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ON A CERTAIN CLASS OF POLYTOPES ASSOCIATED WITH INDEPENDENCE SYSTEMS

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Abstract.

A class of centrally symmetric polytopes associated with independence systems is studied. We characterize the weak Hanner polytopes among them and show that the pair of antiblocking polytopes introduced by D. R. Fulkerson is equivalent to a pair of facets of dual centrally symmetric polytopes. Finally new proofs of results of V. Chvãtal are presented.

1. Introduction.

Let P be a centrally symmetric polytope in R^n. P is called a weak Hanner polytope (WHP) or a CL-polytope if $P = \text{conv}\,(F \cup -F)$ for every facet F of P. In this paper we study a certain class of polytopes: Let J be an independence system (I.S.) on a finite set $X = \{v_1, \ldots, v_n\}$, that is, a set of subsets of X with the property that $T \subset S \in J$ implies $T \in J$. With every $S \in J$ we associate the point $x_S = (1, x_1^S, \ldots, x_n^S) \in \mathbf{R}^{n+1}$ with $x_i^S = 1$, $[x_i^S = -1]$, if $v_i \in S$, $[v_i \notin S$, respectively$]$. By P_J we denote the convex hull of the points $\{\pm x_S,\ S \in J\}$. In section 2 we study some properties of P_J and give a characterization of those P_J's which are weak Hanner polytopes. In section 3 we show that the pair of antiblocking polytopes associated with J, which was introduced by D. R. Fulkerson in [3], is equivalent to a pair of facets of the dual pair P_J and P_J^*. This together with a result of Fulkerson leads to the statement (in Theorem 6) that P_J is a WHP if and only if J is the set of independent subsets of vertices in a perfect graph. Section 4 is devoted to sub-I.S.'s and in the last section we present some new proofs of results due to V. Chvãtal.

2. Weak Hanner polytopes.

Let J be an independence system (I.S.) on a finite set $X = \{v_1, v_2, \ldots, v_n\}$, $n \in \mathbf{N}$, that is J is a set of subsets of X containing all singletons $\{v_i\}$, $i = 1, 2, \ldots, n$, and so that if $S \in J$ and $T \subset S$ then $T \in J$. We denote by P_J the centrally symmetric polytope

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$$(1)\qquad P_J = \operatorname{conv}\left\{\pm\left(-f_0^n+2\sum_{v_i\in S} e_i^n\right) \;\middle|\; S\in J\right\} \subset \mathsf{R}^{n+1},$$

where $f_0^n = {}_0(-1,1,\ldots,1)_n$ and $e_i^n = {}_0(0,0,\ldots,1,\ldots,0)_n$, $i=0,1,\ldots,n$. Let

$$x_S^n = -f_0^n+2\sum_{v_i\in S} e_i^n .$$

Clearly

$$(2)\qquad \partial_e P_J = \{\pm x_S \mid S\in J\} .$$

where $\partial_e P_J$ denotes the set of vertices of P_J. Let $\theta_n\colon \mathsf{R}^{n+1}\to \mathsf{R}^{n+1}$ be the linear transformation defined by

$$(3)\qquad \theta_n f_0^n = e_0^n, \quad \theta_n(-f_0^n+2e_i^n) = e_i^n, \quad i=1,2,\ldots,n .$$

θ_n is clearly non-singular and its inverse is given by the $(n+1)\times(n+1)$ symmetric matrix

$$(\theta_n)^{-1} = \begin{Bmatrix} -1 & 1 & 1 & 1 & \ldots & 1\\ 1 & 1 & -1 & -1 & \ldots & -1\\ 1 & -1 & 1 & -1 & \ldots & -1\\ 1 & -1 & -1 & 1 & \ldots & -1\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & -1 & -1 & -1 & \ldots & 1 \end{Bmatrix}$$

The vertices of $\theta_n P_J$ are the points

$$(4)\qquad \partial_e(\theta_n P_J) = \left\{\pm\left((|S|-1)e_0^n+\sum_{v_i\in S} e_i^n\right) \;\middle|\; S\in J\right\}.$$

Indeed,

$$(5)\qquad \begin{aligned} \theta_n x_S^n &= \theta_n\left(-f_0^n+2\sum_{v_i\in S} e_i^n\right)\\ &= \theta_n\left(-f_0^n+\sum_{v_i\in S}(2e_i^n-f_0^n)+|S|f_0^n\right)\\ &= (|S|-1)e_0^n+\sum_{v_i\in S} e_i^n . \end{aligned}$$

We shall omit the index "n" on θ_n, f_0^n, e_i^n and x_S^n, whenever it won't lead to a misunderstanding. Observe that since $\varnothing$ (the empty set) and all singletons $\{v_i\}$, $i=1,2,\ldots,n$, are in J, the points

(6) $$-x_{\varnothing} = f_0,\ x_{\{v_1\}} = -f_0+2e_i,\ \ldots,\ x_{\{v_n\}} = -f_0+2e_n$$

are vertices of P_J, so the points

(7) $$-\theta x_{\varnothing} = e_0,\ \theta x_{\{v_1\}} = e_1,\ \ldots,\ \theta x_{\{v_n\}} = e_n$$

are vertices of θP_J.

Let Q be any centrally symmetric polytope in $\mathbf{R}^n$. We denote by Q^* the dual polytope

(8) $$Q^* = \{y \in \mathbf{R}^n \mid (x,y) \leqq 1, \forall x \in Q\}$$

where $(\cdot,\cdot)$ is the usual scalar product in $\mathbf{R}^n$. Q is called a weak Hanner polytope (WHP) or a CL-polytope if $Q = \mathrm{conv}(F \cup -F)$ for every facet (maximal proper face) F of Q. It is well known and easy to show that the dual of a WHP is a WHP. We denote by $\partial_e Q$ the set of vertices of Q.

