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PERTURBATIONS OF CENTRE-FIXING DYNAMICAL SYSTEMS

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1. Introduction.

Let α and β be σ-weakly continuous representations of a locally compact group G as automorphisms on a von Neumann algebra $\mathscr{M}$. If the action $\sigma_g = \beta_g \alpha_{-g}$ is pointwise inner, we say that β is an inner perturbation of the dynamical system $(\mathscr{M}, G, \alpha)$. Assuming that α leaves the centre of $\mathscr{M}$ pointwise invariant, we shall investigate under what conditions (on $G, \mathscr{M}$ and α) an inner perturbation is given by a strongly continuous unitary 1-cocycle in $\mathscr{M}$ (that is, $u_{gh} = u_g \alpha_g(u_h)$) in the sense that $\beta = (\operatorname{Ad} u) \circ \alpha$.

If α is the constant mapping onto the identity automorphism this is the wellknown problem concerning implementability of a pointwise inner group of automorphism – in other words, the question of when a projective or ray unitary representation lifts to a (genuine) unitary representation. The strongest result in this direction has been obtained by C. C. Moore ([11, Theorem 5]) who showed that if $\mathscr{M}$ has separable predual and G is a separable group with $H^2(G, \mathsf{T}) = 0$, such an implementation exists. In the more special case of a uniformly continuous representation, it was shown by J. Moffat in [8] that implementability occurs for any von Neumann algebra and any connected topological abelian group.

As far as the situation with non-trivial α is concerned, A. Connes showed in [2, Théoreme 1.2.8] that an implementing unitary 1-cocycle for σ may be found when $G = \mathsf{R}$ and $\mathscr{M}$ a factor with separable predual. When the perturbation is assumed to be bounded – i.e. σ to be uniformly continuous – D. Buchholz and J. R. Roberts proved in [1, Proposition 4.1] that such a 1-cocycle can be found for $G = \mathsf{R}$ and any von Neumann algebra (or simple C*-algebra). In the latter paper, references to the physical relevance of the perturbation problem are amply supplied.

In this paper, we take an approach which stresses that vanishing cohomology of G is sufficient but not always necessary to ensure implementability. We show that if G is abelian connected and $\mathscr{M}$ a factor with separable predual, then an inner perturbation is given by a quasi-1-cocycle

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($u_{g+h}=b(g,h)u_g\alpha_g(u_h)$ with b a continuous bihomomorphism from $G\times G$ into the circle). In case G has vanishing cohomology, the perturbation is given by a regular 1-cocycle.

If A is a simple C*-algebra with unit, α and β strongly continuous representations and β a bounded perturbation of α then there exists a norm-continuous quasi-1-cocycle u in A such that $\beta=(\operatorname{Ad} u)\circ\alpha$.

In the general von Neumann algebra case (assuming that α is centre-fixing) we show that when G is abelian with vanishing cohomology and $\mathcal{M}$ has separable predual, then an inner perturbation is given by a unitary 1-cocycle.

Finally, we obtain a proof of the result that when α is the constant mapping onto the identity automorphism and β is a uniformly continuous representation of a connected abelian group G then β is implemented by a norm continuous unitary group.

2. Notation and preliminaries.

Let $\mathcal{M}$ be a von Neumann algebra and denote by $\operatorname{Aut}\mathcal{M}$ the group of all *-automorphism of $\mathcal{M}$, by $\operatorname{Int}\mathcal{M}$ the subgroup of inner *-automorphisms. In the following we shall be interested in two topologies on $\operatorname{Aut}\mathcal{M}$, both making it a topological group. The *uniform topology* arises by regarding $\operatorname{Aut}\mathcal{M}$ as a subset of the Banach space $B(\mathcal{M})$ of bounded linear operators on $\mathcal{M}$, and the *σ-weak topology* by regarding convergence of the automorphisms pointwise on the elements of $\mathcal{M}$ in the $\sigma(\mathcal{M},\mathcal{M}_*)$-topology, $\mathcal{M}_*$ denoting the predual of $\mathcal{M}$. Thus $\alpha_\lambda \to \alpha$ σ-weakly if $\varrho(\alpha_\lambda(x)) \to \varrho(\alpha(x))$ for every ϱ in $\mathcal{M}_*$ and x in $\mathcal{M}$.

