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AN IMPRIMITIVITY THEOREM FOR HOPF ALGEBRAS

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1. Introduction.

Let B be a subring of A, both equipped with the same unit element. If V is a left (unitary) B-module one calls the left A-module $A \otimes_B V$ *induced* or more precisely adjoint induced. Similarly, the left A-module $\mathrm{Hom}_B(A, V)$ is called *coinduced* in short for coadjoint induced. An imprimitivity theorem is a criterion for a left A-module W to be induced or coinduced.

The best-known imprimitivity theorems are perhaps those for locally compact groups (Mackey [3], [1], [5]). Some time ago Blattner ([2]) established an imprimitivity theorem for Lie algebras which can be viewed as an infinitesimalization of the Mackey theorem. More recently Rieffel ([6]) observed that the Morita duality theorems imply a general imprimitivity theorem for rings, and used this to derive an imprimitivity result for group rings which is close in spirit to the Mackey theorem (in the form [1]).

The statements of the imprimitivity theorems for groups and Lie algebras in [6] and [2] both seem to arise from the same source, namely from the Hopf algebra structure of the group algebra and the universal enveloping algebra of the Lie algebra. Indeed, in this paper we use the Morita duality theorems as in [6] to prove an imprimitivity theorem for Hopf algebras with bijective antipode. When applicable this approach seems to bring out just the basic ingredients of the theory. In particular, we obtain as a corollary the exact analogue of Blattner's theorem for restricted Lie algebras $\mathfrak{g}$, which in fact was the original motivation for this work. A connection between imprimitivity systems and completely reducible restricted $\mathfrak{g}$-modules is also indicated.

2. The statement of the theorem.

Let R be a commutative ring with 1 and A a Hopf algebra over R. We employ the same notation for Hopf algebras as in [8]; in particular, we let $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ denote the diagonal map, ε the counit and S the antipode of A. Let B be a sub-Hopf algebra of A. We may then consider the base ring R as a right B-

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module via $rb = \varepsilon(b)r$, $r \in R$, $b \in B$ and form the R-space $F = \mathrm{Hom}_B(A, R)$ where A is viewed as a right B-module in a natural way.

If $f_1, f_2 \in F$ then, evidently, the product

$$(1) \qquad (f_1 * f_2)(a) = \sum f_1(a_{(2)}) f_2(a_{(1)}), \qquad a \in A$$

turns F into a R-algebra with ε as a unit element. In addition, we consider F as a left A-module with respect to the action

$$(af)(a') = f(S(a)a'), \ a, a' \in A, \ f \in F .$$

DEFINITION. Let W be an A-module. We call an F-module structure on W a system of imprimitivity for W based on B if for all $a \in A$, $f \in F$, $w \in W$ the following consistency condition is satisfied

$$\text{(C)} \qquad a(fw) = \sum (a_{(1)}f)(a_{(2)}w) .$$

THEOREM. *Let A be a Hopf algebra with bijective antipode S. Let B be a sub-Hopf algebra and assume that A as a right B-module is a finitely generated projective generator. Then an A-module W is induced (or coinduced) from a B-module if and only if W admits a system of imprimitivity based on B.*

If G is a group and H a subgroup of finite index in G then the theorem implies the statement in [6]: a G-module W is induced from an H-module if and only if W can be equipped with a $C(G/H)$-module structure such that $x(fw) = (xf)(xw)$ for all $x \in G$, $w \in W$, $f \in C(G/H)$, where $C(G/H)$ is the algebra under pointwise multiplication of all R-valued functions on G/H. As observed in [6] this formulation is very close to the imprimitivity theorem for locally compact groups ([1], [3]).

Next let k be a field of characteristic $p > 0$, $\mathfrak{g}$ a restricted Lie algebra over k and $\mathscr{U}(\mathfrak{g})$ the restricted universal enveloping algebra of $\mathfrak{g}$ ([7]). In this situation the theorem implies the following corollary, which is an exact analogue of Blattner's theorem ([2]).

COROLLARY. *Let $\mathfrak{h}$ be a restricted sub-Lie algebra of $\mathfrak{g}$ such that* $\dim(\mathfrak{g}/\mathfrak{h}) < \infty$. *Then a $\mathfrak{g}$-module W is induced (or coinduced) from an $\mathfrak{h}$-module if and only if W admits an $F = \mathrm{Hom}_{\mathscr{U}(\mathfrak{h})}(\mathscr{U}(\mathfrak{g}), k)$-module structure such that*

$$x(fw) = (xf)w + f(xw) ,$$

for all $x \in \mathfrak{g}$, $f \in F$, $w \in W$.