Since $e_0, e_1, \ldots, e_n \in \theta P_J$, we have that

(9) $$(\theta P_J)^* \subset \{y \in \mathbf{R}^{n+1} \mid |(y,e_i)| \leqq 1,\ i=0,1,\ldots,n\} \stackrel{\mathrm{def}}{=} C^{n+1}.$$

In particular all vertices of $(\theta P_J)^*$ are in the $n+1-$ cube C^{n+1}. We can show

LEMMA 1. *If $y=(y_0,y_1,\ldots,y_n)$ is a vertex of $(\theta P_J)^*$, then $y_0 = \pm 1$.*

PROOF. Assume that $|y_0|<1$. We claim that y can be written

$$y = \tfrac{1}{2}(1+y_0)u + \tfrac{1}{2}(1-y_0)w$$

with $u, w \in (\theta P_J)^*$ and $u_0 = -w_0 = 1$. Indeed, define u and w as follows:

(i) $u_0 = -w_0 = 1$

(ii) If $v_i \in S \in J$ and $(\theta x_S, y) = 1$ put

$$u_i = (1+y_0)^{-1}(2y_i + y_0 - 1), \qquad w_i = 1,$$

(iii) If $v_i \in S \in J$ and $(\theta x_S, y) = -1$, put

$$u_i = -1, \qquad w_i = (1-y_0)^{-1}(2y_i + y_0 + 1).$$

Observe first, that all coordinates of u and w are defined at least once in (i), (ii) and (iii). Indeed, u_0 and w_0 are defined in (i), and since y is a vertex of $(\theta P_J)^*$, y lies in the intersection of $n+1$ linearly independent facets of $(\theta P_J)^*$, defined by $n+1$ linearly independent vertices of θP_J. Hence for every $i=1,2,\ldots,n$ there is an $S \in J$ so that $v_i \in S$ and $(\theta x_S, y) = \pm 1$. To show that (ii) and (iii) give the same u_i and w_i when the conditions overlap, let $S, T \in J$ with $v_i \in S \cap T$, so that $(\theta x_S, y) = -(\theta x_T, y) = 1$. Since $S\setminus\{v_i\},\ T\setminus\{v_i\} \in J$, we have

$$(\theta x_{S\setminus\{v_i\}}, y) \leqq 1 \quad \text{and} \quad (\theta x_{T\setminus\{v_i\}}, y) \geqq -1.$$

Now

$$(10)\qquad 2 = (\theta x_S, y) - (\theta x_T, y)$$

$$= \left((|S|-1)e_0 + \sum_{v_j \in S} e_j, y\right) - \left((|T|-1)e_0 + \sum_{v_j \in T} e_j, y\right)$$

$$= y_0 + y_i + \left((|S|-2)e_0 + \sum_{v_j \in S \setminus \{v_i\}} e_j, y\right) -$$

$$- y_0 - y_i - \left((|T|-2)e_0 + \sum_{v_j \in T \setminus \{v_i\}} e_j, y\right)$$

$$= (\theta x_{S\setminus\{v_i\}}, y) - (\theta x_{T\setminus\{v_i\}}, y)$$

$$\leqq 1 - (-1) = 2$$

so $(\theta x_{S\setminus\{v_i\}}, y) = 1$. This implies that

$$(11)\qquad 0 = (\theta x_S, y) - (\theta x_{S\setminus\{v_i\}}, y) = y_0 + y_i ,$$

so $y_i = -y_0$. Inserting $y_i = -y_0$ either in (ii) or (iii), we obtain $u_i = -1$ and $w_i = 1$. To show that u and w as defined above are in $(\theta P_J)^*$, we only have to check whether

$$|(u, \theta x_T)| \leqq 1 \quad \text{and} \quad |(w, \theta x_T)| \leqq 1$$

for every $T \in J$. By symmetry it suffices to show that $u \in (\theta P_J)^*$. Let now $T \in J$. If $T = \emptyset$, then $\theta x_T = -e_0$ and $(\theta x_T, u) = -1$, so assume that $v_i \in T$. If $u_i = -1$, then

$$(\theta x_T, u) = (\theta x_{T\setminus\{v_i\}}, u) .$$

We may therefore assume that u_i is defined by (ii) for all i with $v_i \in T$. Thus

$$(12)\qquad (\theta x_T, u) = \left((|T|-1)e_0 + \sum_{v_i \in T} e_i, u\right)$$

$$= |T| - 1 + \sum_{v_i \in T} (1+y_0)^{-1}(2y_i + y_0 - 1)$$

$$= (1+y_0)^{-1}\left(|T| - 1 + y_0|T| - y_0 + 2\sum_{v_i \in T} y_i + y_0|T| - |T|\right)$$

$$= (1+y_0)^{-1}\left(2\left((|T|-1)y_0 + \sum_{v_i \in T} y_i\right) + y_0 - 1\right)$$

$$= (1+y_0)^{-1}(2(\theta x_T, y) + y_0 - 1)$$

$$\leqq (1+y_0)^{-1}(1+y_0) = 1$$

We still have to show that $(\theta x_T, u) \geqq -1$. Let $v_i \in T$. Since u_i is defined by (ii), there is an $S \in J$ with $v_i \in S$ and $(\theta x_S, y) = 1$. Then $S \setminus \{v_i\} \in J$, so

$$(13) \qquad \begin{aligned} y_0 + y_i &= (\theta x_S, y) - (\theta x_{S\setminus\{v_i\}}, y) \\ &= 1 - (\theta x_{S\setminus\{v_i\}}, y) \geqq 0 \end{aligned}$$

This gives that

$$(14) \qquad \begin{aligned} (\theta x_T, u) &= |T| - 1 + (1+y_0)^{-1} \sum_{v_i \in T} (2y_i + y_0 - 1) \\ &\geqq |T| - 1 + (1+y_0)^{-1} \sum_{v_i \in T} (-y_0 - 1) \geqq -1 \end{aligned}$$

Since clearly $y = \frac{1}{2}(1+y_0)u + \frac{1}{2}(1-y_0)w$, we are done.

LEMMA 2. *The following two statements are equivalent*

(a) P_J *is a* WHP

(b) $\partial_e(\theta P_J)^* \subset \partial_e C^{n+1}$

PROOF. Assume first (a) and let $y = (y_0, y_1, \ldots, y_n) \in \partial_e(\theta P_J)^*$. Since being a WHP is preserved under linear equivalence and the duality mapping, $(\theta P_J)^*$ is a WHP. Let $0 \leqq i \leqq n$. Since $e_i \in \partial_e \theta P_J$,

$$F_i = \{z \in (\theta P_J)^* \mid (z, e_i) = 1\}$$

is a facet of $(\theta P_J)^*$, so

$$(\theta P_J)^* = \operatorname{conv}(F_i \cup -F_i) .$$

Since $y \in \partial_e(\theta P_J)^*$ this implies that $y \in F_i \cup -F_i$ and therefore $y_i = \pm 1$.

