MATH. SCAND. 41 (1977), 199–217

GEOMETRIC INTERPRETATIONS OF THE GENERALIZED HOPF INVARIANT

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1. Introduction and statement of results.

There is a well-known procedure, due to Pontrjagin and Thom, which identifies the homotopy group $\pi_{r+n+m}(S^{n+m})$ with the bordism group of r-dimensional submanifolds of $\mathbf{R}^{r+n+m}$ with framed normal bundles. Expressed in this language, the m-fold suspension homomorphism

$$E^m\colon \pi_{r+n}(S^n) \to \pi_{r+n+m}(S^{n+m})$$

is induced by the inclusion $\mathbf{R}^{r+n} \subset \mathbf{R}^{r+n+m}$. Thus an element in the image of E^m can be represented by an embedding $g\colon M^r \to \mathbf{R}^{r+n+m}$, together with a framing $\bar{g}\colon \nu(g) \xrightarrow{\cong} \varepsilon^{n+m}$ (=trivial $(n+m)$-bundle) of the normal bundle $\nu(g)$ of g, having the following properties

$$(\mathfrak{E}) \quad g(M^r) \subset \mathbf{R}^{r+n}, \quad \text{and}$$

$(\mathfrak{J})$ the composition

$$\bar{g}_2\colon \varepsilon^m \subset \varepsilon^{r+n} \oplus \varepsilon^m \xrightarrow{\text{proj}} \nu(g) \xrightarrow{\bar{g}} \varepsilon^{n+m}$$

is given by the obvious inclusion id.

In this paper we will study intermediate situations, namely framed embeddings where only one of these additional properties is required, and we will measure to what extent the other property fails to hold. This will lead to two geometric interpretations of the generalized Hopf invariant of Whitehead [13] and James [5] and, more generally, of the EHP-sequence of Whitehead and James.

Most of our results can be summarized as follows.

Main Theorem. *Assume* $0<r\leqq 2n-2$ *and* $m\geqq 0$. *Then there is a commuting diagram*

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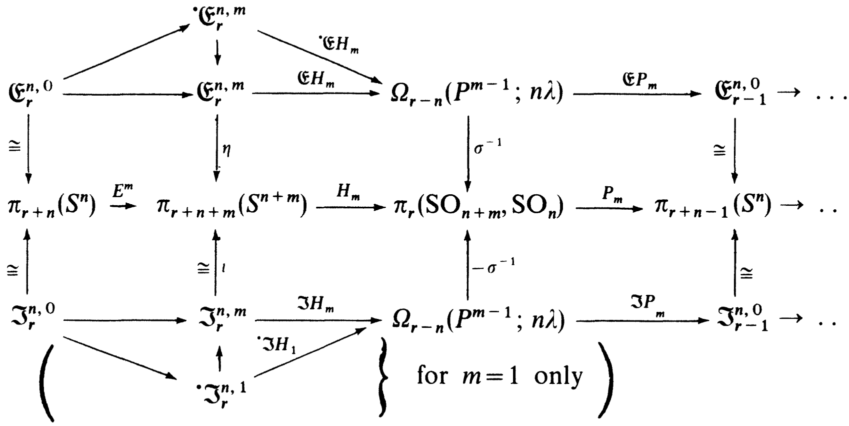
[1] Research partially supported by the Sonderforschungsbereich 40, Mathematisches Institut der Universität Bonn.

Received October 26, 1976.

(1.1)



where all horizontal sequences are exact, and all vertical arrows are isomorphisms.

Here the middle sequence is the EHP-sequence of James [5]. The other terms are bordism groups and homomorphisms (defined below) which give the desired geometric interpretations. For $m=1$, the square involving the geometric Hopf invariant $\mathfrak{E}H_m$ was already established and beautifully applied by R. Wood [15].

We now proceed to describe the terms occuring in the diagram above. $\mathfrak{E}_r^{n,m}$ is the bordism group of pairs $(M, \bar{g})$, where M is a closed smooth r-dimensional submanifold of $\mathbb{R}^{r+n}$, and $\bar{g}$ is a framing of its normal bundle in the higher-dimensional space $\mathbb{R}^{r+n+m}$. As already noted in [15], the J-homomorphism provides examples of such pairs.

In order to determine whether $(M, \bar{g})$ satisfies also the condition $(\mathfrak{J})$, consider the "second component" $\bar{g}_2\colon \varepsilon^m \to \varepsilon^{n+m}$ of $\bar{g}$. Let $\bar{u}\colon \varepsilon^m \to \varepsilon^{n+m}$ be a homomorphism over $M \times I$ which restricts to $\bar{g}_2$ over $M \times \{0\}$, and to the natural inclusion id over $M \times \{1\}$. If $\bar{u}$ has no singularity, then the class $[M, \bar{g}]$ lies in the image of the forgetful homomorphism $\mathfrak{E}_r^{n,0} \to \mathfrak{E}_r^{n,m}$. Hence the (non-degenerate) singularity of $\bar{u}$ provides a measure for the deviation. We can extract the following data (see section 2 for the details):

(i) an $(r-n)$-manifold X (the locus of points where $\bar{u}$ fails to be injective);
(ii) a sub-linebundle λ of ε^m over X (the kernel bundle of $\bar{u}$), or, equivalently, a map $h\colon X \to P^{m-1}$; and
(iii) a (stable) isomorphism $\bar{h}\colon TX \oplus \lambda^n \cong \varepsilon^r$ (derived from the tangent map of $\bar{u}$ and from $\bar{g}$).

Triples of this type give rise to the normal bordism group $\Omega_{r-n}(P^{m-1}; n\lambda)$ (cf.

[8]), and we define the *singularity Hopf invariant* by

$$\mathfrak{E}H_m([M,\bar{g}]) = [X,\lambda,\bar{h}] .$$

To obtain an alternate description, consider $\bar{g}_2$ as a map into the Stiefel manifold $V_{n+m,m}=SO_{n+m}/SO_n$. Up to homotopy, $\bar{g}_2$ factors through the canonical map i,

$$\bar{g}_2 \cong i\circ j\colon M \xrightarrow{j} P^{n+m-1}/P^{n-1} \xrightarrow{i} V_{n+m,m}$$

(see section 2 or [6]). Now recall that P^{n+m-1}/P^{n-1} is the Thom space of the nontrivial bundle λ^n over P^{m-1}. Making j transversal to P^{m-1}, we get the triple

$$(X = j^{-1}(P^{m-1}),\ h = j\,|\,X,\ \bar{h}\colon TX\oplus\lambda^n \cong TM\,|\,X \cong \varepsilon^r)$$

which also represents the singularity Hopf invariant.

Both versions of this construction lead also to the isomorphism σ of (1.1) which is a variation of the singularity isomorphism of [8, § 5]. In particular,

$$\sigma = \mathrm{tp}\circ i_*^{-1}\colon \pi_r(SO_{n+m}, SO_n) = \pi_r(V_{n+m,m}) \cong \pi_r(P^{n+m-1}/P^{n-1})$$
$$\cong \Omega_{r-n}(P^{m-1};\, n\lambda) ,$$

where tp is the obvious Thom–Pontrjagin isomorphism.

