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CHARACTERIZATION OF WEAK CONVERGENCE OF SIGNED MEASURES ON [0,1]

GÖRAN HÖGNÄS

0. Introduction.

In [1, p. 65] Paul Lévy considers the following problem:

By the Riesz representation theorem a bounded linear functional L on the space of continuous real-valued functions on $[0,1]$ can be characterized in terms of a function K of bounded variation on $[0,1]$. The correspondence between K and L is given by the Stieltjes integral

$$Lf = \int_0^1 f(t)\,dK(t)$$

where f is any continuous function on $[0,1]$. Let K now depend on a parameter s

$$L_s f = \int_0^1 f(t)\,dK_s(t) .$$

The problem of Lévy is to find the (necessary and sufficient) conditions on the family $\{K_s\}$ under which $L_s f$ is a continuous function of s.

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1. Notations and preliminary results.

Let X denote the closed unit interval $[0,1]$ and $C=C(X)$ the space of continuous real-valued functions on X. If C is endowed with the usual supremum norm, denoted $\|\cdot\|$, it becomes a Banach space. Let C^* be the dual of C, i.e. the set of all bounded linear functionals on C. C^* is a Banach space when the norm $\|L\|$ of an element $L \in C^*$ is defined by

$$\|L\| = \sup_{f \in C} |Lf|/\|f\| .$$

By the Riesz theorem, see for instance [3, p. 131], C^* is isomorphic to the set $\mathscr{M}$ of bounded signed measures on the measure space $(X,\mathscr{B})$ where $\mathscr{B}$ is the σ-

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algebra of Borel sets. The correspondence between $L \in C^*$ and $\mu \in \mathcal{M}$ is given by

$$Lf = \int f \, d\mu, \quad f \in C .$$

Moreover,

$$\|L\| = \|\mu\|(X)$$

where $\|\mu\|$ is the total variatiin of μ.

Let μ^+ and μ^- denote the positive and negative parts, respectively, in the Jordan–Hahn decomposition of μ:

$$\mu = \mu^+ - \mu^-, \qquad \|\mu\| = \mu^+ + \mu^- .$$

Let K be the *distribution function* of μ defined by

(1) $$K(x) = \mu[0,x], \quad x \in X .$$

If K^+ and K^- denote the corresponding distribution functions of μ^+ and μ^-, respectively, then

$$K = K^+ - K^- .$$

K is a function of bounded variation since K^+ and K^- are non-decreasing on $[0,1]$. The linear functional Lf, also denoted μf or $\int f d\mu$, can then be written as a Stieltjes integral

$$\int_0^1 f(t)\,dK(t) = \int_0^1 f(t)\,dK^+(t) - \int_0^1 f(t)\,dK^-(t) .$$

Conversely, every function of bounded variation on $[0,1]$ defines, by the above formula, a bounded linear functional on C and thus a bounded signed measure $\mu \in \mathcal{M}$.

Two modes of convergence in $\mathcal{M}$ will be considered, strong and weak convergence. The net μ_α in $\mathcal{M}$ is said to converge strongly towards μ iff

$$\|\mu_\alpha - \mu\|(X) \to 0$$

and μ_α converges weakly to μ iff

$$\mu_\alpha f \to \mu f \quad \text{for every } f \in C .$$

The strong and weak convergence of μ_α to μ will be denoted by $\mu_\alpha \xrightarrow{s} \mu$ and $\mu_\alpha \to \mu$, respectively.

From the definition of norm in $C^* = \mathcal{M}$ we immediately get

(2) $$\|\mu f - \mu g\| \leqq \|\mu\|(X)\|f - g\|, \quad \mu \in \mathcal{M};\ f,g \in C .$$

For completeness, we state the following easy result about strong convergence:

PROPOSITION 1. a) $\mu_\alpha \xrightarrow{s} \mu \Rightarrow \mu_\alpha \to \mu$.
b) $\mu_\alpha \xrightarrow{s} \mu \Leftrightarrow (K_\alpha - K)^+(1),\ (K_\alpha - K)^-(1) \to 0$ *where* K_α *and* K *are the distribution functions of* μ_α *and* μ, *respectively.*

PROOF. a) If $f \in C$ and $\mu_\alpha \xrightarrow{s} \mu$ then

$$|\mu_\alpha f - \mu f| \leqq \|\mu_\alpha - \mu\|(X)\,\|f\| \to 0\,.$$

b)

$$\begin{aligned} \mu_\alpha \xrightarrow{s} \mu &\Leftrightarrow \|\mu_\alpha - \mu\|(X) \to 0 \\ &\Leftrightarrow (\mu_\alpha - \mu)^+(X),\ (\mu_\alpha - \mu)^-(X) \to 0 \\ &\Leftrightarrow (K_\alpha - K)^+(1),\ (K_\alpha - K)^-(1) \to 0\,. \end{aligned}$$