Assume next that (b) is fulfilled. We shall show that $(\theta P_J)^*$ is a WHP. Let F be a facet and $y = (y_0, y_1, \ldots, y_n)$ a vertex of $(\theta P_J)^*$. By (2) there is an $S \in J$, so that

$$F = \pm\{z \in (\theta P_J)^* \mid (\theta x_S, z) = 1\} .$$

Hence we need only to show that $|(\theta x_S, y)| = 1$. This will imply that $\partial_e(\theta P_J)^* \subset F \cup -F$, so $(\theta P_J)^* = \operatorname{conv}(F \cup -F)$. By definition of $(\theta P_J)^*$ we have that $|(\theta x_S, y)| \leqq 1$ and from the assumption that $y \in \partial_e C^{n+1}$ we get that

$$(\theta x_S, y) = (|S| - 1) y_0 + \sum_{v_i \in S} y_i$$

is an odd integer. Therefore $(\theta x_S, y) = \pm 1$ and we are done.

If J is an I.S. on X, we denote by $\bar{J}$ the system

$$\begin{aligned}\bar{J} &= \{S\subset X \mid \forall v', v'' \in S,\ v' \neq v'':\ \{v', v''\} \notin J\} \\ &= \{S\subset X \mid \forall T \in J:\ |S\cap T| \leqq 1\}\ .\end{aligned}$$

Clearly $J\subset\bar{\bar{J}}$, but the following example shows that the inclusion may be strict: Let

$$J_\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}\ .$$

Then J_Δ is an I.S. on $\{1,2,3\}$, but $\bar{J}_\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}\}$, so $\bar{\bar{J}}_\Delta = J\cup\{\{1,2,3\}\}$.

We have

THEOREM 4. *The following three statements are equivalent*:

(a) P_J *is a* WHP

(b) $P_{\bar{J}}$ *is a* WHP *and* $J=\bar{\bar{J}}$

(c) $P_{\bar{J}}=(\theta P_J)^*$

PROOF. Let us first prove that (a) implies (c), so let $x_S \in P_{\bar{J}}$ and $\theta x_T \in \theta P_J$. Then

$$(15)\qquad (x_S, \theta x_T) = \Big(-f_0 + 2\sum_{v_i\in S} e_i,\ (|T|-1)e_0 + \sum_{v_j\in T} e_j\Big)$$

$$= (|T|-1)(-f_0, e_0) + \Big(-f_0,\ \sum_{v_j\in T} e_j\Big) + 2\Big(\sum_{v_i\in S} e_i,\ \sum_{v_j\in T} e_j\Big)$$

$$= |T| - 1 - |T| + 2\sum_{v_i\in S\cap T} 1$$

$$= 2|S\cap T| - 1\ .$$

Since $|S\cap T| \in \{0,1\}$, we have $|(x_S, \theta x_T)| \leqq 1$ and therefore $x_S \in (\theta P_J)^*$. To show that $(\theta P_J)^* \subset P_{\bar{J}}$, observe that

$$\partial_e(\theta P_J)^* \subset \partial_e C^{n+1}$$

by Lemma 3. Hence every point in $\partial_e(\theta P_J)^*$ is on the form x_S for some $S\subset X$. Now let $x_S \in \partial_e(\theta P_J)^*$. We need only to show that $S \in \bar{J}$, so let $v_i, v_j \in S$ with $i \neq j$. Then

$$(\theta x_{\{v_i, v_j\}}, x_S) = (e_0 + e_i + e_j, x_S) = 3\ ,$$

so $\{v_i, v_j\} \notin J$, and we are done.

(a) follows easily from (c) and Lemma 2, because

$$\partial_e(\theta P_J)^* = \partial_e P_{\bar{J}} \subset \partial_e C^{n+1}\ .$$

Assume now that (a) holds. Then by (c), $(\theta P_J)^* = P_{\bar{J}}$ so $P_{\bar{J}}$ is a WHP. This proves the first part of (b).

To show the second part of (b), observe that θ is symmetric, so by (c),

$$(16) \qquad P_{\bar{J}} = (\theta P_J)^* = (\theta^*)^{-1} P_J^* = \theta^{-1} P_J^*$$

(16) together with the fact that $P_{\bar{J}}$ is a WHP gives that

$$P_{\bar{\bar{J}}} = (\theta P_{\bar{J}})^* = (\theta\theta^{-1} P_J^*)^* = P_J^{**} = P_J .$$

Assume finally that (b) holds. Then, by the equivalence of (a) and (c), $P_{\bar{J}}$ and therefore P_J is a WHP.

3. Anti-blocking polytopes and graphs.

In [3] Fulkerson introduced the powerful notion of anti-blocking polytopes: Let A be a non-negative $m \times n$-matrix in which no column consists entirely of zeros. Let P_A be the convex hull in R^n of the rows in A. The antiblocker of P_A is the polytope

$$(17) \qquad \bar{P}_A = \{y \in \mathsf{R}_+^n \mid \forall x \in P_A\colon\ (x,y) \leqq 1\}$$

If J is an I.S. on $X = \{v_1, \ldots, v_n\}$, the incidence matrix of J is the matrix

$$A = \{a_{Si} \mid S \in J,\ i = 1,2,\ldots,n\} ,$$

where $a_{Si} = 1$ if $v_i \in S$ and $a_{Si} = 0$ if $v_i \notin S$.

To simplify the notation, we regard R^n as the subspace of R^{n+1} spanned by $e_1^n, \ldots, e_n^n$ and take $(e_1^n, \ldots, e_n^n)$ as the standard basis of R^n. Let now $\varphi_n\colon \mathsf{R}^{n+1} \to \mathsf{R}^{n+1}$ be the affine transformation defined by $\varphi_n(x) = 2x - \mathbf{1} + 2e_0^n$, where $\mathbf{1} = (1,1,\ldots,1)$. Then we have:

LEMMA 5. *Let J be an* I.S. *on $X = \{v_1, \ldots, v_n\}$ and let A be the incidence matrix of J. Then*

(a) $\varphi_n P_A = \{x \in P_J \mid (x, e_0^n) = 1\}$

(b) $\varphi_n \bar{P}_A = \{y \in (\theta_n P_J)^* \mid (y, e_0^n) = 1\}$

(c) $\bar{P}_A$ *has integral vertices if and only if* $\partial_e(\theta_n P_J)^* \subset \partial_e C^{n+1}$.

PROOF. (a) Let $S \in J$. Then

$$(18) \qquad \varphi_n \sum_{v_i \in S} e_i^n = 2 \sum_{v_i \in S} e_i^n - \sum_{i=0}^{n} e_i^n + 2e_0^n$$

$$= e_0^n + \sum_{v_i \in S} e_i^n - \sum_{v_i \in X \setminus S} e_i^n = x_S^n$$

This proves (a).