$\mathfrak{E}P_m$ is defined as follows. If $(X,\lambda,\bar{h})$ represents an arbitrary class in $\Omega_{r-n}(P^{m-1};\, n\lambda)$, embed X into $\mathbf{R}^r$ and identify a tubular neighbourhood with λ^n via $\bar{h}$. Each element y in the sphere bundle $S(\lambda^n)$ determines in an obvious way a line in $\mathbf{R}^n$ and hence a reflection l_y flipping this line around; thus we get an automorphism l of ε^n. Also, realize λ as a subbundle of ε^n, and use the resulting reflections to define another automorphism l_X of ε^n first over X, and then, by pullback, over $S(\lambda^n)$. Now consider $S(\lambda^n)$ as a submanifold of $\mathbf{R}^{r+n-1}$, framed by

$$\bar{g}\colon \nu(S(\lambda^n), \mathbf{R}^r)\oplus\varepsilon^{n-1} = \varepsilon^n \xrightarrow{l\circ l_X} \varepsilon^n ,$$

where the very first bundle to the left is trivialized via outward pointing vectors. Put

$$\mathfrak{E}P_m([X,\lambda,\bar{h}]) = [S(\lambda^n),\bar{g}] .$$

Note that $\mathfrak{E}P_m$ factors through $\mathfrak{E}_{r-1}^{1,n-1}$.

Finally, let η denote the forgetful homomorphism

$$\eta\colon \mathfrak{E}_r^{n,m} \to \mathfrak{E}_r^{n+m,0} \cong \pi_{r+n+m}(S^{n+m}) .$$

For identifications such as the one to the right, we use the following general procedure. Represent an element of $\pi_p(S^q)$ by a smooth map $f\colon S^p \to S^q$ and choose a regular value $z \in S^q - \{*\}$ to obtain the framed manifold $f^{-1}(z)$ in $S^p - \{*\} \cong \mathbf{R}^p$. Here the stereographic projection from the north pole $*$ is also used to orient S^q.

The $\mathfrak{C}$-sequence has the following variant. Define $^{\cdot}\mathfrak{C}_r^{n,m}$ to be the bordism set of triples $(M, B, \bar{g})$, where M and $\bar{g}$ are as before and $B \subset M$ is an embedded r-ball outside of which $\bar{g}_2$ is given by id. A bordism of M is required to contain an isotopy of B. Thus $[\bar{g}\,|\,B] \in \pi_r(\mathrm{SO}_{n+m}, \mathrm{SO}_n)$ is a well-defined bordism invariant, and we denote its image under σ by $^{\cdot}\mathfrak{C}H_m([M, B, \bar{g}])$. The map from $\mathfrak{C}_r^{n,0}$ to $^{\cdot}\mathfrak{C}_r^{n,m}$ is obtained from unlinked disjoint union with a trivial sphere which contains the needed ball.

Now we come to the terms in the lower half of diagram (1.1). $\mathfrak{J}_r^{n,m}$ denotes the bordism group of triples $(M, g, \bar{g})$, where

(i) M is closed r-manifold;
(ii) $g = (g_1, g_2)\colon M \to \mathbf{R}^{(r+n)+m}$ is a smooth embedding such that $g_1\colon M \to \mathbf{R}^{r+n}$ is an immersion; and
(iii) $\bar{g}_1\colon \nu(g_1) \xrightarrow{\cong} \varepsilon^n$.

This corresponds to the requirement ($\mathfrak{J}$) of the introduction.

We want to measure to what extent $(M, g, \bar{g}_1)$ fails to satisfy also the requirement ($\mathfrak{C}$). Clearly the trouble lies at the selfintersections of g_1. We may assume that they occur only at transverse double points. Then their locus forms a closed $(r-n)$-manifold X. For each $x \in X$, the two points $x', x'' \in g_1^{-1}(x)$ have different images under g_2 and hence determine a line $\lambda_x \subset \mathbf{R}^m$. Thus we get a bundle $\lambda \subset \varepsilon^m$ over X. Furthermore, the normal space of X has an natural splitting

$$\nu_x(X, \mathbf{R}^{r+n}) = \mathbf{R}^n(x') \oplus \mathbf{R}^n(x'') .$$

Here e.g. $\mathbf{R}^n(x')$ denotes the normal space of X in the sheet through x'; it gets a frame $e'_1, \ldots, e'_n$ from the normal framing of the other sheet. Now we map $e'_i - e''_i$ to the unit vector in λ_x on the side of x', and we map $e'_i + e''_i$ to 1, $i = 1, \ldots, n$. This leads to canonical isomorphisms $\nu(X, \mathbf{R}^{r+n}) \cong \lambda^n \oplus \varepsilon^n$ and

$$\bar{h}\colon TX \oplus \lambda^n \oplus \varepsilon^n \cong \varepsilon^{r+n} .$$

We define the *double point Hopf invariant* by

$$\mathfrak{J}H_m([M, g, \bar{g}_1]) = [X, \lambda, \bar{h}] .$$

Next recall that $[\iota_n, \iota_n]$ is represented by

$$L = S^{n-1} \times \{0\} \cup \{0\} \times S^{n-1} \subset S^{2n-1} \quad (\subset \mathbf{R}_1^n \times \mathbf{R}_2^n)$$

with standard framing. Further observe that this framed manifold remains unchanged when we switch the factors $\mathbf{R}_i^n = \mathbf{R}^n$, $i = 1, 2$. Now, given an arbitrary class $[X, \lambda, \bar{h}] \in \Omega_{r-n}(P^{m-1}; n\lambda)$ and $x \in X$, each orientation of λ_x determines a diagonal of $\lambda_x^n \times \mathbf{R}^n$, and we obtain an isomorphism $\lambda_x^n \times \mathbf{R}^n \cong \mathbf{R}_1^n \times \mathbf{R}_2^n$, well defined up to switches. Thus we can realize L first in $\lambda_x^n \times \mathbf{R}$ and then in

$\lambda_x^n \times \mathsf{R}^{n-1}$, using the stereographic projection from $(0,\ldots,0;\sqrt{2},0,\ldots,0)$. Performing this construction in each fiber, we obtain a framed submanifold of $\lambda^n \times \mathsf{R}^{n-1}$ and hence of R^{r+n-1}, since we can use $\bar{h}$ to embed the total space λ^n as open set in R^r. This *generalized composition with* $[\iota_n, \iota_n]$ defines the homomorphism $\mathfrak{J}P_m$.

Finally, the homomorphism ι is obtained by completing the framing $\bar{g}_1$ to a framing of $\nu(g)$ in the obvious fashion. According to the main theorem, ι is an isomorphism. In particular, up to bordism, framed embeddings into R^{r+n+m} can be compressed to immersions into R^{r+n} in a way compatible with the framing. When the target group of ι is in the stable range, this follows readily from immersion theory. However, for $r \geqq n+m$, it is not obvious why an immersion should be the projection of an embedding.

For $m=1$, we have the following variant of $\mathfrak{J}_r^{n,m}$. Let ${}^{\bullet}\mathfrak{J}_r^{n,1}$ be the bordism set of quadruples $(M, B, g_1, \bar{g}_1)$, where $g_1\colon M \to \mathsf{R}^{r+n}$ is an immersion framed by $\bar{g}_1$, and $B \subset M$ is a closed embedded r-ball such that $g_1 | B$ and $g_1 | M-B$ are embeddings. A bordism of M is again required to contain an isotopy of B.

In this setting the double point Hopf invariant ${}^{\bullet}\mathfrak{J}H_1$ is simply given by the transversal intersection $g_1(B) \cap g_1(M-B)$, framed in the ball $g_1(B)$ by $-\bar{g}_1 | M-B$. Furthermore, ${}^{\bullet}\mathfrak{J}_r^{n,1}$ is mapped to $\mathfrak{J}_r^{n,1}$ by slightly lowering the x_{r+n+1}-values of $g_1(B)$ in R^{r+n+1}.