REMARK. $\|K_\alpha^+ - K^+\|,\ \|K_\alpha^- - K^-\| \to 0$ does not suffice to guarantee $\mu_\alpha \xrightarrow{s} \mu$ (and hence, of course, $\|K_\alpha - K\| \to 0$ does not either) as is shown by the following example:

$$\begin{aligned} K_{2n}^- &= K^- = 0 \qquad (n=1,2,\ldots) \\ K(x) &= x, \qquad x \in [0,1]\,, \\ K_{2n}(x) &= k/n, \qquad x \in [(2k-1)/2n, k/n[\qquad (k=1,2,\ldots,n) \\ K_{2n}(x) &= 2(x-(k-1)/n)+(k-1)/n, \qquad x \in [(k-1)/n, (2k-1)/2n[\\ K_{2n}(0) &= 0, \qquad K_{2n}(1) = 1\,. \end{aligned}$$

In this case, $(K_n - K)^+(1) = \frac{1}{2}$ for all n.

2. Weak convergence of signed measures on X.

If μ_α is a weakly convergent net in $\mathcal{M}$ then the net of norms $\|\mu_\alpha\|(X)$ is uniformly bounded (Banach–Steinhaus theorem). In this chapter we shall therefore assume that *the nets of signed measures considered are uniformly bounded in norm* by some constant, which we shall take to be 1. As we shall see in the course of this chapter this assumption is crucial to many of the results.

Consider a net μ_α in $\mathcal{M}$ with $\mu_\alpha \to 0$. The aim of the subsequent discussion is to find the implications of this convergence to the distribution functions K_α. The assumptions of uniform boundedness made above imply $K_\alpha^+(1), K_\alpha^-(1) \leqq 1$.

PROPOSITION 2. *Put* $\|K_\alpha\| = \sup\{|K_\alpha(x)| \mid 0 \leqq x \leqq 1\}$. *Suppose* $\|\mu_\alpha\|(X) \leqq 1$.

Then

$$\|K_\alpha\| \to 0 \Rightarrow \mu_\alpha \to 0 .$$

PROOF. Let $\varepsilon > 0$ and $f \in C$. Since f is uniformly continuous there is a $\delta > 0$ such that

$$\sup_{|x-y|<\delta} |f(x) - f(y)| < \varepsilon . \tag{3}$$

Construct a partition of X into N disjoint intervals I_k:

$$I_1 = [x_0, x_1],\ I_2 =]x_1, x_2], \ldots,\ I_N =]x_{N-1}, x_n]$$

where $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$ and

$$x_k - x_{k-1} < \delta \quad (k = 1, 2, \ldots, N) .$$

Note that $\mu_\alpha I_k = K_\alpha(x_k) - K_\alpha(x_{k-1})$.

Approximate f with a step function g constant on the I_k's such that

$$\|f - g\| < \varepsilon \quad \text{and} \quad \|g\| \leqq \|f\| ,$$

e.g.

$$g = \inf\{f(x) \mid x \in I_k\} .$$

The assumption $\|K_\alpha\| \to 0$ implies that $\|K_\alpha\| < \varepsilon/N$ for α larger than some α_0. Then,

$$|\mu_\alpha g| \leqq \sum_{k=1}^{N} \|g\| \, |\mu_\alpha(I_k)| \leqq 2\varepsilon \|f\|$$

since

$$\|g\| \leqq \|f\| \quad \text{and} \quad |\mu_\alpha(I_k)| \leqq 2\|K_\alpha\| .$$

(2), the triangle inequality and the uniform boundedness of $\|\mu_\alpha\|(X)$ now yield

$$|\mu_\alpha f| \leqq \varepsilon(2\|f\| + \|\mu_\alpha\|(X)) \leqq \varepsilon(2\|f\| + 1)$$

for $\alpha \geqq \alpha_0$. Hence $\mu_\alpha f \to 0$.

PROPOSITION 3. *Let* $\|\mu_\alpha\|(X) \leqq 1$. *If* $\int_0^1 |K_\alpha| \, dx \to 0$ *and* $K_\alpha(1) \to 0$ *then* $\mu_\alpha \to 0$.

PROOF. Choose $\varepsilon > 0$ and $f \in C$ and construct the same partition of X into N intervals as in the proof of the preceding proposition.

