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UNIFORMITY IN WEAK CONVERGENCE WITH RESPECT TO BALLS IN l^∞, c, $C([0,1])$

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1. Introduction.

Let X be a Banach space and $M_+(X)$ be the space of non-negative, totally finite Radon measures on X equipped with the weak topology. Let $\mu \in M_+(X)$ be such that $\mu(\partial B)=0$ for each ball $B \subset X$. If $\mu_n \to \mu$, $\mu_n \in M_+(X)$ then, obviously $\mu_n(B) \to \mu(B)$, for each B. F. Topsøe [1] has shown, that there is no uniformity when $X=c_0$ and that there is uniformity in the case $X=l^p$ $1 \leqq p < \infty$.

In the present note we shall show that in the space l^∞, $C([0,1])$ and c weak convergence implies uniform convergence over balls of bounded radii. Therefore the property under investigation in this paper does not divide Banach space into classes, which are usually "good" and "bad" spaces.

2. Basis definition and facts (cf. [1]).

When X and $M^+(X)$ are as in the introduction, $\mathscr{F}(X)$ stands for the family of all closed subsets in X. Let $\mathscr{A} \subset \mathscr{F}(X)$, $\mu \in M_+(X)$. We shall say that $\mathscr{A}$ is μ-continuous or equivalenty, that μ is $\mathscr{A}$-continuous if $\mu(\partial A)=0$ for each $A \in \mathscr{A}$. $\mathscr{A}$ is said to be an μ-uniformity class if for each sequence $\mu_n \to \mu$, $\mu_n \in M_+(X)$

$$\limsup_{n\to\infty} \sup_{A \in \mathscr{A}} |\mu_n(A) - \mu(A)| \;=\; 0\,.$$

If $\mathscr{A}$ is an μ-uniformity class for each $\mathscr{A}$-continuous measure μ, then $\mathscr{A}$ is called an ideal uniformity class. $B[x,r]$ will denote the closed and $B(x,r)$ the open ball with radius r and center x. For $A \subset X$ we denote by $\partial_\delta A$ the δ-boundary of A:

$$\partial_\delta A \;=\; \{x \;:\; B(x,\delta) \cap A \neq \emptyset\}\,.$$

$\mathscr{B}_r = \mathscr{B}_r(X)$ denotes the class of all closed balls with radii $\leqq r$.

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The problem we shall investigate is for which $r>0$ $\mathscr{B}_r$ is an ideal uniformity class. In what follows we shall need the following lemma due to F. Topsøe [1]:

LEMMA 1. *Let $\mathscr{A}\subset\mathscr{F}(X)$, $\mu\in M_+(X)$. The following conditions are equivalent*:

(i) *$\mathscr{A}$ is a μ-uniformity class.*
(ii) $\lim_{\delta\to 0}\sup_{A\in\mathscr{A}}\mu(\partial_\delta A)=0$.
(iii) $\forall\,(A_n)_{n\geqq 1}$ *in* $\mathscr{A}$, $\mu(\bigcap_{n=1}\partial_{1/n}A_n)=0$.

A proof of the above lemma can be found in [2] and [3].

3. The space l^∞.

THEOREM 1. *In the Banach space l^∞, for each $0<r<\infty$, $\mathscr{B}_r$ is an ideal uniformity class.*

LEMMA 2. *For each sequence $(B[x_n, r_n])_{n\geqq 1}$ in $\mathscr{B}_r$ there exist $z, y\in l^\infty$, such that*

$$\bigcap_{n=1}^{\infty}\partial_{1/n}B[x_n r_n]\subset\partial B[z,r]\cup\partial B[y,r]\,.$$

PROOF. We may and do assume that $\bigcap_{n=1}\partial_{1/n}B[x_n, r_n]\neq\emptyset$. Let

$$A=\bigcap_{n=1}B\left[x_n, r_n+\frac{1}{n}\right]\cap$$

$$\cap\left\{x\in l^\infty : \forall n\,\exists i,\ r_n-\frac{1}{n}<|x_n(i)-x(i)|<r_n+\frac{1}{n}\right\}.$$