Our results give two quite distinct interpretations of the (generalized) Whitehead product and especially of the (generalized) Hopf invariant. Consider e.g. the situation in which Hopf originally defined his invariant, i.e. $m=1$ and $r=n$, and represent an element $[f] \in \pi_{2n+1}(S^{n+1})$ by the framed submanifold $M^n = f^{-1}$ (regular value) in R^{2n+1}. The singularity interpretation describes $H_1([f])$ as the mapping degree of $\bar{g}_2\colon M \to S^n$, where we assume that M is already embedded in R^{2n}, and $\bar{g}_2(x)$ is the normal vector $(0,\ldots,0,1) \in \mathsf{R}^{2n+1}$, expressed in terms of the normal $(n+1)$-frame at $x \in M$ (see also [7, Lemma 6.1], and [15]). On the other hand, in the double point interpretation we have to compress M^n to an immersed manifold in R^{2n}. Then $H_1([f])$ is the sėlfintersection number; it is a well defined integer even for n odd, since the last component in R^{2n+1} distinguishes an "upper" and a "lower" sheet at each double point. (See Figure 1).

Note that the particular example in Figure 1 is quite exceptional: e.g. a framed immersion of an n-manifold in R^{2n} with odd intersection number can occur only in dimension 1, 3 or 7. This is equivalent to Adams' result on elements of Hopf invariant 1.

If we drop the metastable range condition $r \leqq 2n-2$, then many of the vertical arrows in (1.1) may fail to be bijective or even defined. (Actually, we

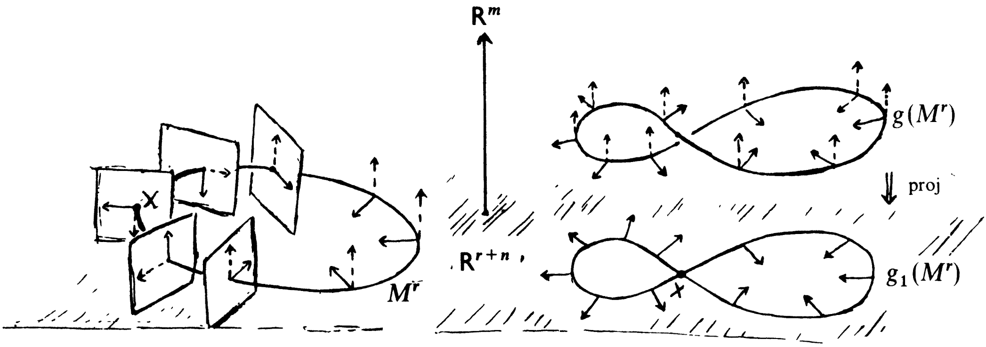


Figure 1. Singularity versus double point Hopf invariant for the same homotopy class.

can show that precisely those arrows continue to be bijective for all $n, r > 0$ which are marked as isomorphisms in (1.1).) However, all five horizontal sequences continue to be exact and to have a specific geometric interpretation, provided we replace the third terms by suitable substitutes (which will be different for each sequence). For example, if we replace $\Omega_{r-n}(P^{m-1}; n\lambda)$ by $\pi_r(\mathrm{SO}_{n+m}, \mathrm{SO}_n)$ in the very top line, we obtain an exact sequence (at least for $n \geqq 3$) which can be identified canonically with a homotopy sequence involving the spaces F_{n+m}, F_n, SO_{n+m} and SO_n. On the other hand, replacing $\Omega_{r-n}(P^{m-1}; n\lambda)$ by $\mathfrak{I}_{r-n}^{n,0}$ in the very bottom line, we get

$$(1.2) \qquad \ldots \to \mathfrak{I}_r^{n,0} \to {}^{\bullet}\mathfrak{I}_r^{n,1} \to \mathfrak{I}_{r-n}^{n,0} \to \mathfrak{I}_{r-1}^{n,0} \to \ldots$$

which is canonically isomorphic (at least for $n \geqq 3$) to the homotopy sequence of the fibration $F_n \subset G_{n+1} \xrightarrow{p} S^n$. Here F_n denotes the space of based maps $S^n \to S^n$ of degree 1, and the subspaces SO_n and G_n are given by suspensions of orthogonal, resp. arbitrary unbased, maps on S^{n-1} of degree 1. (Observe that G_n is written as "G_{n-1}" in [5]). Spheres are based by their north poles $(0, \ldots, 0, 1)$, and function spaces by their unit.

Our proofs use only differential topology and very basic methods of homotopy theory, with the exception of the following slightly strengthened form of James' key result in [5].

THEOREM 1.3. *The homomorphism (induced by the inclusion)*

$$\varphi\colon \pi_r(\mathrm{SO}_{n+m}, \mathrm{SO}_n) \to \pi_r(F_{n+m}, F_n)$$

is bijective for $r \leqq 2n-2$ and onto for $r = 2n-1$.

We will outline a geometric proof in section 3. There we will also discuss the first and the fifth sequences of (1.1). The other two geometric sequences are developped very parallelly in sections 2 and 4. However, for the $\mathfrak{I}$-sequence we need an additional crucial piece of information: we have to know how singularities can be deformed into double points. This is provided by the isotopy theorem in section 5, which seems to be of independent interest.

In this paper we have restricted our attention mainly to the metastable range. In a further paper [9] we will remove this restriction. Then triple, quadruple, etc. points of immersions (as well as higher kernel dimensions at singularities) will enter the picture and lead e.g. to higher Hopf invariants. Our starting point will be the observation that many of the well-established models for loop-spaces can be viewed in a certain sense as "Thom spaces for immersions".

2. Singularities of frame deformations.

Throughout the paper we will assume that $m \geqq 0$ and $n, r > 0$. Additional assumptions will be stated explicitly in the propositions. In the text of sections 2 and 4, however, we will often assume $r \leqq 2n-2$ without specifically mentioning it; we need this condition for various embedding, isotopy and destabilization arguments.

First we discuss the $\mathfrak{E}$-sequence. Consider pairs $(M, \bar{g})$ where M is a compact r-dimensional submanifold of

$$\mathbf{R}_+^{r+n} = \{x \in \mathbf{R}^{r+n} \mid x_{r+1} \geqq 0\},$$

possibly with boundary $\partial M = M \pitchfork \partial \mathbf{R}_+^{r+n}$, and

$$\bar{g}: v(M, \mathbf{R}_+^{r+n}) \oplus \varepsilon^m \xrightarrow{\cong} \varepsilon^{n+m}$$

is a framing of the normal bundle of M in $\mathbf{R}_+^{r+n+m}$ such that the "second component"

$$\bar{g}_2: \varepsilon^m \to \varepsilon^{n+m} \quad (= \varepsilon^n \oplus \varepsilon^m)$$

restricts, over ∂M, to the identical inclusion id. Two such pairs $(M, \bar{g})$ and $(M', \bar{g}')$ are called *bordant* if there is an $(r+1)$-dimensional submanifold N of $\mathbf{R}_+^{r+n} \times \mathbf{R}$ with a framing $\bar{G}$ of its normal bundle in $\mathbf{R}_+^{r+n+m} \times \mathbf{R}$ such that (i) $\partial N = N \pitchfork \partial \mathbf{R}_+^{r+n} \times \mathbf{R}$ (the symbol $\pitchfork$ means transversal intersection); (ii) $\bar{G}$ restricts to id on $\varepsilon^m \mid \partial N$; (iii) $N \cap (\mathbf{R}_n^{r+n} \times I)$ is compact and (iv) at each level $\mathbf{R}_+^{r+n} \times \{t\}$, where $t < \frac{1}{5}$, respectively $t > \frac{4}{5}$, N and $\bar{G}$ are given by $(M, \bar{g})$, respectively $(M', \bar{g}')$. The resulting set of bordism classes is made into a group $\operatorname{rel} \mathfrak{E}_r^{n,m}$ via unlinked disjoint union. The corresponding absolute group $\mathfrak{E}_r^{n,m}$ is defined the same way, but with no boundaries allowed.