(b) Let $z=(0,z_1,\ldots,z_n)\in\mathbf{R}^n$. Then $\varphi_n z=(1,2z_1-1,\ldots,2z_n-1)$. Also,

$$(19)\quad z\in\bar{P}_A \Leftrightarrow \forall S\in J:\left(z,\sum_{v_j\in S}e_j^n\right)\leqq 1 \quad\text{and}\quad \forall i=1,\ldots,n:\ z_i\geqq 0\,.$$

Now, let $S\in J$ and $1\leqq i\leqq n$. Then

$$(20)\quad \left(z,\sum_{v_j\in S}e_j^n\right)\leqq 1 \Leftrightarrow \sum_{v_j\in S}z_j\leqq 1$$

$$\Leftrightarrow |S|-1+\sum_{v_j\in S}(2z_j-1)\leqq 1$$

$$\Leftrightarrow (\varphi_n z,\theta_n x_S^n)\leqq 1$$

and

$$(21)\quad z_i\geqq 0\Leftrightarrow 2z_i-1\geqq -1$$

$$\Leftrightarrow (\varphi_n z,\theta_n x_{\{vi\}}^n)\geqq -1\,.$$

Note that if $2z_i-1\geqq \dot{-}1$ for all i, then $(\varphi_n z,\theta_n x_S^n)\geqq -1$ for all $S\subset X$, so (20) and (21) give that $z\in\bar{P}_A$ if and only if $\varphi_n z\in(\theta_n P_J)^*$. Since every

$$y_0\in\{y\in(\theta_n P_J)^*\mid (y,e_0^n)=1\}$$

can be written $y_0=\varphi_n z$ for some $z\in\mathbf{R}^n$, we are done.

(c) Follows easily from (b).

The above lemma shows that when A is the incidence matrix for an I.S., the notion of antiblockong pairs of polytopes is equivalent to the usual duality of centrally symmetric polytopes. Observe also that in view of Lemma 5 the fact that if P_J is a WHP then $\bar{\bar{J}}=J$ (see Theorem 4) has been observed by Padberg [6]. Fulkersons theory of antiblocking pairs of polytopes has been very useful in graph theory: Let $X=\{v_1,\ldots,v_n\}$ be a finite set and $G=(X,E)$ a loopless non-directed graph on X without multiple vertices, i.e., $E\subset X^{(2)}$, where $X^{(2)}$ is the set of two-points subsets of X. G is said to be a clique (complete) if $E=X^{(2)}$. A subset $S\subset X$ is called independent if $E\cap S^{(2)}=\varnothing$. The graph $(S,E\cap S^{(2)})$ is called the subgraph of G induced by S. The complement of G is the graph $\bar{G}=(X,X^{(2)}\setminus E)$. The stability number $\alpha(G)$ is the maximal cardinality of a stable set of vertices of G and the covering number $\Theta(G)$ is the minimum number of cliques in G covering X. G is called (α-)perfect if $\alpha(G')=\Theta(G')$ for every induced subgraph G' of G. In [4] Lovasz proved that if G is perfect, then $\bar{G}$ is perfect, and in [5] he proved the following characterization of perfect graphs: G is perfect if and only if $\omega(G')\omega(\bar{G}')\geqq|G'|$ for every induced subgraph G' of G. Here

$\omega(G)$ denotes the maximal cardinality of a clique in G. Using the theory of antiblocking polytopes Fulkerson proved (see [7]) that if A is the incidence matrix for the independent subsets of G, then G is perfect if and only if $\bar{P}_A$ has integral vertices. An alternative proof based on results of Lovatz is given by Chvãtal in [2]. Observe that if J is an I.S. on $X=\{v_1,\ldots,v_n\}$ then $J=\bar{\bar{J}}$ if and only if J can be represented as theset of independent sets in a graph on X. Using Lemmas 2 and 5 and Theorem 4 we have therefore,

THEOREM 6. *Let J be an* I.S. *on X. The following two statements are equivalent*:

(a) P_J *is a* WHP.

(b) *J can be represented as the set of independent subsets of vertices of a perfect graph on X.*

4. Induced sub-I.S.'s.

Let J be an I.S. on $X=\{v_1,\ldots,v_n\}$ and let $X'\subset X$. The I.S.

$$\{S \mid S \subset X',\ S \in J\}$$

is called the sub-I.S. of J induced by X' and is denoted by $J\downarrow X'$. We note for future reference the identity

$$\overline{J\downarrow X'} = \bar{J}\downarrow X' \tag{22}$$

LEMMA 7. *Let J be an* I.S. *on $X=\{v_1,\ldots,v_n\}$ and J' the sub-*I.S. *of J induced by $X'=\{v_1,\ldots,v_k\}$, $1\leqq k<n$. Then*

(a) *If $y'=(1,y_1,\ldots,y_k)\in\partial_e(\theta_k P_{J'})^*$ then $y=(1,y_1,\ldots,y_k,-1,\ldots-1)\in\partial_e(\theta_n P_J)^*$.*

(b) *If $y=(y_0,y_1,\ldots,y_n)\in(\theta_n P_J)^*$, then $y'=(y_0,y_1,\ldots,y_k)\in(\theta_k P_{J'})^*$.*

PROOF. Let $y'=(1,y_1,\ldots,y_k)\in\partial_e(\theta_k P_{J'})^*$ and put $y=(1,y_1,\ldots,y_k,-1,\ldots,-1)$. To show that $y\in(\theta_n P_J)^*$, let $S\in J$. Then $S\cap X'\in J'$ and

$$\begin{aligned}(\theta_n x^n_S, y) &= |S|-1+\sum_{v_i\in S} y_i \\ &= |S|-1+\sum_{v_i\in S\cap X'} y_i+\sum_{v_i\in S\setminus X'} y_i \\ &= |S|-1-|S\setminus X'|+\sum_{v_i\in S\cap X'} y_i\end{aligned} \tag{23}$$

$$= |S \cap X'| - 1 + \sum_{v_i \in S \cap X'} y_i$$

$$= (\theta_k x^k_{S \cap X'}, y')$$

Since $y' \in (\theta_k P_{J'})^*$, it follows that $|(\theta_n x^n_S, y)| \leqq 1$. Next we show that y is a vertex of $(\theta_n P_J)^*$. Since $y_0 = 1$, there exist $k+1$ linearly independent points $\{x^k_{S_i}\}^k_{i=0}$ in $P_{J'}$ with $S_0 = \varnothing$ and

$$|(\theta_n x^k_{S_i}, y')| = 1, \quad i = 0, 1, \ldots, k .$$

Then the points $\{x^n_{S_i}\}^k_{i=0} \cup \{x^n_{\{v_j\}}\}^n_{j=k+1}$ are linearly independent vertices of P_J and

$$|(y, \theta_n x^n_{S_i})| = |(y', \theta_k x^k_{S_i})| = 1 \quad \text{for } i = 0, 1, \ldots, k$$

and

$$|(y, \theta_n x^n_{\{v_j\}})| = |y_j| = 1 \quad \text{for } j = k+1, \ldots, n ,$$

so $y \in \partial_e(\theta_n P_J)^*$.