Let α be a σ-weakly continuous homomorphism from the locally compact abelian (henceforth abbreviated l.c.a.) group G into $\operatorname{Aut}\mathcal{M}$. We shall call the triple $(\mathcal{M},G,\alpha)$ a dynamical system. In the following we shall always assume $(\mathcal{M},G,\alpha)$ to be *centre-fixing*, by which we mean that α leaves the centre $\mathcal{Z}$ of $\mathcal{M}$ pointwise invariant – a condition automatically fulfilled when $\mathcal{M}$ is a factor.

Let $(\mathcal{M},G,\alpha)$ be a dynamical system. A σ-weakly continuous map σ: $G\to\operatorname{Aut}\mathcal{M}$ is called a 1-cocycle (with respect to $\operatorname{Ad}\alpha$) if for all g and h in G

$$\sigma_{g+h} = \sigma_g\alpha_g\sigma_h\alpha_{-g} \quad (=\sigma_g \operatorname{Ad}\alpha_g(\sigma_h)) .$$

Let $(\mathcal{M},G,\beta)$ be another dynamical system. The map σ defined by $\sigma_g=\beta_g\alpha_{-g}$ is then a 1-cocycle. Especially, σ is a homomorphism if and only if α and β commute. Conversely, given α and a 1-cocycle $\sigma,\beta=\sigma\alpha$ is a representation of G. We say that β is a *perturbation* of α by σ. When σ is pointwise inner, we speak of σ as an *inner cocycle* and of $\beta=\sigma\alpha$ as an *inner perturbation* of α. If σ is uniformly continuous, i.e.

$$\|\sigma_g-\iota\| \to 0 \quad \text{as } g\to 0$$

we call $\beta=\sigma\alpha$ a *bounded perturbation* of α (see also [1]).

A unitary map $u\colon G \to \mathcal{M}$ is called a 1-*cocycle with respect to* α if it satisfies the equation

$$u_{g+h} = u_g\alpha_g(u_h)$$

for all g and h in G. If b is a bihomomorphism from $G \times G$ into the unitary group in $\mathcal{Z}$, u is said to be a *quasi*-1-*cocycle with respect to* α if

$$u_{g+h} = b(g,h)\,u_g\alpha_g(u_h)$$

for all g and h in G.

We shall call a 1-cocycle $\sigma\colon G \to \operatorname{Aut}\mathcal{M}$ *implementable* if there exists a (suitably continuous) unitary 1-cocycle in $\mathcal{M}$ such that $\sigma = \operatorname{Ad} u$, and *quasi-implementable* if $\sigma = \operatorname{Ad} u$ with u a unitary quasi-1-cocycle. In terms of the perturbation β we shall say that β is *given by* a (quasi-)1-cocycle u when σ defined by $\sigma_g = \beta_g\alpha_{-g}$ is (quasi-)implementable.

In this paper, we list a number of sufficient conditions for (quasi-) implementability.

3. The bihomomorphism associated to an inner perturbation.

DEFINITION 3.1. Let $(\mathcal{M}, G, \alpha)$ be a centre-fixing dynamical system and σ: $G \to \operatorname{Aut}\mathcal{M}$ a pointwise inner cocycle with respect to $\operatorname{Ad}\alpha$. Assume that u: $G \to \mathcal{M}$ is a unitary map such that $\sigma = \operatorname{Ad} u$. We define c to be the map from $G \times G$ into the unitary group in the centre $\mathcal{Z}$ of $\mathcal{M}$ given by

$$c(g,h) = u_g\alpha_g(u_h)\alpha_h(u_g^*)u_h^* .$$

PROPOSITION 3.2. *The map c as defined in* 3.1 *is a skew-symmetric bihomomorphism which is independent of the specific choice of implementing family u for σ. When $\mathcal{M}$ has separable predual and G is separable* (*i.e. second countable*), *c is σ-weakly* (*strongly*) *continuous.*