There is a homomorphism

$$\mathfrak{E}H_m^{\mathrm{rel}}: \operatorname{rel} \mathfrak{E}_r^{n,m} \to \Omega_{r-n}(P^{m-1}; n\lambda)$$

defined as follows. Given a class $[M,\bar{g}]$, consider a (non-degenerate) morphism $\bar{u}\colon \varepsilon^m \to \varepsilon^{n+m}$ over $M\times I$ which restricts to $\bar{g}_2$ over $M\times\{0\}$ and to id over $M\times\{1\}\cup\partial M\times I$. The singularity X of $\bar{u}$ comes with the kernel bundle $\lambda\subset\varepsilon^m$ and an isomorphism $\bar{h}\colon TX\oplus\lambda^n\cong\varepsilon^r$ extracted from the tangent map of $\bar{u}$ and from the stable framing of TM (for details see [8]. To specify $\bar{h}$ more precisely, deform the canonical isomorphism

$$\begin{aligned} TX\oplus\lambda^{n+m} &= TX\oplus(\lambda\otimes\mathbf{Coker})\oplus(\lambda\otimes\mathbf{Im}) \cong TM\,|\,X\oplus TI\,|\,X\oplus(\lambda\otimes\mathbf{Ker}^{\perp}) \\ &\cong \varepsilon^r \oplus(\lambda\otimes\lambda)\oplus(\lambda\otimes\mathbf{Ker}^{\perp}) \\ &= \varepsilon^r \oplus\lambda^m \end{aligned}$$

until it is of the form $\bar{h}'\oplus\mathrm{Id}_{\lambda^m}$, and define

$$\bar{h} = \bar{h}'\circ(\mathrm{Id}\oplus-\mathrm{Id})\colon TX\oplus\lambda^n \xrightarrow{\mathrm{Id}\oplus-\mathrm{Id}_{\lambda^n}} TX\oplus\lambda^n \xrightarrow{\bar{h}'} \varepsilon^r;$$

the correction term $-\mathrm{Id}_{\lambda^n}$, which matters only when n is odd, would be quite unnatural in [8], but it is very convenient here.) Another characterization of these data is given below. The normal bordism class $\mathfrak{E}H_m^{\mathrm{rel}}([M,\bar{g}])$ of $(X,\lambda,\bar{h})$ depends only on $[M,\bar{g}]$.

This procedure also defines the isomorphism σ in (1.1); just replace M by D^r and $\bar{g}_2$ by elements of

$$[(D^r,S^{r-1}),(V_{n+m,m},\mathrm{id})] = \pi_r(\mathrm{SO}\,(n+m),\mathrm{SO}\,(n))\,.$$

Returning to the previous situation, let us use the singularity data to describe the following standard form of $\bar{g}_2$. There is an embedding identifying the total space of λ^n with an open subset of $M-\partial M$ such that $\bar{g}_2=l\,|\,\varepsilon^m$ over λ^n and $\bar{g}_2=\mathrm{id}$ everywhere else; here we define the automorphism l of ε^{n+m} (over $\lambda^n\cong\lambda^{*n}$ as *base* space, λ^* the dual of λ) as follows: for any $y\in\lambda_x^{*n}$, where $x\in X$, l_y is the reflection of $\mathbf{R}^{n+m}$ determined by the line $\{(y(v),v)\in\mathbf{R}^n\oplus\mathbf{R}^m \mid v\in\lambda_x\}$. In other words, on a fiber λ_x^n (of the tubular neighbourhood λ^n of X in M) l is given by a very natural map into the projective space of $\mathbf{R}^n\oplus\lambda_x\subset\mathbf{R}^{n+m}$: at the center of λ_x^n this map takes λ_x as its value, and along each ray it rotates towards the line in $\mathbf{R}^n$ determined by the ray.

It follows from the homotopy classification theorem of [8] (cf. also [6], 8.1) that we can always deform $\bar{g}_2$ into standard form. The triple $(S,\lambda,\bar{h})$ involved in the construction (where $\bar{h}$ destabilizes to identify $\nu(X,M)$ with λ^n) represents $[M,\bar{g}]$.

Now we can isotop M in $\mathbf{R}_+^{r+n}$ until the closure of λ^n lies at a certain fixed $(x_{r+1},0,\ldots,0)$-level and the rest of M takes lower x_{r+1}-values.

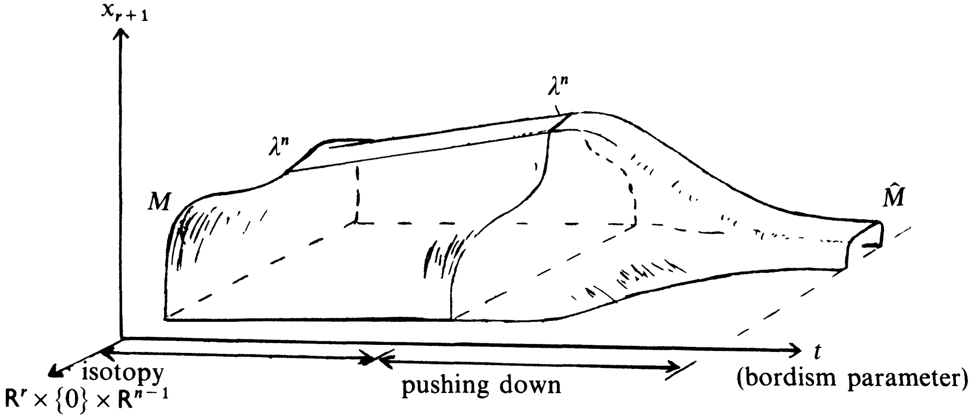


Figure 2. A bordism from M to the model $\hat{M}$.

Pushing the complement of this closure down to negative x_{r+1}-values (cf. Fig. 2) then leads to a bordism from $(M, \bar{g})$ to the following model $(\hat{M}, \hat{\bar{g}})$. $\hat{M}$ is a closed tubular neighbourhood of some embedding of X into R^r. Using $\bar{h}$, we can identify its interior with the total space of λ^n. Thus, we have in particular an automorphism l of ε^{n+m} over $\hat{M}$ as above. Moreover, we can represent the bundle λ over X as a subbundle of ε^n; the resulting reflections extend to an automorphism l_X of ε^n over all of $\hat{M}$. Now we define the framing $\hat{\bar{g}}$ by

$$\hat{\bar{g}}\colon \nu(\hat{M}, \mathsf{R}^{r+n+m}) = \varepsilon^{n+m} \xrightarrow{l \circ (l_X \times \mathrm{id})} \varepsilon^{n+m};$$

the correction term l_X guarantees that $\hat{\bar{g}} \mid X \sim \mathrm{Id}$. Finally we lift all but $\partial\hat{M}$ to positive x_{r+1}-levels.

This model construction yields an inverse homomorphism of $\mathfrak{C}H_m^{\mathrm{rel}}$. Hence we have

Proposition 2.2. *The relative singularity homomorphism* $\mathfrak{C}H_m^{\mathrm{rel}}$ *is an isomorphism for* $r \leqq 2n-2$.