For α large enough $(\alpha \geqq \alpha_0)$

$$\int |K_\alpha| \, dx < \varepsilon\delta/2N \quad \text{and} \quad K_\alpha(1) < \varepsilon/N$$

whence

$$m\{|K_\alpha| > \varepsilon/N\} \; < \; \delta/2 \tag{4}$$

where m denotes Lebesgue measure.

Consider the partition of X into N half-open (with the exception of I_1) intervals. For every $\alpha \geqq \alpha_0$ we are able to construct a new partition consisting of at most $2N$ half-open intervals in the following way: Let x be a point of division for the original partition. If $|K_\alpha(x)| \leqq \varepsilon/N$ then x will also be taken as a point of division for the new partition. If $|K_\alpha(x)| > \varepsilon/N$, then, by (4), there are points to the left and to the right of x, respectively, where $|K_\alpha| \leqq \varepsilon/N$. By (4), it is possible to choose these two new points of division within $]x-\delta/2, x+\delta/2[$. Thus the new partition is made up of halfopen intervals of length $\leqq \delta$. Choose g as in the proof of Proposition 2. Then

$$|\mu_\alpha g| \; \leqq \; 4\varepsilon \|g\| \; \leqq \; 4\varepsilon \|f\|$$

and hence

$$|\mu_\alpha f| \; \leqq \; \varepsilon(4\|f\|+1) \, .$$

Let m denote Lebesgue measure and $\varepsilon_{\{1\}}$ the point mass at 1. Put

$$m' \; = \; m + \varepsilon_{\{1\}} \, . \tag{5}$$

Then Proposition 3 may be written

COROLLARY. $K_\alpha \to 0$ *in* $L_1(m') \; \Rightarrow \; \mu_\alpha \to 0$.

REMARK. $K_\alpha(1)$ is just the total mass $\mu_\alpha(X)$.

Proposition 3 shows that $\int |K_\alpha| \, dm' \to 0$ is a sufficient condition for $\mu_\alpha \to 0$. We shall now try to see whether it is also a necessary condition. (Trivially, $K_\alpha(1) \to 0$ is a necessary condition).

LEMMA 1. *Let a be a positive real number. Suppose that there is a net K_α of distribution functions such that the sets $\{x \mid K_\alpha(x) > a\}$ all contain an interval of length greater than some positive constant b. Then there is an interval I, of positive length, such that*

$$I \subseteqq \{x \mid K_{\alpha'}(x) > a\}$$

for a subnet $K_{\alpha'}$ of K_α.

PROOF. Let y_α be the midpoint of an interval of length $\geqq b$ contained in $\{K_\alpha > a\}$. Let y be a limit point of the net y_α. Then there is a subnet K_α of K_α such that

$$\left[y-\frac{b}{4},\ y+\frac{b}{4}\right] \cap [0,1] \subseteqq \{K_{\alpha'} > a\} .$$

LEMMA 2. *Let a be a positive constant and μ_α a net of (uniformly bounded) signed measures on X such that all $\{K_\alpha > a\}$ contain an interval I of positive length. Then $\mu_\alpha \not\to 0$.*

PROOF. If $\mu_\alpha(X) = K_\alpha(1) \not\to 0$ there is nothing to prove. We shall therefore assume that $\mu_\alpha(X) \to 0$.

Put $I =]c,d]$. Then, for α greater than some α_0, we have: $\mu_\alpha[0,x] > a$ for all $x \in I$ and $\mu_\alpha]d,1] < -3a/4$.

Take $N > 2/a$. There exist $f_k \in C$ with $0 \leqq f_k \leqq 1$, $f_k = 1$ on the interval $[0, x_{k-1}]$ and $f_k = 0$ on $[x_k, 1]$, where

$$x_k = c + (k-1)(d-c)/N \qquad (k = 1,2,\ldots,N) .$$

Put $J_k =]x_{k-1}, x_k]$.

Suppose that $\mu_\alpha \to 0$. Then there is an α_1 such that

$$\alpha > \alpha_1 \Rightarrow |\mu_\alpha f_k| < a/4 \qquad (k = 1,2,\ldots,N)$$

$$\mu_\alpha f_k = \int_{[0,x_{k-1}]} f_k \, d\mu_\alpha + \int_{J_k} f_k \, d\mu_\alpha$$

$$= K_\alpha(x_{k-1}) + \int_{J_k} f_k \, d\mu_\alpha .$$

Since $K_\alpha(x) > a$ for all $x \in I$ we get

$$-\tfrac{3}{4}a > \int_{J_k} f_k \, d_{\mu_\alpha} \geqq -\int_{J_k} f_k \, d_{\mu_\alpha^-} \geqq -\mu_\alpha^-(J_k)$$

hence

$$\mu_\alpha^-(J_k) > \tfrac{3}{4}a$$

and therefore

$$\mu_\alpha^-(I) \geqq \mu_\alpha^-\left(\bigcup_{k=1}^{N} J_k\right) = \sum_{k=1}^{N} \mu_\alpha^-(J_k) > \tfrac{3}{4}aN > \tfrac{3}{2}$$

contradicting $\|\mu_\alpha\| \leqq 1$.