PROOF. c does not depend on the map u, since another unitary map w: $G \to \mathcal{M}$ for which $\sigma = \operatorname{Ad} w$ is the pointwise product of a unitary map z: $G \to \mathcal{Z}$ and u. c is skew-symmetric since

$$c(h,g) = u_h\alpha_h(u_g)\alpha_g(u_h^*)u_g^* = (u_g\alpha_g(u_h)\alpha_h(u_g^*)u_h^*)^* = c(g,h)^*$$

of α_g and the fact that $u_h\alpha_h(u_k)$ and u_{h+k} both implement σ_{h+k}, thus $u_h\alpha_h(u_k) = u_{h+k}z$ with z a central unitary, we have that

$$\begin{aligned} v_h\beta_h(v_k) &= (\alpha_g(u_h)u_h^*)\beta_h(\alpha_g(u_k)u_k^*) \\ &= \alpha_g(u_h)(\sigma_h^{-1}\circ\beta_h)(\alpha_g(u_k)u_k^*)u_h^* \\ &= \alpha_g(u_h)\alpha_{h+g}(u_k)\alpha_h(u_k)^*u_h^* \\ &= \alpha_g(u_h\alpha_h(u_k))(u_h\alpha_h(u_k))^* \\ &= \alpha_g(u_{h+k}z)(u_{h+k}z)^* \\ &= \alpha_g(u_{h+k})zz^*u_{h+k}^* \\ &= \alpha_g(u_{h+k})u_{h+k}^* = v_{h+k}\,. \end{aligned}$$

Now a standard argument shows that if v is a 1-cocycle with respect to β, then for any fixed unitary w we have that

$$h \mapsto wv_h\beta_h(w^*)$$

is also a cocycle with respect to β. Thus for every fixed g

$$h \mapsto c(g,h) = u_g(\alpha_g(u_h)u_h^*)\beta_h(u_g^*)$$

is a cocycle with respect to β. But $c(g,h) \in \mathscr{Z}$ and β leaves $\mathscr{Z}$ pointwise invariant, thus $h \mapsto c(g,h)$ is a homomorphism. Since

$$c(g,h)^* = c(h,g)$$

we conclude that also $g \mapsto c(g,h)$ for fixed h is a homomorphism.

Note that the weak, σ-weak, strong and σ-strong topologies on $\mathscr{M}$ all coincide on the unitary group $\mathscr{U}$, and that $\mathscr{U}$ equipped with this topology is a Polish space when the predual $\mathscr{M}_*$ of $\mathscr{M}$ is separable.

Given $\sigma\colon G \to \operatorname{Int}\mathscr{M}$, $\mathscr{M}_*$ separable, we know by e.g. [6, p. 47] that there exists a Borel function $u\colon G \to \mathscr{U}$ such that $\sigma = \operatorname{Ad} u$. Thus c is a (jointly) Borel bihomomorphism from $G \times G$ into $\mathscr{U} \cap \mathscr{Z}$, and by an argument similar to that employed for a homomorphism in [14, p. 67] c is then (jointly) continuous when G is separable l.c.a. and $\mathscr{U} \cap \mathscr{Z}$ separable.

REMARK 3.3. If $(\mathscr{M}, G, \alpha)$ is not centre-fixing, an inner perturbation β need not give rise to a bihomomorphism.

To see this, take e.g. $\mathscr{M} = L^\infty(\mathbf{R})$, $G = \mathbf{R}$ and $\alpha = \beta$ to be the translation group $(\alpha_t(f(s)) = f(s-t))$. Then any unitary function u_t in $L^\infty(\mathbf{R})$ implements σ_t (where $\sigma_t = \beta_t\alpha_{-t}$) so take e.g. $t \mapsto u_t$ to be the constant map onto some u. Then

$$c(s,t)(p) = u(p)u(p-s)\overline{u(p-t)\,u(p)} = u(p-s)\overline{u(p-t)},$$

so $c(1,1)\equiv 1$, $c(1,0)(p)=u(p-1)\overline{u(p)}$ and $c(1,-1)(p)=u(p-1)\overline{u(p+1)}$. Thus choosing e.g.

$$u(p) = \begin{cases} 1 & -\infty<p<0 \\ -1 & 0\leqq p<\infty \end{cases}$$

we see that $c(1,0)\not\equiv c(1,1)c(1,-1)$ and so c is not homomorphic in t.