The relevance of this result stems from the fact that our relative bordism group occurs in the diagram (2.3) of horizontal exact sequences.

Here it is convenient (and possible for arbitrary $n, r > 0$) to identify all terms in the middle sequence with bordism groups of framed embeddings. E.g., given $[f] \in \pi_r(F_{n+m}, F_n)$, we get a map $f_0\colon D^r \times \mathsf{R}^{n+m} \to S^{n+m}$, which may be assumed to have $-* \in S^n$ as a regular value. Now we interprete D^r as the positive half-sphere in R^{r+1}, with tubular neighbourhood $D^r \times \mathsf{R}^{n+m}$ in R_+^{r+n+m} in the obvious way, and in each normal slice R^{n+m} we push a neighbourhood of $-*$ in S^n into a neighbourhood of 0 in $\mathsf{R}^n \times \{0\}$ by an ambient isotopy. Thus

$$\begin{array}{ccccccc}
 & & & \Omega_{r-n}(P^{m-1}; n\lambda) & & & \\
 & & \overset{\mathfrak{E}H_m}{\nearrow} & \uparrow \mathfrak{E}H_m^{\mathrm{rel}} \cong & \overset{\mathfrak{E}P_m}{\searrow} & & \\
\dots \to \mathfrak{E}_r^{n,0} & \longrightarrow & \mathfrak{E}_r^{n,m} & \longrightarrow \mathrm{rel}\,\mathfrak{E}_r^{n,m} & \longrightarrow & \mathfrak{E}_{r-1}^{n,0} \to \dots \\
\cong\downarrow & & \downarrow & \downarrow & & \cong\downarrow \\
\dots \to \pi_r(F_n) & \longrightarrow & \pi_r(F_{n+m}) & \longrightarrow \pi_r(F_{n+m}, F_n) & \longrightarrow & \pi_{r-1}(F_n) \to \dots \\
\cong\updownarrow & & \cong\updownarrow & \uparrow\varphi & & \cong\updownarrow \\
\pi_{r+n}(S^n) & \xrightarrow{E^m} & \pi_{r+n+m}(S^{n+m}) & \xrightarrow{H_m} \pi_r(\mathrm{SO}_{n+m}, \mathrm{SO}_n) & \xrightarrow{P_m} & \pi_{r+n-1}(S^n) \to \dots
\end{array}$$

Diagram (2.3)

$f_0^{-1}(-*)$ becomes a framed submanifold of R_+^{r+n+m} whose boundary lies, and is nicely framed, in $\partial \mathsf{R}_+^{r+n}$. (Here we orient all spheres via the stereographic projections from their north poles.) The resulting relative bordism class gives the required geometric description of $[f]$ (compare also [3, 10.5]). Note that the corresponding construction for $\pi_r(\mathrm{SO}_{n+m}, \mathrm{SO}_n)$ factors through $\mathrm{rel}\,\mathfrak{E}_r^{n,m}$.

All unnamed arrows in the diagram above stand for obvious forgetful or boundary homomophisms. Clearly everything commutes here, in part due to the very definition of James' EHP-sequence (we identify $\pi_r(F_k) \cong \mathfrak{E}_r^{k,0} \cong \pi_{r+k}(S^k)$ according to our conventions). Moreover, both triangles in the square

$$(2.4)\qquad \begin{array}{ccc}
\Omega_{r-n}(P^{m-1}; n\lambda) & \xleftarrow[\cong]{\mathfrak{E}H_m^{\mathrm{rel}}} & \mathrm{rel}\,\mathfrak{E}_r^{n,m} \\
\sigma \big\updownarrow \cong & \nearrow & \big\downarrow \\
\pi_r(\mathrm{SO}_{n+m}, \mathrm{SO}_n) & \xrightarrow{\varphi} & \pi_r(F_{n+m}, F_n)
\end{array}$$

commute. The statements of the main theorem concerning the $\mathfrak{E}$-sequence follow now from Theorem 1.3, diagrams (2.4) and (2.3), and the five lemma. Furthermore, we obtain that $\eta\colon \mathfrak{E}_r^{n,m} \to \pi_{r+n+m}(S^{n+m})$ is still onto for $r = 2n-1$.

REMARK 2.5. The straight top line in (2.3) is an exact sequence of group homomorphisms for $n > 0$ and arbitrary r. Moreover, $\mathfrak{E}H_m^{\mathrm{rel}}$ (and hence $\mathfrak{E}H_m$), as well as further singularity invariants corresponding to higher kernel dimensions, are also well-defined group homomorphisms for arbitrary r (compare [8, 2.15]).

3. The homotopy of certain function spaces.

In this section we briefly outline a geometric proof of Theorem 1.3, based on the work in [11]. Then we discuss the sequences in (1.1) which involve ${}^{\bullet}\mathfrak{E}_r^{n,m}$ and ${}^{\bullet}\mathfrak{J}_r^{n,1}$. Some familiarity with the homotopy groups of squares (and cubes) of maps will be helpful, see e.g. § 0 of [11] or [14].

Proof of Theorem 1.3. Considering an exact sequence of the square

$$Q_{n,m} = \begin{pmatrix} F_{n+m} & F_n \\ SO_{n+m} & SO_n \end{pmatrix}$$

we can reword 1.3 as follows:

$$\pi_r(Q_{n,m}) = 0 \quad \text{for } r<2n . \tag{3.1}$$

Now we know from [11, Remark (2) on p. 454] that

$$\pi_r(F_{n+1}, G_{n+1}) = 0 \quad \text{for } r<2n . \tag{3.2}$$

Moreover, we have the commutative diagram

$$\begin{array}{ccccccc}
\ldots \to & \pi_r(SO_{n+1}, SO_n) & \to & \pi_r(F_{n+1}, F_n) & \to & \pi_r(Q_{n,1}) & \to \ldots \\
& \downarrow \cong & & \| & & \downarrow & \\
\ldots \to & \pi_r(G_{n+1}, F_n) & \longrightarrow & \pi_r(F_{n+1}, F_n) & \to & \pi_r(F_{n+1}, G_{n+1}) & \to \ldots
\end{array}$$

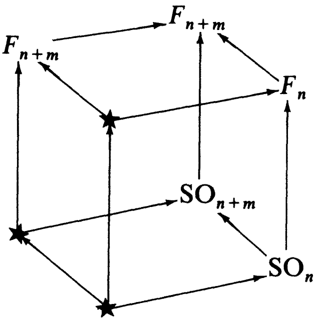
Thus we get $\pi_r(Q_{n,1})=0$ from the five lemma. The full result follows now by induction over m from the exact sequence

$$\pi_r(Q_{n,1}) \to \pi_r(Q_{n,m}) \to \pi_r(Q_{n+1,m-1}) .$$

Remark 3.3. In [11] the proof that $\pi_r(F_{n+1}, G_{n+1})=0$ for $r<2n$ is purely geometric, involving first surgery and then engulfing. Thus we can establish Theorem 1.3, and hence the EHP-sequence, in purely geometric fashion. In [5], James proves that $\pi_r(F_{n+1}, G_{n+1})=0$ for $r<2n-1$ essentially by first assuming the exactness of an EHP-sequence.