Then

(1) $$A\supset\bigcap_{n=1}^{\infty}\partial_{1/n}B[x_n, r_n]$$

and

$$\bigcap_{n=1}^{\infty}B\left[x_n, r_n+\frac{1}{n}\right]=\prod_{i=1}^{\infty}[a_i, b_i]$$

where

$$a_i=\sup_{n\geqq 1}\left(x_n(i)-r_n-\frac{1}{n}\right)\quad b_i=\inf_{n\geqq 1}\left(x_n(i)+r_n+\frac{1}{n}\right).$$

For each $n, b_i - a_i \leqq 2r_n + 2/n$ so that $b_i - a_i \leqq 2r$. Let

$$I_1 = \left\{x \in l^\infty : \exists i \exists n_k \to \infty r_{n_k} - \frac{1}{n_k} < x_{n_k}(i) - x(i) < r_{n_k} + \frac{1}{n_k}\right\}$$

$$I_2 = \left\{x \in l^\infty : \exists i \exists n_k \to \infty, r_{n_k} - \frac{1}{n_k} < x(i) - x_{n_k}(i) < r_{n_k} + \frac{1}{n_k}\right\}$$

$$I_3 = \left\{x \in l^\infty : \exists i_k \to \infty \exists n_k \to \infty, r_{n_k} - \frac{1}{n_k} < x_{n_k}(i_k) - x(i_k) < r_{n_k} + \frac{1}{n_k}\right\}$$

$$I_4 = \left\{x \in l^\infty : \exists i_k \to \infty \exists n_k \to \infty, r_{n_k} - \frac{1}{n_k} < x(i_k) - x_{n_k}(i_k) < r_{n_k} + \frac{1}{n_k}\right\}$$

then

$$A \subset I_1 \cup I_2 \cup I_3 \cup I_4 . \tag{2}$$

Put

$$z(i) = b_i - r, \qquad y(i) = a_i + r .$$

Then $z, y \in l^\infty$.

Evidently $A \subset B[z,r] \cup B[y,r]$. If $x \in A \cap I_1$, then

$$r \geqq \|x-y\|_{l^\infty} = \sup_j (x(j)-y(j)) \geqq \sup_j |x(j)-a_j-r|$$

$$\geqq |x(i)-a_i-r| = \left|x(i) - \sup_n\left(x_n(i) - r_n - \frac{1}{n}\right) - r\right| \geqq r .$$

Similarly if $x \in A \cap I_2$, then $\|x-z\|_{l^\infty} = r$.

If $x \in A \cap I_3$, then

$$r \geqq \|x-y\|_{l^\infty} = \sup_j |x(j)-y(j)| \geqq \lim_{k\to\infty} |x(i_k)-y(i_k)| = r$$

and alike if $x \in A \cap I_4$, then $\|x-z\|_{l^\infty} = r$.

From the above consideration it follows that

$$A \cap (I_1 \cup I_3) \subset \partial B[y,r] \quad \text{and} \quad A \cap (I_2 \cup I_4) \subset \partial B[z,r]$$

where from, in view of (1), (2), the assertion of the lemma follows.

Now, the proof of Theorem 1 follows in a straightforward manner from Lemma 1 and Lemma 2.

REMARK 1. The above proof shows that Theorem 1 is true for $\mathscr{B}_\infty$ (the class of all closed balls in X) as well.

4. The space $C(Z)$ of continuous real functions over the compact metric set Z with the sup-norm.

THEOREM 2. *In the Banach space $C(Z)$ for each $0<r<\infty$ $\mathscr{B}_r$ is an ideal uniformity class.*

LEMMA 3. *If $K\subset C(Z)$ is compact, then the two functions $\check{k}$, $\hat{k}$ defined by*

$$\hat{k}(x) = \sup_{f\in K} f(x) \quad \textit{and} \quad \check{k}(x) = \inf_{f\in K} f(x)$$

are continuous.