REMARK 3.4. In general, we have to use a "measurable choice" argument to prove continuity of c, and this presupposes separability of $\mathscr{M}_*$.

In two important special cases we are, however, able to prove some kind of continuity without such a type of argument, thus without separability conditions on $\mathscr{M}_*$ and G. One case is where the cocycle σ is uniformly continuous. For this, we refer to the lemma below. The other case is where α is the constant mapping onto the identity automorphism, and $\sigma=\beta$ a pointwise inner homomorphism from G into $\operatorname{Aut}\mathscr{M}$. Indeed, here we have that

$$c(g,h) = u_g(\alpha_g(u_h)u_h^*)\beta_h(u_g^*) = u_g\beta_h(u_g^*)\,.$$

Thus the continuity of $h\mapsto c(g,h)$ for fixed g is immediate and the skew-symmetry of c then implies that $g\mapsto c(g,h)$ for fixed h is continuous. Thus c is separately continuous. It now follows from e.g. [7, Lemma 9.2] that c is jointly measurable, thus by e.g. [14, p. 67] jointly continuous, whenever $\mathscr{M}_*$ and G are separable.

LEMMA 3.5. *Let $(\mathscr{M},G,\alpha)$ be a centre-fixing dynamical system, G a connected* l.c.a. *group and $\sigma\colon G\to\operatorname{Aut}\mathscr{M}$ a uniformly continuous cocycle with respect to* $\operatorname{Ad}\alpha$. *Then σ is pointwise inner and the map c defined as in* 3.1 *is norm-continuous.*

PROOF. By [5, Lemma 5, Theorem 7] we may for each g in G such that $\|\sigma_g-\iota\|<2$ choose u_g in $\mathscr{M}$ so $\sigma_g=\operatorname{Ad}u_g$ and

$$\operatorname{sp}(u_g) \subseteqq \{z\in\mathbf{C} \mid \operatorname{Re}z\geqq\tfrac{1}{2}\sqrt{4-\|\sigma_g-\iota\|^2}\}\,.$$

Since G is connected, the fact that σ is inner on a neighbourhood of 0 in G implies that it is inner everywhere. Furthermore, any family u chosen as above on this neighbourhood is continuous at 0, thus the map c is continuous at (0,0). But then the bihomomorphic property of c implies that it is everywhere continuous.

REMARK 3.6. The connection between c and the 2-cocycle most often considered in implementation studies is simple. In fact, when a measurable implementing unitary map $u\colon G \to \mathcal{M}$ is chosen, the central 2-cocycle is defined to be the map

$$\omega\colon (g,h) \mapsto u_g \alpha_g(u_h) u_{g+h}^*$$

(see e.g. [2, p. 151]). It is thus immediate that

$$c(g,h) = \omega(g,h)\omega(h,g)^* .$$

Assume $\mathcal{M}$ to be a factor. Then $\omega \in H^2(G,\mathsf{T})$, and if G has trivial cohomology, ω is symmetric, thus $c \equiv 1$. If $\mathcal{M}$ is a general von Neumann algebra, it follows from [11, Theorem 1] that this is also true. Thus trivial cohomology of G suffices to ensure $c \equiv 1$.

4. The factor case: quasi-implementability and implementability.

In this section we take $\mathcal{M}$ to be a factor with separable predual $\mathcal{M}_*$ and show that a quasi-implementation can be obtained even in cases where the cohomology of G is non-trivial (notably $G = \mathsf{R}^n$). When G has trivial cohomology, implementability occurs.

When dealing with bounded perturbations the separability assumption on $\mathcal{M}_*$ can be dropped and a norm-continuous implementing (quasi-)1-cocycle can be found.