Next consider the cube

$C_{n,m}$ =



The double rôle of F_{n+m} here allows us to simplify the description of $\pi_{r+1}(C_{n,m})$, and to interprete its elements as homotopy classes of maps

$$f\colon (D^r; D^r_+, D^r_-, D^{r-1}, S^{r-1}) \to (F_{n+m}; SO_{n+m}, F_n, SO_n, *) ,$$

where e.g. $D^r_+ = \{x \in D^r \mid x_r \geqq 0\}$. Now identify D^r with the northern hemisphere of S^r and extend f trivially. We obtain a map f_0 defined on the tubular neighbourhood $S^r \times \mathrm{R}^{n+m}$ of S^r in R^{r+n+m} and with values in S^{n+m}. After a suitable canonical deformation, we may assume that $-* \in S^{n+m}$ is a regular value of f_0, and, moreover, that $f_0^{-1}(-*)$ lies already in R^{r+n} and the second component $\bar{g}_2$ of the induced framing is trivial outside of the (half-)ball $D^r_+ \subset f_0^{-1}(-*)$. This construction defines a map

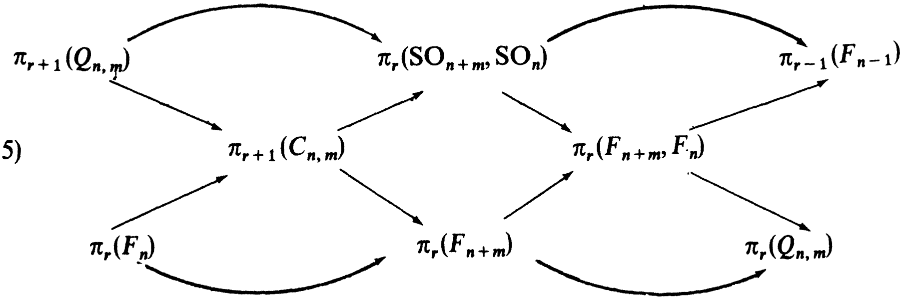
$$\mu\colon \pi_{r+1}(C_{n,m}) \to {}^{\bullet}\mathfrak{E}_r^{n,m}$$

which commutes with the two obvious maps into $\pi_r(F_{n+m}) = \mathfrak{E}_r^{n+m,0}$.

PROPOSITION 3.4. *μ is bijective for $n>2$ and onto for $n=2$.*

This result relates ${}^{\bullet}\mathfrak{E}_r^{n,m}$ to the following braid of exact sequences, valid at least for $r \geqq 2$, where the arrows stand for the obvious forgetful or boundary homomorphisms.

(3.5)



If $r \leqq 2n-2$, the homotopy groups of $Q^{\bullet}_{n,m}$ vanish according to (3.1), and hence the remaining sequences are isomorphic (compare this with diagram (2.3)). In particular, it can be seen that the forgetful map

$$^{\bullet}\eta\colon {}^{\bullet}\mathfrak{E}_r^{n,m} \to \pi_{r+n+m}(S^{n+m})$$

is an isomorphism even for $r=1$.

Finally, we discuss the bottom sequence in (1.1). Embed S^n as the unit sphere in $(x_r, x_{r+1}, \ldots, x_{r+n})$-space, such that $*$ corresponds to $(1, 0, \ldots, 0)$. Identify D^r with a small ball in $(x_1, \ldots, x_r)$-space centered at $-*$, and extend in the obvious way to identify $D^r \times S^n$ with a tubular neighbourhood of S^n in $\mathbf{R}^{r+n}$. Now, given $[f] \in \pi_r(G_{n+1})$, f defines a map from this tube to S^n, and we may assume that $-* \in S^n$ is a regular value. Fit $f^{-1}(-*)$ together with a large annulus in $\mathbf{R}^r$ around $D^r \times \{-*\}$, and close by a faraway half-sphere in $\{x \in \mathbf{R}^{r+1} \mid x_{r+1} \leqq 0\}$, to obtain an r-manifold M which is immersed and framed in $\mathbf{R}^{r+n}$. The only double points lie at

$$f^{-1}(-*) \cap D^r \times \{*\} \approx (p \circ f)^{-1}(-*)\,.$$

Define $\varrho[f]$ as the bordism class of M with $B = D^r \times \{*\}$.

Proposition 3.6. *The diagram of canonical maps*

$$\begin{array}{ccc} \pi_r(G_{n+1}) & \xrightarrow{\ \varrho\ } & {}^{\bullet}\mathfrak{I}_r^{n,1} \\ \Big\downarrow i_* & & \downarrow \\ & & \downarrow \\ \pi_r(F_{n+1}) & \xrightarrow{\ \cong\ } & \mathfrak{I}_r^{n+1,0} \end{array}$$

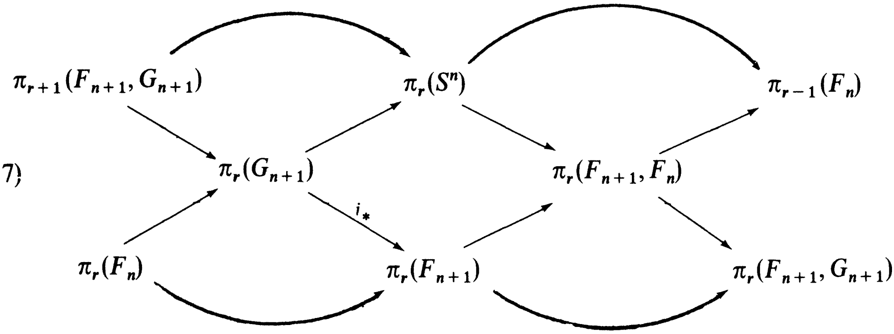
commutes. Furthermore, ϱ is bijective for $n \geqq 3$ and onto for $n \geqq 2$.

Proof. Given $x \in D^r \times \{-*\}$, rotate $\{x\} \times S^n$ in $\mathbf{R}^{r+n+1}$ around the axis $\mathbf{R}^{r-1} \times \{x_r\} \times \mathbf{R}^n$ by 90° towards positive x_{r+n+1}-levels. This leads to an isotopy from the embedding $D^r \times S^n \subset \mathbf{R}^{r+n} \subset \mathbf{R}^{r+n+1}$ used above to the standard embedding. This isotopy allows to deform the manifold M representing $\varrho[f]$, to a framed submanifold of $\mathbf{R}^{r+n+1}$ which corresponds to $i_*[f]$ under the identification of section 2.

Isotopy arguments, together with the Thom construction, also yield the second statement.

Now consider the braid of exact sequences of the triple (F_{n+1}, G_{n+1}, F_n)

(3.7)



If $r \leqq 2n-2$, the two outside groups in the antidiagonal vanish according to (3.2), and hence the two diagonal sequences are isomorphic. In particular, i_* is bijective, and so is ϱ by Proposition 3.6, even for $n=2$. Now clearly all claims concerning the $^{\bullet}\mathfrak{J}$-sequence in the main theorem are proved as soon as the corresponding facts for the $\mathfrak{J}$-sequence are established.

REMARK 3.8. In the non-metastable range, the antidiagonal sequences in diagram (3.7) also allow interesting (PL-)geometric interpretations.

4. Double points of immersions.

It remains to discuss the $\mathfrak{J}$-sequence. With one important exception the development will be entirely parallel to the one in section 2.

Let rel $\mathfrak{J}_r^{n,m}$ be the relative bordism group of triples $(M, g, \bar{g}_1)$, where the compact r-manifold M is embedded by $g=(g_1, g_2)$: $M \to \mathbf{R}_+^{r+n+m}$ such that $g_1: M \to \mathbf{R}_+^{r+n}$ is an immersion with framing $\bar{g}_1$, and $g(\partial M) = g(M) \pitchfork \partial \mathbf{R}_+^{r+n+m}$ lies already in $\mathbf{R}_+^{r+n}$. The definition of $\mathfrak{J}H_m$ in section 1 extends naturally to this relative group. Along the lines of section 2 we can prove:

PROPOSITION 4.1. *The relative double point map*

$$\mathfrak{J}H_m^{\mathrm{rel}}: \mathrm{rel}\, \mathfrak{J}_r^{n,m} \to \Omega_{r-n}(P^{m-1}; n\lambda)$$

is an isomorphism for $r \leqq 2n-2$.