Let $]b, b+a[$ be an interval $\subseteqq [-1, 1]$ $(a>0)$. Let $\gamma_{b,b+a}(K_\alpha)$ denote the number of downcrossings of K_α through the interval $]b, b+a[$, see [2, p. 127].

LEMMA 3. $\gamma_{b,b+a}(K) \leqq a^{-1}$.

PROOF. Suppose that there exist $0 \leqq t_1 < t_2 < \ldots < t_{2N} \leqq 1$ such that $K_\alpha(t_{2k-1}) > b+a$ and $K_\alpha(t_{2k}) < b$ $(k=1, \ldots, N)$. Consider

$$\mu_\alpha\left(\bigcup_{k=1}^{N}]t_{2k-1}, t_{2k}]\right) < -Na .$$

Since $\|\mu_\alpha\| \leqq 1$, we get $Na < 1$, that is, $N < a^{-1}$. Hence $\gamma_{b,b+a}(K_\alpha) \leqq a^{-1}$.

LEMMA 4. *If* $\int |K_\alpha|\, dx \nrightarrow 0$, *then there exist positive constants a and b and a subnet* $K_{\alpha'}$ *of* K_α *such that the sets* $\{K_{\alpha'} > a\}$ *contain intervals of length* $\geqq b$.

PROOF. If $\int |K_\alpha|\, dx \nrightarrow 0$, then either

$$\int \max\{K_\alpha, 0\}\, dx \nrightarrow 0 \quad \text{or} \quad \int \max\{-K_\alpha, 0\}\, dx \nrightarrow 0 .$$

Suppose that $\int \max\{K_\alpha, 0\}\, dx \nrightarrow 0$. Then there exists a positive number a and a subnet $K_{\alpha'}$ of K such that $\int \max\{K_{\alpha'}, 0\}\, dx > 5a$, which guarantees that the set $\{K_{\alpha'} > 4a\}$ has positive measure $2c > a$.

Put $N = \gamma_{2a, 3a}(K_{\alpha'})$. $N \leqq a^{-1}$ by Lemma 3. Then there exist $0 \leqq t_1 < t_2 < \ldots < t_{2N} \leqq 1$ such that

$$K_{\alpha'}(t_{2k-1}) > 3a \quad \text{and} \quad K_{\alpha'}(t_{2k}) < 2a, \quad k=1, \ldots, N .$$

Consider $u_1 = \inf\{t > 0 \mid K_{\alpha'}(t) > 3a\}$. We have $u_1 \leqq t_1$. If $u_2 = \inf\{t > u_1 \mid K_{\alpha'}(t) < 2a\}$, then $u_2 \leqq t_2$. The right continuity of $K_{\alpha'}$ implies $u_1 < u_2$ and also $t_1 < u_2$ (since otherwise $\gamma_{2a,3a}(K_{\alpha'}) \geqq N+1$). Proceeding in this way, we define (with $u_0 = 0$)

$$u_{2k-1} = \inf\{t > u_{2k-2} \mid K_{\alpha'}(t) > 3a\}$$

$$u_{2k-2} = \inf\{t > u_{2k-1} \mid K_{\alpha'}(t) < 2a\}, \quad k=1, \ldots, N$$

which have the property

$$0 \leqq u_i \leqq t_i < u_{i+1}, \quad i=1, \ldots, 2N-1 ,$$

$$u_{2N} \leqq t_{2N} \leqq 1 .$$

The right continuity of $K_{\alpha'}$ now enables us to choose the t_i's in such a way as to satisfy (with $t_0 = 0$)

$$t_{2k-2} \leqq t < t_{2k-1} \Rightarrow K_{\alpha'}(t) \leqq 4a$$

and

$$t_{2k-1} \leqq t < t_{2k} \Rightarrow K_{\alpha'}(t) > a, \quad k=1,\ldots,N\,.$$

(Take e.g. $t_{2k-1}=u_{2k-1}$ if $K_{\alpha'}(u_{2k-1})>3a$, otherwise $t_{2k-1}=u_{2k-1}+\varepsilon$, where $\varepsilon>0$ is small enough.)