Finally it is perhaps worth mentioning that for a class of algebras, including all classical Lie algebras, the simple, finite dimensional algebra modules can be described in a unified fashion as submodules of coinduced modules (cf. [10], [4], [9]). It would be of interest to find an imprimitivity theorem of the type above to describe these submodules, since it might be applied, for example, to the study of the algebra radical (cf. [4]). At this point, however, we content ourselves with the following observation.

If $\mathfrak{g}$ is a classical Lie algebra over a field k of characteristic $p>3$, let $\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$ denote its root space decomposition with respect to a fixed classical Cartan subalgebra $\mathfrak{h}$. Let W be a finite dimensional simplet restricted $\mathfrak{g}$-module and $W^{\mathfrak{n}^+}$ the space of $\mathfrak{n}^+$-invariants in W. Let $P(W^{\mathfrak{n}^+})$ denote the $\mathfrak{g}$-module coinduced from the 1-dimensional $\mathscr{U}(\mathfrak{h})$-module $W^{\mathfrak{n}^+}$. As shown in [4], W may be identified with a submodule of $P(W^{\mathfrak{n}^+})$. Since $P(W^{\mathfrak{n}^+})$ in turn may be taken to lie inside the k-dual of $\mathscr{U}(\mathfrak{g})$, we may form the product $F*W$ where $*$ is as in the formula (1) above.

The space $F*W$ is a $\mathfrak{g}$-module, and it is seen at once that $F*W\subset P(W^{\mathfrak{n}^+})$. Conversely, by looking at an explicit choice of k-bases for $P(W^{\mathfrak{n}^+})$ and F one proves $P(W^{\mathfrak{n}^+})\subset F*W$. From this we get the following

COROLLARY. *Let $\mathfrak{g}$ be a classical Lie algebra and $\mathfrak{h}$ a classical Cartan subalgebra of $\mathfrak{g}$. If W is a restricted, completely reducible, finite dimensional $\mathfrak{g}$-module, then the $\mathfrak{g}$-module $F*W$ admits a system of imprimitivity based on $\mathscr{U}(\mathfrak{h})$.*

3. The proof of the theorem.

If A is any bialgebra over R and F an A-module algebra then the space $F\otimes A$ (tensor product over R) becomes an associative algebra with respect to the product

$$(f\otimes a)(f'\otimes a') = \sum f(a_{(1)}f')\otimes a_{(2)}a', \qquad f,f' \in F,\ a,a' \in A\ .$$

It is called the smash product of F and A and is denoted by $F\sharp A$ ([8]). If A is a Hopf algebra and B a sub-Hopf algebra then $F=\mathrm{Hom}_B(A,R)$ is clearly an A-module algebra with respect to the A-action $(af)(a')=f(S(a)a')$, $f\in F$, $a,a'\in A$. In the following central lemma $\mathrm{End}_B(A)$ is viewed as an R-algebra with respect to the composition of maps.

LEMMA. *Let A and B be as in the theorem and let $F=\mathrm{Hom}_B(A,R)$. Then the map*

$$\eta\colon F\sharp A \to \mathrm{End}_B(A)$$

defined by $\eta(f\sharp a)(a')=\sum f(a_{(1)}a'_{(1)})a_{(2)}a'_{(2)}$ for all $f\in F$, $a,a'\in A$ is an R-algebra isomorphism.

PROOF. A standard application of the basic properties of the antipode S shows that η is an algebra homomorphism.

To prove that η is bijective we proceed as follows. First let A' denote A as a right B-module with respect to the action $ab = a\varepsilon(b)$ for all $a \in A$, $b \in B$. Let $*$ denote the convolution product in End (A) and define

$$\beta\colon \operatorname{End}_B(A) \to \operatorname{Hom}_B(A, A')$$

by setting $\beta(\psi) = \psi * S$ for each $\psi \in \operatorname{End}_B(A)$. This is clearly a bijective map; indeed, if I is the identity map of A then $\beta^{-1}(\lambda) = \lambda * I$. In addition, the map $\alpha\colon F \otimes A \to F \sharp A$,

$$\alpha(f \otimes a) = \sum a_{(1)} f \sharp a_{(2)}, \quad f \in F,\ a \in A,$$

also has an inverse

$$\alpha^{-1}(f \sharp a) = \sum S^{-1}(a_{(1)}) f \otimes a_{(2)},$$

since S was assumed to be bijective. Let η' be the composite $\eta' = \beta\eta\alpha$.