An inverse is provided by the following model construction (see also Figure 3). Given $[X, \lambda, \bar{h}] \in \Omega_{r-n}(P^{m-1}; n\lambda)$, embed X into $\mathbf{R}^r$ and use $\bar{h}$ to identify a tubular neighbourhood with λ^n. Next fix $x \in X$ and note that every orientation of λ_x determines an n-dimensional diagonal in $\lambda_x^n \times \mathbf{R}^n$, a vector $z = (z_0, 0, \ldots, 0; 1, 0, \ldots, 0)$ in this diagonal, and hence a "diagonal" n-ball with center $\frac{1}{5} \cdot z$ and radius 1. Translating everything by the first base vector of $\mathbf{R}^n$, we obtain two balls B' and B'' which intersect each other transversally at precisely one point. Now attach the two cylinders which join the boundary spheres to their projection in $\lambda_x^n \times \{0\} \times \mathbf{R}^{n-1}$, and smooth corners. This does not create any new selfintersections. Finally, frame the balls by the standard basis of $\mathbf{R}^n$, and extend this normal framing to the cylinders. Performing this construction over each slice λ_x^n of the tubular neighbourhood of X, we obtain a framed immersion $\hat{g}_1$ into $\mathbf{R}_+^{r+n}$ with selfintersection at $X \times \{e_{r+1}\}$. Moreover, we use the inclusion $\lambda \subset \varepsilon^m$ in the obvious way to decompress $\hat{g}_1$ to an embedding in $\mathbf{R}_+^{r+n+m}$.

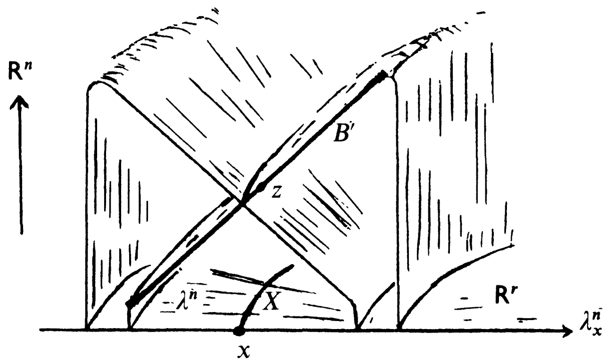


Figure 3. The model construction for double points.

The proof of that part of the main theorem which involves the $\mathfrak{I}$-sequence now proceeds along the lines of section 2 (use in particular the analogue of the Diagram 2.2). However, there is one important additional difficulty: it is not at all obvious that $-\Phi \circ \sigma^{-1} \circ \mathfrak{I}H_m^{\mathrm{rel}}$ coincides with the natural homomorphism from rel $\mathfrak{I}_r^{n,m}$ to $\pi_r(F_{n+m}, F_n)$. We also need to know that this homomorphism is still onto for $r=2n-1$. We will fill these last gaps in the next section.

Remark 4.2. For $n>0$ (and r arbitrary) we have the obvious long exact sequence involving the group rel $\mathfrak{I}_r^{n,m}$. Also $\mathfrak{I}H_m^{\mathrm{rel}}$ (and hence $\mathfrak{I}H_m$), as well as the corresponding invariants measuring triple, etc. points, are always well-defined group homomorphisms.

5. Singularities versus double points.

Recall from section 2 that we can identify $\pi_r(F_{n+m}, F_n)$ canonically with a relative bordism group of framed r-manifolds embedded in $\mathbf{R}_+^{r+n+m}$.

Proposition 5.1. *For* $r \leqq 2^n - 2$, *the diagram*

$$\begin{array}{ccc}
\Omega_{r-n}(P^{m-1};\, n\lambda) & \xleftarrow[\cong]{\mathfrak{E}H_m^{\mathrm{rel}}} & \mathrm{rel}\,\mathfrak{E}_r^{n,m} \\
{\scriptstyle -\mathfrak{I}H_m^{\mathrm{rel}}}\Big\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\Big\downarrow \\
\mathrm{rel}\,\mathfrak{I}_r^{n,m} & \xrightarrow{\qquad\qquad} & \pi_r(F_{n+m}, F_n)
\end{array}$$

commutes (where the unnamed arrows denote the natural forgetful homomorphisms).

PROOF. Given any triple $(X, \lambda, \bar{h})$ as in the discussion preceeding Proposition 2.2, consider the corresponding model $(\hat{M}, l)$; since we drop the correction term $(l_X \times \mathrm{id})$, this model has the singularity data $(X, \lambda, \bar{h} \circ (\mathrm{Id} \times (-\mathrm{Id}_\lambda)))$. We will isotop the framed manifold $\hat{M}$ in $\mathbf{R}_+^{r+n+m}$ until at the end the complicated framing l will have been "untwisted", but in turn the projection of $\hat{M}$ into $\mathbf{R}_+^{r+n}$ will have gained double points with data $(X, \lambda, \bar{h} \circ (\mathrm{Id} \times (-\mathrm{Id}_{\lambda \oplus \mathbf{R}})))$. Both forms of $\hat{M}$ determine the same element in $\pi_r(F_{n+m}, F_n)$, and the proposition will follow.

It suffices to describe the required isotopy on a typical fiber of the disk-bundle $\hat{M}$. This is done below. Clearly, the whole procedure works for arbitrary r as long as we start out from the model situation. Thus Theorem 1.3 implies that the bottom arrow in 5.1 is still onto for $r = 2n-1$.

Now decompose $\mathbf{R}^{2n+1} = \mathbf{R}^n \times {'\mathbf{R}^n} \times \mathbf{R}$ into two copies $\mathbf{R}^n$ and $'\mathbf{R}^n$ of n-space, and a real line. Elements of these factors are denoted by (x, y), (x', y') or z respectively, where x, x', z are real numbers, and y, y' are $(n-1)$-dimensional vectors. Also let c: $\mathbf{R}^n \cong {'\mathbf{R}^n}$ be the identical map.

We frame the obvious embedding g^0: $\mathbf{R}^n \subset \mathbf{R}^{2n+1}$ via the isomorphism

$$\bar{g}^0 \colon \nu(g^0) = \varepsilon^{n+1} \to \varepsilon^{n+1},$$

which at $(x, y) \in \mathbf{R}^n$ is the reflection given by the line through $(c(x, y), 1) \in \mathbf{R}^{n+1}$.

ISOTOPY THEOREM 5.2. *For any $n \geq 1$ there is a smooth deformation of g^0 through embeddings g^t: $\mathbf{R}^n \to \mathbf{R}^{2n+1}$, $t \in I$, accompanied by a deformation $\bar{g}^t$ of normal framings such that*

(i) *the projection of g^1 into $\mathbf{R}^{2n} = \mathbf{R}^n \times {'\mathbf{R}^n}$ is an immersion; its only double point lies at the origin of $\mathbf{R}^{2n}$, and in a neighbourhood the two sheets coincide with the graphs of c and $-c$; moreover, the two points of selfintersection have the form $(\pm x_0, 0)$, and the sign of the x-value equals the sign of the z-component of $g^1(\pm x_0, 0)$;*

(ii) *the framing $\bar{g}^1$ maps the last component of $\mathbf{R}^{2n+1}$ identically into the last component of ε^{n+1}; in addition, at $0 \in \mathbf{R}^{2n}$ both sheets are framed, via $\bar{g}^1$, by $-e_{n+1}, e_{n+2}, \ldots, e_{2n}$, where e_i is the standard basis vector;*

(iii) *for all* $t \in I$, g^t *and* $\bar{g}^t$ *are invariant under the involution of* R^n, R^{2n+1} *and* ε^{n+1} *given by*

$$((x,y),(x'y'),z) \to ((-x,-y),(x',y'),-z);$$

(iv) *for all* $t \in I$, g^t *and* g^0 *coincide outside of the unit ball of* R^n; *also* $\bar{g}^t$ *and* $\bar{g}^0$ *have the same limit behaviour along the rays through* 0.