(b) Obvious.

Assume now that P_J is not a WHP and choose a vertex $y = (1, y_1, \ldots, y_n) \in \partial_e(\theta_n P_J)^* \backslash \partial_e C^{n+1}$ by Lemma 2. Let

$$X_y = \{v_i \in X \mid |y_i| < 1\} .$$

We shall assume that $X_y = \{v_1, \ldots, v_m\}$, $1 \leqq m \leqq n$. Let $S_0 = \varnothing$ and $S_i = \{v_i\}$ for $i = m+1, \ldots, n$. Then

$$|(\theta x_{S_i}, y)| = 1 \quad \text{for } i = 0, m+1, \ldots, n .$$

Since $y \in \partial_e(\theta P_J)^*$, the linearly independent set $x_{S_0}, x_{S_{m+1}}, \ldots, x_{S_n}$ can be extended to a linearly independent set $\{x_{S_i}\}^n_{i=0}$ of vertices of P_J, so that $|(\theta x_{S_i}, y)| = 1$ for $i = 0, 1, \ldots, n$.

LEMMA 8. *Let J, y, X_y, m and $\{S_i\}^n_{i=0}$ be as above. Then*

(a) *$S_1, \ldots, S_m$ can be chosen in $J \downarrow X_y$.*

(b) *When $S_1, \ldots, S_m$ have been choosen as in* (a), *then they are maximal elements of $J \downarrow X_y$.*

(c) *$P_J \downarrow X_y$ is not a* WHP.

PROOF. (a) Choose $\{S_i\}^m_{i=1}$ so that $|S_1| + \ldots + |S_m|$ is as small as possible. Suppose $v_i \in S_j$ for some $1 \leqq j \leqq m < i \leqq n$. Then $y_i = \pm 1$, by definition.

If $y_i = -1$, then

$$\theta x_{S_j\setminus\{v_i\}} = \theta x_{S_j} - e_0 - e_i = \theta x_{S_j} + \theta x_{S_0} - \theta x_{S_i}$$

and

$$|(\theta x_{S_j\setminus\{v_i\}}, y)| = |(\theta x_{S_j}, y) - y_0 - y_i| = |(\theta x_{S_j}, y)| = 1,$$

so we can replace S_j by $S_j\setminus\{v_i\}$ and still satisfy the above specifications, but this contradicts the minimality of $|S_1|+\ldots+|S_m|$.

If $y_i=1$, then we will show that $S_j \subset X\setminus X_y$, and therefore x_{S_j} is a linear combination of $x_{S_0}, x_{S_{m+1}}, \ldots, x_{S_n}$, which contradicts the linear indepence of $\{x_{S_i}\}_{i=0}^n$. Indeed, if $v_k \in S_j$, $i \neq k$, then $\{v_i, v_k\} \in J$. Therefore

$$|(\theta x_{\{v_i, v_k\}}, y)| = |y_0 + y_i + y_k| = |2 + y_k| \leqq 1 ,$$

so since $|y_k| \leqq 1$, $y_k = -1$ and therefore $k \in X\setminus X_y$.

(b) Choose $i,j \in \{1,2,\ldots,m\}$ such that $v_i \notin S_j$. We will show that $S_j \cup \{v_i\} \notin J$ or, equivalently, that $x_{S_j\cup\{v_i\}} \notin P_J$. It suffices to show that $(\theta x_{S_j\cup\{v_i\}}, y) > 1$, since $y \in (\theta P_J)^*$. Now

$$(\theta x_{S_j\cup\{v_i\}}, y) = (\theta x_{S_j}, y) + y_0 + y_i > (\theta x_{S_j}, y)$$

because $y_0=1$ and $|y_i|<1$, so we need only to show that $(\theta x_{S_j}, y)=1$. We know already that $(\theta x_{S_j}, y) = \pm 1$. Since

$$(\theta x_{S_j}, y) = |S_j| - 1 + \sum_{v_k \in S_j} y_k$$

and $y_k > -1$ for $v_k \in S_j$, it follows that $(\theta x_{S_j}, y) > -1$ and therefore $(\theta x_{S_j}, y) = 1$.

(c) This follows easily from (a) and Lemma 7(b).

THEOREM 9. *Let J be an* I.S. *on X and $X' \subset X$. If P_J is a* WHP, *then $P_{J\downarrow X'}$ is a* WHP.

PROOF. Follows immediately from Lemma 7(a).

5. Intersection, union and substitution of I.S.'s.

Let J_1 and J_2 be I.S.'s on X_1 and X_2 respectively. Then $J_1 \cap J_2$ is an I.S. on $X_1 \cap X_2$ and $J_1 \cup J_2$ is an I.S. on $X_1 \cup X_2$. Observe that if $X' \subset X_1 \cup X_2$, then

$$(24) \qquad (J_1 \cup J_2)\downarrow X' = (J_1 \downarrow (X_1 \cap X')) \cup (J_2 \downarrow (X_2 \cap X')) .$$

The following example shows that if P_{J_1} and P_{J_2} are WHP's, $P_{J_1 \cap J_2}$ need not be a WHP:

Let $X_1 = X_2 = \{1,2,3,4,5\}$ and

$$J_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}, \{1,3\}, \{1,2,3\}\} ,$$

$$J_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}, \{1,4\}, \{1,4,5\}\} .$$

Then it is easy to show that P_{J_1} and P_{J_2} are WHP's and that $(1,0,0,0,0,0) \in \partial_e(\theta_5 P_{J_1 \cap J_2})^*$.

In view of Lemma 5(c) the following two theorems are equivalent to results that have been proved in [2] (See also [1] and [4]). We shall however furnish them with easy geometric proofs based on Lemma 2.

THEOREM 10. *Let J_1 and J_2 be* I.S.'s *on X_1 and X_2 respectively. If P_{J_1} and P_{J_2} are* WHP's *and $X_1 \cap X_2 \in J_1 \cap J_2$, then $P_{J_1 \cup J_2}$ is a* WHP.