This follows from the Arzela–Ascoli Theorem (cf. [4, IV 6 Corollary 8]).

LEMMA 4. *For each sequence $r_n\leqq r$ and each sequence $f_n \in C(Z)$ there exist $\check{f}$ lower semicontinuous and $\hat{f}$ upper semicontinuous functions such that* $\sup|\hat{f}(x)-\check{f}(x)|\leqq 2r$ *and*

$$\bigcap_{n=1}^{\infty} \partial_{1/n}B[f_n, r_n] \subset A(\hat{f}, \check{f}) \tag{3}$$

where

$$A(\hat{f},\check{f}) = \{g \in C(Z) : \check{f}\leqq g\leqq\hat{f} \textit{ and } \exists\, (x_n)_{n\geqq 1}$$
$$\lim_{n\to\infty} |\hat{f}(x_n)-g(x_n)|=0 \textit{ or } \lim_{n\to\infty} |\check{f}(x_n)-g(x_n)|=0\} .$$

PROOF. We may assume that $\bigcap_{n=1}^{\infty} \partial_{1/n}B[f_n, r_n] \neq \emptyset$. Put

$$\hat{f}(x) = \inf_n\left(f_n(x)+r_n+\frac{1}{n}\right),$$

$$\check{f}(x) = \sup_n\left(f_n(x)-r_n-\frac{1}{n}\right).$$

It is easy to see that $\hat{f}$ and $\check{f}$ have the desired properties.

PROOF OF THEOREM 2. Let $(B[f_n, r_n])_{n\geqq 1}$ be given. We may assume that $\bigcap_{n=1}^{\infty} \partial_{1/n}B[f_n, r_n] \neq \emptyset$. Let $\hat{f}$ and $\check{f}$ be as in Lemma 4. It is sufficient to show, that for each compact $K\subset A(\hat{f},\check{f})$ there exist two functions $k_1, k_2 \in C(Z)$ such that

$$K \subset \partial B[k_1, r] \cup \partial B[k_2, r] .$$

Once this is proved, in view of (3) the same is true for each compact set contained in $\bigcap_{n=1}^{\infty} \partial_{1/n}B[f_n, r_n]$, Lemma 1 will complete the proof of Theorem 2.

Let $K \subset A(\hat{f}, \check{f})$ be compact. From Lemma 3

$$\hat{k}(x) = \sup_{f \in K} f(x) \quad \text{and} \quad \check{k}(x) = \inf_{f \in K} f(x)$$

are continuous.

$$\sup_{x \in Z} |\hat{k}(x) - \check{k}(x)| \leqq \sup_{x \in Z} |\hat{f}(x) - \check{f}(x)| \leqq 2r .$$

Put

$$k_1(x) = \hat{k}(x) - r, \qquad k_2(x) = \check{k}(x) + r .$$

If $f \in K$, then

$$\check{f}(x) \leqq \check{k}(x) \leqq f(x) \leqq \hat{k}(x) \leqq \hat{f}(x)$$

and

$$\exists\, (x_n)_{n \geqq 1} \lim_{n \to \infty} |\check{k}(x_n) - f(x_n)| = 0 \text{ or } \lim_{n \to \infty} |\hat{k}(x_n) - f(x_n)| = 0 .$$

Then

$$\|f - k_1\| = r \quad \text{or} \quad \|f - k_2\| = r .$$

This completes the proof.

REMARK 2. The above proof shows that Theorem 2 is true for $\mathscr{B}_\infty$ as well.

COROLLARY 1. *In the Banach space* $C([0,1])$ *for each* $0 < r \leqq \infty$, $\mathscr{B}_r$ *is an ideal uniformity class.*

COROLLARY 2. *In the Banach space* c *of convergent sequences for each* $0 < r \leqq \infty$, $\mathscr{B}_r$ *is an ideal uniformity class.*

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