then define

$$(\mu_\perp)_s = \mu_1+\mu_2$$

where

$$\mu_1 = (\mu_\perp)_s|_{T(\beta_0)} \quad \text{and} \quad \mu_2 = (\mu_\perp)_s|_{S\backslash T(\beta_0)} .$$

Notice $\mu_2 \neq 0$ by (4).

Let $\tau \in M_\varphi(S)$ and consider

(5) $$|\tau|\hat{\ }(|\beta_0|^2) = \int_{S\backslash T(\beta_0)} |\beta_0|^2(s)\, d|\tau|(s) .$$

Now for fixed k we have $\lim_j \varphi(\gamma_j-\gamma_k)=\infty$, so since the Gelfand transform is continuous on $\hat{S}$ and multiplication of semi-characters separately continuous we may conclude that $|\tau|\hat{\ }(|\beta_0|^2)=0$. Thus, we gather from (5) that $M_\varphi(S)$ is carried by $T(\beta_0)$. Recall for any $\omega_1, \omega_2 \in M(S)$ that

(6) $$\text{carrier}\,(\omega_1 * \omega_2) \subset (\text{carrier}\,\omega_1)(\text{carrier}\,\omega_2) .$$

Inasmuch as $\boldsymbol{T}(\beta_0)$ is an ideal and $\boldsymbol{S}\backslash\boldsymbol{T}(\beta_0)$ is a semigroup we obtain via (6) that for any $\omega \in M(\boldsymbol{S})$ and $\tau \in M_\varphi(\boldsymbol{S})$ the condition:

$$\mu_2 * \prod_{i=1}^{m} (\mu_2 - (\delta_i)_s) \perp \tau + \omega * \mu_1 . \tag{7}$$

We gather from (3) and (7) that

$$\mu_2 * \prod_{i=1}^{m} (\mu_2 - (\delta_i)_s) = 0 . \tag{8}$$

Pulling back, we have

$$\mu_\perp = \varrho_1 + \varrho_2, \quad \varrho_1 \perp \varrho_2 \tag{9}$$

where $(\varrho_i)_s = \mu_i$, $i=1,2$. As a consequence of (8)

$$\varrho_2 * \prod_{i=1}^{m} (\varrho_2 - \delta_i) = 0, \quad (\varrho_2 \neq 0) . \tag{10}$$

Since $\varrho_i \in M_\varphi^\perp(G)$ $(i=1,2)$ we see from (10) that $\|\mu - \varrho_2\| \leqq \|\mu\| - 1$. So if $\varrho_1 \neq 0$ we apply this finite descent argument to $\mu - \varrho_2$ and therefore conclude that

$$\hat{\mu}_\perp(\Gamma) \subset \mathbf{Z} . \tag{11}$$

This completes the proof.

Theorem 1 has an application to semi-idempotent measures which we now give. A subset E of Γ is said to be a Sidon set if $f \in L^\infty(G)$ with $\operatorname{supp} \hat{f} \subset E \Rightarrow \sum |\hat{f}(\gamma)| < \infty$. For any subset A of Γ put

$$\boldsymbol{F}(A) = \{\mu \in M(G) : \hat{\mu} \text{ is integer-valued on } A\}$$

and

$$\boldsymbol{I}(A) = \{\mu \in M(G) : \hat{\mu} = 0 \text{ or } 1 \text{ on } A\} .$$

Assume φ is a non-trivial isomorphism of Γ into $\mathbf{R}$. The following result is an analogue of a result announced by I. Kessler [1]. See also Y. Meyer [3, pp. 206–211].

THEOREM 2. *Let E be a Sidon subset of Γ. Suppose $\mu \in \boldsymbol{F}(\Gamma \backslash -\mathscr{P} \cup E)$. Then there is a $\nu \in \boldsymbol{F}(\Gamma)$ such that $\hat{\mu} = \hat{\nu}$ off $-\mathscr{P} \cup E$. In particular, if $\mu \in \boldsymbol{I}(\Gamma \backslash -\mathscr{P} \cup E)$ then $\nu \in \boldsymbol{I}(\Gamma)$.*

PROOF. Suppose $\mu \in \boldsymbol{F}(\Gamma \backslash -\mathscr{P} \cup E)$ and let N_i $(i=1,2,\ldots,m)$ be the distinct non-zero integer-values of $\hat{\mu}$ off $-\mathscr{P} \cup E$. It is apparent that

$$\text{supp}\left\{\mu * \prod_{i=1}^{m} (\mu-\delta_i)\right\}^{\wedge} \subset -\mathscr{P} \cup E \,. \tag{1}$$

where $\delta_i = N_i\delta_0$. By Theorem 2 of [7, p. 368] we see that (1) gives:

$$\left\{\mu * \prod_{i=1}^{m} (\mu-\delta_i)\right\}^{\wedge} \in C_0(\mathscr{P}) \,. \tag{2}$$