PROOF. We need to specify g^t only on the half-space

$$H = \{(x,y) \in \mathsf{R}^n \mid x \geqq \delta\}$$

for some small fixed $\delta > 0$. Then, by (iii), g^t is determined also for $x \leqq -\delta$, and we extend it over $|x| \leqq \delta$ by joining the values of $(\pm\delta, y)$, $y \in \mathsf{R}^{n-1}$, linearly, with subsequent smoothing of corners. (See also Figure 4).

Let φ: $\mathsf{R}^n \to [0,1]$ be a symmetric smooth function with support in D^n, which takes the value 1 in a neighbourhood of $[-2\delta, +2\delta] \times \{0\}$ and is constant on each segment $[-2\delta, +2\delta] \times \{y\}$, $y \in \mathsf{R}^{n-1}$. Consider the graph of

$$\varphi \cdot (c,1)\colon \mathsf{R}^n \to {}'\mathsf{R}^n \times \mathsf{R}\ .$$

We use the parameter values $0 \leqq t \leqq \frac{1}{3}$ to deform H vertically into this graph.

Next choose an isotopy of R^n in the negative x-direction which pushes (δ, y) into $((1-2\varphi(\delta,y))\delta, y)$. Exploit the induced isotopy of our graph to deform $g^{\frac{1}{3}} \mid H$ into $g^1 \mid H$. We may assume that $g^t(H)$ intersects the hyperplane $x=0$ for the first time at $t=\frac{1}{2}$.

If we use the natural smoothings, the resulting family g^t of embeddings has the stated properties.

In order to cover it by an isotopy of framings, we need only to deform the normal field $v^0 = (\bar{g}^0)^{-1}(0,0,1)$ suitably, and apply the homotopy covering principle.

Along each ray $\{s \cdot (x,y) \mid s \geqq 0\}$, (x,y) a unit vector, v^0 rotates from $-e_{2n+1}$ (at $s=0$) *through* $-c(x,y)$ (at $s=1$) towards $+e_{2n+1}$. Our deformation will be given essentially by the family of fields w^t: $\mathsf{R}^n \to \mathsf{R}^{2n+1}$, $t \in I$, such that, along the same ray, w^t rotates first from $-\cos(\pi t) \cdot e_{2n+1} + \sin(\pi t) \cdot e_1$ to $-c(x,y)$, and then towards e_{2n+1} as before. However, we have to reparametrize this family with respect to the variables x and t. Let f: $\mathsf{R}^n \to \mathsf{R}^n$ be a symmetric homeomorphism which preserves y-values and orientations, and which stretches the set

$$D = \{(x,y) \in D^n \mid |x| \leqq 2\delta\}$$

onto D^n. Using the time interval from 0 to $\frac{1}{3}$, we deform v^0 into $v^0 \circ f$ via a homotopy $\mathrm{Id} \sim f$. For $\frac{1}{3} \leqq t \leqq \frac{2}{3}$, we define $v^t = w^{3t-1} \circ f$. Finally, we use the time

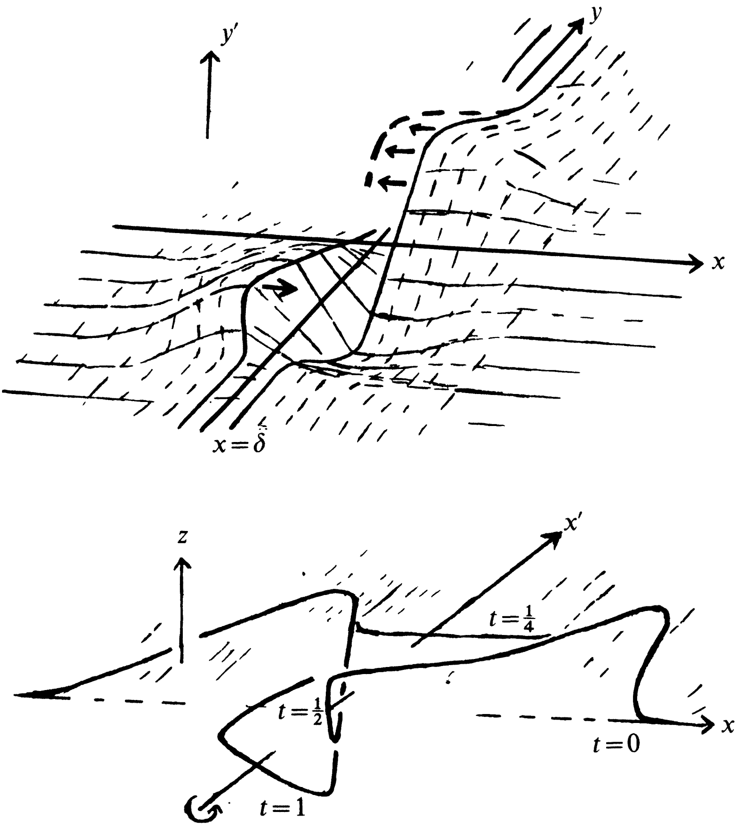


Figure 4. Construction of the isotopy g^t. At the right-hand side, the image of the x-axis under g^t.

parameters between $\frac{2}{3}$ and 1 in order to deform $w^1 \circ f$ linearly to the constant normal field with value e_{2n+1}.

It remains to show that v^t is never tangential to g^t. Note that g^t preserves y-values and that, on the other hand, the values of v^t do not involve $e_2, \ldots, e_n$. Therefore, we have to check only that v^t is nowhere a multiple of $G^t = \partial g^t / \partial x$. For $t \leqq \frac{1}{3}$ or outside of $\mathring{D}$ this is clear: G^t involves e_1, but v^t does not. Also, because of symmetries, we need to deal only with the case $x \geqq 0$. Thus fix $t > \frac{1}{3}$ and consider a segment $[0, 2\delta) \times \{y\}$ for small y. Here g^t is obtained, via smoothing, from affine maps on the two sub-segments $[0, \delta] \times \{y\}$ and $[\delta, 2\delta) \times \{y\}$. Thus G^t is a linear combination, with non-negative coefficients, of the two vectors

$$G^t_+ = ((1,0),\ (\varphi(\delta, y), 0),\ \ 0)$$
$$G^t_- = ((\alpha, 0),\ (0, 2\varphi(\delta, y) \cdot y),\ 2\varphi(\delta, y))$$

where $\alpha>0$ if $t<\frac{1}{2}$. On the other hand, v^t has the form

$$v^t = ((\beta,0),(-\gamma_1\cdot x, -\gamma_2\cdot y), \gamma_3\cdot(2t-1)),$$

where $\gamma_1,\gamma_2,\gamma_3>0$ and, for $t<\frac{2}{3}$, $\beta>0$. The linear independence of G^t and v^t follows easily.

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