PROOF. Suppose $X = X_1 \cup X_2 = \{v_1,\ldots,v_n\}$ and assume $P_{J_1 \cup J_2}$ is not a WHP. By Lemma 1 and Lemma 2 choose a vertex

$$y = (1, y_1,\ldots,y_n) \in \partial_e(\theta P_{J_1 \cup J_2})^* \backslash \partial_e C^{n+1},$$

and let

$$X_y = \{v_i \in X \mid |y_i| < 1,\ 1 \leqq i \leqq n\}.$$

Then $P_{J_1 \downarrow (X_1 \cap X_y)}$ and $P_{J_2 \downarrow (X_2 \cap X_y)}$ are WHP's and

$$(X_1 \cap X_y) \cap (X_2 \cap X_y) \in J_1 \downarrow (X_1 \cap X_y) \cap J_2 \downarrow (X_2 \cap X_y),$$

but

$$P_{J_1 \downarrow (X_1 \cap X_y) \cup J_2 \downarrow (X_2 \cap X_y)} = P_{(J_1 \cup J_2) \downarrow X_y}$$

is not a WHP (see Lemma 8(c)). We can therefore assume that $X_y = X$.

Now let $S_0, S_1,\ldots,S_n$ be as in Lemma 8 (with $m = n$). Assume that

$$X_1 \backslash X_2 = \{v_1,\ldots,v_p\}, \quad X_2 \backslash X_1 = \{v_{p+1},\ldots,v_{p+q}\}$$

and

$$X_1 \cap X_2 = \{v_{p+q+1},\ldots,v_n\}$$

and let $t = |X_1 \cap X_2| = n - p - q$. Arrange the sets $\{S_i\}_{i=1}^n$ so that $S_1,\ldots,S_r \in J_1$ and $S_{r+1},\ldots,S_n \in J_2$, and let M be the incidence matrix for $S_1,\ldots,S_n$. Since $\theta x_{S_0}, \theta x_{S_1},\ldots,\theta x_{S_n}$ are linearly independent, M is non-singular. In particular, the first p and the next q columns of M are linearly independent, so we must have $p \leqq r$ and $q \leqq n - r$. Let now $y' = (1, y_1,\ldots,y_p, y_{p+q+1},\ldots,y_n)$. By Lemma 7(b),

$$y' \in (\theta P_{J_1})^* \stackrel{\text{def}}{=} Q.$$

(Note that $(J_1 \cup J_2) \downarrow X_1 = J_1$, since $X_1 \cap X_2 \in J_1$).

Furthermore, for $i = 0,1,\ldots,r$ we have $S_i \in J_1$ and

$$|(\theta_{p+t} x_{S_i}^{p+t}, y')| = |(\theta_n x_{S_i}^n, y)| = 1,$$

so y' lies in the intersection F of $r+1$ independent facets of Q. Therefore,

(25) $$\dim F \leqq 1+p+t-(1+r) = t+p-r .$$

If $r=p+t$, then $F=\{y'\}$, so $y' \in \partial_e Q$, but this contradicts the assumption that P_{J_1} is a WHP. Hence $r<p+t$. By symmetry we have $n-r<q+t$, so

(26) $$p < r < p+t .$$

Since $\partial_e F \subset \partial_e Q \subset \partial_e C^{p+t+1}$ and $|y_i|<1$, $i=1,2,\ldots,n$, the coordinate functionals $e_{p+q+1},\ldots,e_n$ cannot be constant on F and their maximal values on F must be 1. Choose $z_1,\ldots,z_t \in \partial_e F$ with

$$(e_{p+q+i}, z_i) = 1, \quad i=1,2,\ldots,t .$$

Since $X_1 \cap X_2 \in J_1$ we have $|(\theta_{p+t}x^{p+t}_{X_1\cap X_2}, z_i)| \leqq 1$. Now

(27) $$(\theta_{p+t}x^{p+t}_{X_1\cap X_2}, z_i) = t-1+\sum_{j=1}^{t} (e_{p+q+j}, z_i)$$

$$= t+\sum_{\substack{j=1\\ j\neq i}}^{t} (e_{p+q+j}, z_i) \geqq 1$$

because $|(e_{p+q+j}, z_i)| \leqq 1$. Hence $(e_{p+q+j}, z_i) = -1$ for $j=1,2,\ldots,t$, $j\neq i$, so the points $z_1,\ldots,z_t$ are affinely independent. Hence $\dim F \geqq t-1$. By (25) and (26) we also have $\dim F \leqq t-1$, so

(28) $$\dim F = t-1 .$$

(25), (26) and (28) give now that $r=p+1$. By symmetry we have $n-r=q+1$, so since $n=p+q+t$,

(29) $$r = p+1 \quad \text{and} \quad t = 2 .$$

This implies that $F=[z_1,z_2]$. Since $(\theta_{p+t}x^{p+t}_{X_1\cap X_2}, z_i)=1$ (see 27), $i=1,2$, also

$$(\theta_{p+t}x^{p+t}_{X_1\cap X_2}, y') = (\theta_n x^{n}_{X_1\cap X_2}, y) = 1 .$$

We may then assume that one of the sets $S_1,\ldots,S_n$, say S_a, equals $X_1 \cap X_2 \in J_1 \cap J_2$. If $a \leqq r$ [or $a>r$] we can arrange the sets S_i so that the first $r-1$ [or $r+1$ respectively] sets belong to J_1 and the remaining ones to J_2. This leads to a contradiction, since we have shown that in any particular partion of $\{S_1,\ldots,S_n\}$ into two classes $D_1 \subset J_1$ and $D_2 \subset J_2$, $|D_1|$ must equal $p+1$.