$$\begin{array}{ccc} F\sharp A & \xrightarrow{\ \eta\ } & \operatorname{End}_B(A) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ F\otimes A & \xrightarrow[\eta']{} & \operatorname{Hom}_B(A, A') \end{array}$$

It is enough to show that η' is bijective. But for each $f \in F$, $a \in A$, $x \in A$ we get explicitly

$$\begin{aligned} \eta'(f \otimes a)(x) &= \sum (\eta(a_{(1)} f \sharp a_{(2)}) * S)(x) \\ &= \sum (a_{(1)} f)(a_{(2)} x_{(1)}) a_{(3)} x_{(2)} S(x_{(3)}) \\ &= \sum f(S(a_{(1)}) a_{(2)} x_{(1)}) a_{(3)} \varepsilon(x_{(2)}) \\ &= \sum f(x_{(1)} \varepsilon(x_{(2)})) \varepsilon(a_{(1)}) a_{(2)} \\ &= f(x) a . \end{aligned}$$

Now to see that η' is bijective we first let M be a finitely generated free B-module with basis $\{m_1, \ldots, m_s\}$. In this case the map

$$\eta_M\colon \operatorname{Hom}_B(M, R) \otimes A \to \operatorname{Hom}_B(M, A')$$

which is given by $\eta_M(f \otimes a)(x) = f(x)a$ for all $a \in A$, $x \in M$, $f \in \operatorname{Hom}_B(M, R)$ is seen to have an inverse

$$\eta_M^{-1}(\lambda) = \sum_{i=1}^{s} f_i \otimes \lambda(m_i), \qquad \lambda \in \operatorname{Hom}_B(M, A'),$$

where $\{f_1, \ldots, f_s\}$ is the basis of $\operatorname{Hom}_B(M, R)$ dual to $\{m_1, \ldots, m_s\}$.

Finally, since A is a finitely generated projective B-module, A appears as a direct summand of some finitely generated free B-module M. The bijective map

η_M restricted to the component $\mathrm{Hom}_B(A,R)\otimes A$ of $\mathrm{Hom}_B(M,R)\otimes A$ is just η' and it follows easily that η' is bijective too. The lemma is thus proved.

To prove the theorem let W be an arbitrary A-module, and view A as a subalgebra of $\mathrm{End}_B(A)$ via the left regular representation, i.e. for all $a \in A$, a and l_a, the left multiplication by a, are identified. Assume first that W admits a system of imprimitivity based on B. In this case the consistency condition (C) implies that the rule

$$(f\sharp a)w = f(aw), \qquad a \in A,\ f \in F$$

turns W into an $F\sharp A$-module. Using the algebra isomorphism η^{-1} we get an $\mathrm{End}_B(A)$-module structure on W. This is an extension of the original A-module structure on W since

$$l_a w = \eta^{-1}(l_a)w = (\varepsilon\sharp a)w = \varepsilon(aw) = aw$$

for each $a \in A$, $w \in W$.

Conversely, assume that the A-module structure on W could be extended to an $\mathrm{End}_B(A)$-module structure. Again, via η, an $F\sharp A$-module action on W is obtained. This in turn is easily seen to equip W with both A- and F-module structures via

$$a\circ w = (\varepsilon\sharp a)w,\ w \in W,\ a \in A$$

$$f\circ w = (f\sharp 1)w, f \in F .$$

These satisfy the consistency condition (C), and moreover

$$a\circ w = (\varepsilon\sharp a)w = \eta(\varepsilon\sharp a)w = l_a w = aw ,$$

i.e. the old and new A-module structures coincide. As in [6] the theorem is now a consequence of the following general result, which is an immediate corollary to the Morita duality theorems: if a ring A with 1 is a finitely generated projective generator over a subring B, then an A-module W is induced (or coinduced) from a B-module if and only if the A-action on W can be extended to an $\mathrm{End}_B(A)$-action.

Remark. If A is a Hopf algebra, B a sub-Hopf algebra and V a B-module then $A\otimes_B V$ always admits a system of imprimitivity based on B. This is seen by defining $f(a\otimes v)=\sum f(a_{(1)})a_{(2)}\otimes v$ for each $a \in A$, $v \in V$, $f \in F$. But we do not know to what extent the condition "A a finitely generated projective generator" in the theorem can be weakened.

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