As a consequence of (2) we gather that

$$\mu * \prod_{i=1}^{m} (\mu-\delta_i) \in M_\varphi(G) \,. \tag{3}$$

Now (3) in combination with Theorem 1 permits the conclusion:

$$\mu_\perp * \prod_{i=1}^{m} (\mu_\perp-\delta_i) = 0 \,. \tag{4}$$

where $\mu_\perp \in M_\varphi^\perp(G)$. Since $\mu_\perp \in \boldsymbol{F}(\Gamma)$ it is evident that $\mu_0 \in \boldsymbol{F}(\Gamma \setminus -\mathscr{P} \cup E)$. Consider

$$\mathscr{F} = \{\gamma \notin -\mathscr{P} \cup E : |\hat{\mu}_0(\gamma)| \geqq 1\} \,.$$

We claim $\mathscr{F}$ is a finite subset of $\mathscr{P}$. To establish our claim, we shall assume $\mathscr{F}$ is infinite and force a contradiction:

Suppose $\mathscr{F}$ is infinite. Clearly $0 \leqq \varphi(\mathscr{F}) \leqq M$ for some $M \in \mathsf{R}^+$ since $\mu_0 \in M_\varphi(G)$. Let r_0 be the largest accumulation point of the set $\varphi(\mathscr{F})$ and let $\gamma_j \in \mathscr{F}$ be a sequence of distinct elements such that $\varphi(\gamma_j) \to r_0$. Then without loss of generality,

$$\bar{\gamma}_j \mu_0 \to \nu \quad \text{weak}-* \tag{5}$$

where ν is singular with respect to Haar measure on G. As a consequence of $\gamma_j \in \mathscr{F}$, (5) gives:

$$\hat{\nu}(0) \neq 0 \,. \tag{6}$$

Now by Theorem 1.4 of [2, p. 8]

$$\underline{\lim}\, (E-\gamma_j) \text{ is a finite subset of } \Gamma \,. \tag{7}$$

Thus, except for a possible finite set of positive γ's,

$$\lim_j \hat{\mu}(\gamma+\gamma_j) = \hat{\nu}(\gamma) = 0 \,. \tag{8}$$

because $\gamma+\gamma_j$ eventually does not belong to $\mathscr{F}$. Appeal to Theorem 1 of [7] yields $\hat{\nu}(0)=0$ and this contradicts (6).

Thus, $\mathscr{F}$ is a finite set so there is a trigonometric polynomial p on G such that $\hat{p} = \hat{\mu}_0$ off $-\mathscr{P} \cup E$ and $\hat{p} = 0$ on $-\mathscr{P} \cup E$. Well, for the ν of our Theorem, take $\nu = \mu_\perp + p$. This concludes the proof.

The assumption that Γ be countable in our paper is of course inessential. The assumption that φ is a non-trivial isomorphism in Theorem 2 is equivalent to Γ having an archemedian order.

Let G be a non-discrete LCA group. The method of proof of Theorem 1 yields the following theorem.

THEOREM 3. *If*

$$\mu * \prod_{i=1}^{m} (\mu - \delta_i) \in M_0(G)$$

then μ *has a decomposition* $\mu = \mu_0 + \mu_\perp$ *where* $\mu_0 \in M_0(G)$, $\mu_\perp \in M_0^\perp(G)$ *and* $\hat{\mu}_\perp(\Gamma) \subset \mathbf{Z}$.

For discrete Γ we call $\mathfrak{R} \subset \Gamma$ *a weak Rajchman set* if $\operatorname{supp} \hat{\mu} \subset \mathfrak{R} \Rightarrow \mu \in M_0(G)$. For examples of Rajchman sets, the reader is referred to [6]. An easy consequence of Theorem 3 is:

If $\mu \in \boldsymbol{F}(\Gamma \backslash \mathfrak{R})$ *then there is a* $v \in \boldsymbol{F}(\Gamma)$ *such that* $\hat{\mu} = \hat{v}$ *off* $\mathfrak{R}$. *In particular, if* $\mu \in \boldsymbol{I}(\Gamma \backslash \mathfrak{R})$ *then* $v \in \boldsymbol{I}(\Gamma)$.

We remark that it is possible to prove a result which encompasses Theorem 1. For Γ discrete suppose Φ is any family of non-trivial homomorphisms from Γ into $\mathbf{R}$. *We designate by* $M_\Phi(G)$ *the set of those* $\mu \in M(G)$ *with the following property*: $\{\gamma_n\}_1^\infty \subset \Gamma$ *with* $\varphi(\gamma_n) \to \infty$ *for all* $\varphi \in \Phi \Rightarrow \hat{\mu}(\gamma_n) \to 0$. The proof of Theorem 1 can be adapted to obtain our final theorem.

THEOREM 4. *If* $\mu * \prod_{i=1}^{m} (\mu - \delta_i) \in M_\Phi(G)$ *then* μ *has a decomposition* $\mu = \mu_0 + \mu_\perp$ *where* $\mu_0 \in M_\Phi(G)$, $\mu_\perp \in M_\Phi^\perp(G)$ *and* $\hat{\mu}_\perp(\Gamma) \subset \mathbf{Z}$.

The author takes pleasure in thanking Professor Y. Domar of Uppsala University for helpful correspondence. The results of this paper were announced in [4].

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