Let now J_1 and J_2 be I.S.'s on X_1 and X_2 respectively and assume that $X_1 \cap X_2 = \emptyset$ and $v \in X_1$. We define an I.S. on $(X_1\backslash\{v\}) \cup X_2$ as follows:

$$(J_1, J_2 \to v) = \{S \subset (X_1\backslash\{v\}) \cup X_2 \mid \text{(a) } S \in J_1 \downarrow (X_1\backslash\{v\}) \text{ or} \\ \text{(b) } S \cap X_2 \in J_2 \text{ and } (S \cap X_1) \cup \{v\} \in J_1\} .$$

$(J_1, J_2 \to v)$ is called the I.S. obtained from J_1 by substituting J_2 for v. Observe the following combinatorial lemma:

LEMMA 11. *Let J_1, J_2, X_1, X_2 and v be as above.*

(a) *If $X' \subset X_1 \backslash \{v\} \cup X_2$, then*

$$(J_1, J_2 \to v) \downarrow X' = (J_1 \downarrow (X_1 \cap X' \cup \{v\}), \quad J_2 \downarrow (X_2 \cap X') \to v) .$$

(b) $\overline{(J_1, J_2 \to v)} = (\bar{J}_1, \bar{J}_2 \to v)$

(c) *Suppose $\bar{\bar{J}}_1 = J_1$ and $\bar{\bar{J}}_2 = J_2$ and let $X_1^1 = \{v' \in X_1 \backslash \{v\} \mid \{v', v\} \in J_1\}$. Then*

$$(J_1, J_2 \to v) = \overline{\overline{J_1 \downarrow X_1^1 \cup \bar{J}_2}} \cup (J_1 \downarrow (X_1 \backslash \{v\})) .$$

PROOF. (a). Follows immediately from the definition of $(J_1, J_2 \to v)$.

(b). Suppose first that $S \in \overline{(J_1, J_2 \to v)}$. Then $S = S_1 \cup S_2$ with $S_1 \subset X_1 \backslash \{v\}$ and $S_2 \subset X_2$. Let $T \in J_1$. Since $v \notin S_1$ and $T \backslash \{v\} \in (J_1, J_2 \to v)$,

$$|S_1 \cap T| \leqq |S \cap (T \backslash \{v\})| \leqq 1 ,$$

so $S_1 \in \bar{J}_1 \downarrow (X_1 \backslash \{v\})$. If $T \in J_2 \subset (J_1, J_2 \to v)$, then

$$|S_2 \cap T| \leqq |S \cap T| \leqq 1 ,$$

so $S_2 \in \bar{J}_2$. Assume now that $S_2 \neq \emptyset$ and let $w \in S_2$ and $T \in J_1$. If $v \notin T$,

$$T \in (J_1, J_2 \to v) \quad \text{and} \quad |(S_1 \cup \{v\}) \cap T| \leqq |S \cap T| \leqq 1 .$$

If $v \in T$,

$$(T \backslash \{v\}) \cup \{w\} \in (J_1, J_2 \to v) ,$$

so

$$|(S_1 \cup \{v\}) \cap T| = |(S_1 \cup \{w\}) \cap ((T \backslash \{v\}) \cup \{w\})|$$
$$\leqq |S \cap ((T \backslash \{v\}) \cup \{w\})| \leqq 1 .$$

We conclude that $S \in (\bar{J}_1, \bar{J}_2 \to v)$.

Next, let $S \in (\bar{J}_1, \bar{J}_2 \to v)$ and $T \in (J_1, J_2 \to v)$. In order to show that $S \in \overline{(J_1, J_2 \to v)}$ we must show that $|S \cap T| \leqq 1$. Let $S = S_1 \cup S_2$ and $T = T_1 \cup T_2$ with $S_1, T_1 \subset X_1 \backslash \{v\}$ and $S_2, T_2 \subset X_2$. If $S_2 = \emptyset$ or $T_2 = \emptyset$, then clearly $|S \cap T| \leqq 1$, because

$$S_1 \in \bar{J}_1 \downarrow (X_1 \backslash \{v\}) \quad \text{and} \quad T_1 \in J_1 \downarrow (X_1 \backslash \{v\}) .$$

If $S_2 \neq \emptyset$ and $T_2 \neq \emptyset$, then $S_1 \cup \{v\} \in \bar{J}_1$ and $T_1 \cup \{v\} \in J_1$. Hence

$$|S_1 \cap T_1| = |(S_1 \cup \{v\}) \cap (T_1 \cup \{v\})| - 1 \leqq 1 - 1 = 0,$$

so

$$|S \cap T| = |S_1 \cap T_1| + |S_2 \cap T_2| \leqq 0 + 1 = 1.$$

(c). Let first $S \in (J_1, J_2 \to v)$. Then $S = S_1 \cup S_2$ with $S_1 \subset X_1$ and $S_2 \subset X_2$. If $S_2 = \emptyset$, then $S \in J_1 \downarrow (X_1 \backslash \{v\})$, so assume that $S_2 \neq \emptyset$. Then $S_1 \cup \{v\} \in J_1$ and therefore

$$S \subset X_1^1 \cup X_2.$$

Let now $T \in \overline{J_1 \downarrow X_1^1} \cup \bar{J}_2$. If $T \in \overline{J_1 \downarrow X_1^1}$, then

$$|S \cap T| = |S_1 \cap T| \leqq 1,$$

and if $T \in \bar{J}_2$, then $|S \cap T| = |S_2 \cap T| \leqq 1$. Hence

$$S \in \overline{\overline{J_1 \downarrow X_1^1} \cup \bar{J}_2}.$$

Next let $S \in \overline{\overline{J_1 \downarrow X_1^1} \cup \bar{J}_2} \cup (J_1 \downarrow (X_1 \backslash \{v\}))$. If $S \in J_1 \downarrow (X_1 \backslash \{v\})$, then $S \in (J_1, J_2 \to v)$ by definition, so assume that

$$S \in \overline{\overline{J_1 \downarrow X_1^1} \cup \bar{J}_2}.$$

Write $S = S_1 \cup S_2$with $S_1 \subset X_1^1$ and $S_2 \subset X_2$. If $T \in \bar{J}_2$, then

$$|S_2 \cap T| \leqq |S \cap T| \leqq 1,$$

so $S_2 \in \bar{\bar{J}}_2 = J_2$. Now let $T \in \bar{J}_1$. If $v \notin T$, then

$$|T \cap (S_1 \cup \{v\})| = |T \cap S_1| \leqq 1$$

because $T \cap X_1^1 \in \bar{J}_1 \downarrow X_1^1 = \overline{J_1 \downarrow X_1^1}$. If $v \in T$, then $T \cap X_1^1 = \emptyset$ by definition of X_1^1, so

$$|T \cap (S_1 \cup \{v\})| = 1.$$

Hence $S \in (J_1, J_2 \to v)$.