Let u' be the infimum of the set $\{t>t_{2N} \mid K_{\alpha'}(t)>3a\}$; if the set is empty, put $u'=1$. Our choice of N guarantees that $K_{\alpha'}\geqq 2a$ on $[u',1]$.

If $1-u'\geqq c$, then $\{K_{\alpha'}\geqq 2a\}$ contains an interval of length c, and $c>b$ where $b=a^2/2$. If, however, $1-u'<c$, then we use

$$\{K_{\alpha'}>4a\} \subseteqq \bigcup_{k=1}^{N} [t_{2k-1},t_{2k}[\,\cup\, [u',1]$$

and

$$m\{K_{\alpha'}>4a\} = 2c$$

to conclude that at least one of the intervals $[t_{2k-1},t_{2k}[$ must have length $>c/N$.

But $[t_{2k-1},t_{2k}[\subseteqq\{K_{\alpha'}>a\}$. Thus $\{K_{\alpha'}>a\}$ contains an interval of length $>c/N>a^2/2=b$.

PROPOSITION 4. $\int|K_\alpha|\,dx\not\to 0 \Rightarrow \mu_\alpha \not\to 0$.

PROOF. Follows directly from Lemmas 4 and 2.

THEOREM. Let m' be the measure defined in (5). Then the net μ_α of uniformly bounded signed measures converges weakly to 0 if and only if $\int|K_\alpha|\,dm' \to 0$, where K_α is the distribution function corresponding to μ_α.

PROOF. Follows directly from Propositions 3 and 4 and from the fact that $K_\alpha(1) \not\to 0 \Rightarrow \mu_\alpha \not\to 0$.

COROLLARY 1. *Let S be a topological space and $\{K_s \mid s \in S\}$ a family of functions of bounded variation on* $[0,1]$ *with $\{\mu_s \mid s\in S\}$ the corresponding family of signed measures on* $([0,1],\mathscr{B})$. *Then the following statements are equivalent*:

(i) *The function $s \curvearrowright \mu_s$ is weakly continuous.*

(ii) *For every $f\in C[0,1]$ the function*

$$s \curvearrowright \int_0^1 f(t)\,dK_s(t)$$

is continuous.

(iii) $\forall s \in S$: $\lim_{r \to s} \int |K_s - K_r|\, dm' = 0$ *and the variation of* K_r *is bounded in a neighbourhood of* s.

With the aid of Corollary 1 we shall also prove the following well-known result

COROLLARY 2. *Let* $\{K_s\}$ *be a family of increasing functions (i.e.* $\{\mu_s\}$ *is a family of measures). Then the following statements are also equivalent*:

(i) *The function* $s \curvearrowright \mu_s$ *is weakly continuous.*

(ii) $\forall s \in S$: $\lim_{r \to s} K_r(t) = K_s(t)$ *for every point of continuity for* K_s *and* $\lim_{r \to s} K_r(1) = K_s(1)$.

PROOF. We shall show that (ii) is equivalent to the following alternative version of (iii) in Corollary 1:

$$\int |K_r - K_s|\, dx \to 0, \qquad K_r(1) \to K_s(1)$$

and the variation of K_r is bounded in a neighbourhood of s.

Let t be a point of continuity for K_s and suppose that $K_r(t) \not\to K_s(t)$ when $r \to s$. In every neighbourhood U of s there is an $r = r(U)$ such that $|K_r(t) - K_s(t)| > a$ for some positive a. Take, for example, $K_r(t) - K_s(t) > a$. Then $K_r - K_s > a/2$ on the interval I where $K_s < K_s(t) + a/2$ and which has t as its left endpoint. (Recall that K_s is continuous at t and K_r is non-decreasing.) Hence

$$\int |K_r - K_s|\, dx \geqq \frac{a}{2} m(I),$$

that is, $\int |K_r - K_s|\, dx \not\to 0$ as $r \to s$.

On the other hand $\int |K_r - K_s|\, dx \not\to 0$ and the variation of K_r ($= K_r(1)$ in this case) bounded in a neighbourhood of s implies, by Lemma 4, that there is an $a > 0$ and an interval I such that, in any neighbourhood U of s,

$$u \in I \Rightarrow |K_s(u) - K_r(u)| > a$$

for some $r = r(U)$. Hence (ii) cannot hold since the set of points of continuity for K_s is a dense subset of $[0,1]$.

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ÅBO AKADEMI
ÅBO
FINLAND