We can now show the following theorem:

THEOREM 12. *Let J_1 and J_2 be* I.S.'s *on X_1 and X_2 respectively. If P_{J_1} and P_{J_2} are* WHP's, *$X_1 \cap X_2 = \emptyset$ and $v \in X_1$, then $P_{(J_1, J_2 \to v)}$ is a WHP.*

PROOF. Let $X = (X_1 \backslash \{v\}) \cup X_2 = \{v_1, \ldots, v_n\}$ and suppose $P_{(J_1, J_2 \to v)}$ is not a WHP. Let $y = (1, y_1, \ldots, y_n)$ be a vertex of $(\theta P_{(J_1, J_2 \to v)})^*$, with $|y_i| < 1$ for some i, and let

$$X_y = \{v_i \in X \mid |y_i| < 1, 1 \leqq i \leqq n\}.$$

Then $P_{J_1\downarrow(X_1\cap X_y)\cup\{v\})}$ and $P_{J_2}\downarrow(X_2\cap X_y)$ are WHP's, $((X_1\cap X_y)\cup\{v\})\cap(X_2\cap X_y)=\emptyset$ and $v\in(X_1\cap X_y)\cup\{v\}$, but

$$P_{(J_1\downarrow((X_1\cap X_y)\cup\{v\}),(J_2\downarrow(X_2\cap X_y))\to v)} = P_{(J_1,J_2\to v)\downarrow X_y}$$

is not a WHP (see Lemma 8 (c)). We can therefore assume that $X_y=X$. We split X_1 into three disjoint sets, $X_1=\{v\}\cup X_1^1\cup X_1^2$, where

$$X_1^1 = \{v_i\in X_1\backslash\{v\} \mid \{v,v_i\}\in J_1\}\ .$$

Let $X_1^1=\{v_1,\ldots,v_p\}$, $X_1^2=\{v_{p+1},\ldots,v_{p+q}\}$ and $X_2=\{v_{p+q+1},\ldots,v_{p+q+t}\}$, with $n=p+q+t$. By Lemma 11(c),

$$(J_1,J_2\to v)\downarrow(X_1^1\cup X_2) = \overline{\overline{J_1\downarrow X_1^1}\cup\bar{J}_2}\ ,$$

so $P_{(J_1,J_2\to v)\downarrow(X_1^1\cup X_2)}$ is a WHP by Theorems 4 and 10. Now let

$$Q = (\theta^{p+t}P_{(J_1,J_2\to v)\downarrow(X_1^1\cup X_2)})^*\ ,$$

and let $S_0,\ldots,S_n$ be chosen in $(J_1,J_2\to v)$, so that $S_0=\emptyset$ and

$$|(\theta_n x_{S_i}^n, y)| = 1 \qquad \text{for } i=0,1,\ldots,n\ .$$

We can arrange $S_1,\ldots,S_n$, so that $S_i\cap X_2\neq\emptyset$ for $i=1,2,\ldots,r$ and $S_i\cap X_2=\emptyset$ for $i=r+1,\ldots,n$. Put

$$y' = (1,y_1,\ldots,y_p,y_{p+q+1},\ldots,y_n)\ .$$

By Lemma 7(b), $y'\in Q$. Furthermore, for $i=0,1,\ldots,r$ we have $S_i\in(J_1,J_2\to v)\downarrow(X_1^1\cup X_2)$ and

$$(\theta_{p+t}x_{S_i}^{p+t},y')| = |(\theta_n x_{S_i}^n,y)| = 1\ ,$$

so y' lies in the intersection F of $r+1$ linearly independent facets of Q. Since $\partial_e F\subset\partial_e Q\subset\partial_e C^{p+t+1}$ and $|y_i|<1$ for $i=1,2,\ldots,n$, the coordinate functionals $e_1,\ldots,e_p,e_{p+q+1},\ldots,e_n$ cannot be constant on F, and their maximal [minimal] values on F must be 1 [-1 respectively]. Now let

$$z = (1,z_1,\ldots,z_p,z_{p+q+1},\ldots,z_n)\in\partial_e F\ .$$

Assume first, that $z_i=(z,e_i)=1$ for some $1\leqq i\leqq p$ and let $p+q+1\leqq j\leqq n$. Then $\{v_i,v_j\}\in(J_1,J_2\to v)\downarrow(X_1^1\cup X_2)$, so

$$|(\theta^{p+t}x_{\{v_i,v_j\}}^{p+t},z)| = |1+z_i+z_j| = |2+z_j| \leqq 1\ .$$

Hence $z_j=-1$. It follows that $z_{p+q+1}=\ldots=z_n=-1$ for every $z\in\partial_e F$ with $z_i=1$ for some $1\leqq i\leqq p$. Since $e_{p+q+1},\ldots,e_n$ are not constant on F, there is for each $p+q+1\leqq j\leqq n$ a vertex z^j of F with $z_j^j=1$ and $z_1^j=\ldots=z_p^j=-1$. We shall show that the vertices z^j $(p+q+1\leqq j\leqq n)$ all coincide with z^n, and therefore

$z_i^n=1$ for $p+q+1 \leqq i \leqq n$. Let $p+q+1 \leqq j \leqq n$ and define $w=(w_0,w_1,\ldots,w_n)$ as follows: $w_i=0$ for $i=p+1,\ldots,p+q$ and $w_i=z_i^j-z_i^n$ for $i=0,1,\ldots,p$, $p+q+1,\ldots,n$. Then $w_i=0$ for $0 \leqq i \leqq p+q$, so

$$(\theta_n x_{S_k}^n, w) = 0 \qquad \text{for } r+1 \leqq k \leqq n .$$

For $1 \leqq k \leqq r$ we have

$$(\theta_n x_{S_k}^n, w) = (\theta_{p+t} x_{S_k}^{p+t}, z^j - z_n) = 0 .$$

It follows that $w=0$, hence $z^j=z^n$. Assume now that $\{v_i,v_j\} \in J_2$ for some $p+q+1 \leqq i < j \leqq n$. Then $\{v_i,v_j\} \in (J_1,J_2 \to v)\downarrow(X_1^1 \cup X_2)$, so

$$|(\theta_{p+t} x_{\{vi,vj\}}^{p+t}, z^n)| \leqq 1 .$$

But

$$|(\theta_{p+t} x_{\{vi,vj\}}^{p+t}, z^n)| = 1+z_i+z_j = 3 .$$

Therefore $J_2=\{\emptyset,\{p+q+1\},\ldots,\{n\}\}$. This together with Theorem 4 and Lemma 11(b) gives that $\bar{J}_2=\{\emptyset,\{p+q+1\},\ldots,\{n\}\}$, but then we must have that $J_2=\{\varphi,\{p+q+1\}\}$ (that is $t=1$), so $P_{(J_1,J_2\to v)}=P_{J_1}$. Since P_{J_1} is a WHP, we are led to a contradiction.

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