LEMMA 4.1. *Let $(\mathcal{M}, G, \alpha)$ be a dynamical system, σ an inner cocycle with respect to* Ad α. *Define c as in* 3.1. *If $\mathcal{M}$ is a factor with separable predual and G is separable connected or the integers there exists a continuous bihomomorphism d from $G \times G$ into* R *such that for all g and h in G*

$$c(g,h) = \exp(id(g,h)) .$$

PROOF. (i) Assume $G = \mathsf{R}^n$. By 3.2 c is continuous, thus there is an open ball $\mathscr{V}$ around 0 such that

$$c(g,h) \in \left\{ z \in \mathsf{T} \mid -\frac{\pi}{2} < \operatorname{Arg} z < \frac{\pi}{2} \right\}$$

for all g and h in $\mathscr{V}$, Arg denoting the principal value of the argument. Set

$$d(g,h) = \operatorname{Arg} c(g,h)$$

for g and h in $\mathscr{V}$, then d is continuous, skew-symmetric and bihomomorphic on $\mathscr{V} \times \mathscr{V}$, that is, a continuous skew-symmetric bilinearform into R.

Now $\mathscr{V}$ generates $\mathbf{R}^n$, thus we may extend d to $\mathbf{R}^n \times \mathscr{V}$ by setting $d(g,h) = md(g',h)$ where g is arbitrary in $\mathbf{R}^n$, $g = mg'$ with $m \in \mathbf{Z}$ and $g' \in \mathscr{V}$ and $h \in \mathscr{V}$. This extension is easily checked to be well-defined and a bihomomorphism of $\mathbf{R}^n \times \mathscr{V}$. Analogously, we may extend d to a bihomomorphism d of $\mathbf{R}^n \times \mathbf{R}^n$. Then $\exp(id)$ defines a bihomomorphism from $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{T}$ which coincides with c on $\mathscr{V} \times \mathscr{V}$, thus is identical to c.

(ii) Assume $G = K$, K compact connected separable. Then c defines a continuous homomorphism Φ from G into $\hat{G}$ by

$$\Phi(g)(h) = c(g,h) .$$

Now $\Phi(G)$ is a compact connected subgroup of the discrete group $\hat{G}$, thus $\Phi(G) = \{\hat{0}\}$. But this implies that c is identically 1.

(iii) Assume G to be connected. By the structure theorem for l.c.a. groups, $G = \mathbf{R}^n \times K$, K compact connected. When G is separable, so is K. We factorize $c \colon G \times G \to \mathbf{T}$

$$\begin{aligned} &c((g,k),(g',k')) \\ &= c((g,0),(g',0))c((g,0),(0,k'))c((0,k),(g',0))c((0,k),(0,k')) \end{aligned}$$

and note that

$$\begin{aligned} &c((g,0),(g',0)) \text{ is a bihomomorphism from } \mathbf{R}^n \times \mathbf{R}^n \text{ into } \mathbf{T} \\ &c((g,0),(0,k')) \text{ is a bihomomorphism from } \mathbf{R}^n \times K \text{ into } \mathbf{T} \\ &c((0,k),(g',0)) \text{ is a bihomomorphism from } K \times \mathbf{R}^n \text{ into } \mathbf{T} \\ &c((0,k),(0,k')) \text{ is a bihomomorphism from } K \times K \text{ into } \mathbf{T} \end{aligned}$$

By (ii), $c((0,k),(0,k'))$ is identically 1. Arguing as in (ii) we see the bihomomorphisms from $\mathbf{R}^n \times K$ (or $K \times \mathbf{R}^n$) define continuous homomorphisms from K into $\mathbf{R}^n$, thus are identically 1. So we conclude that

$$c((g,k),(g',k')) \equiv c((g,0),(g',0)) = \exp\big(id((g,0),(g',0))\big)$$

with $d \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ bilinear.

(iv) $G = \mathbf{Z}$. A direct computation shows that

$$c(n,m) = c(1,m)^n = c(1,1)^{nm} = 1 .$$

By lemma 4.1, all connected separable l.c.a. groups G fall into the class of groups for which the map c has a continuous square-root. The following holds for all such groups:

THEOREM 4.2. *Let $(\mathscr{M}, G, \alpha)$ be a dynamical system, σ an inner cocycle with respect to* $\operatorname{Ad}\alpha$. *Assume $\mathscr{M}$ to be a factor with separable predual, G to be* *separable. Whenever c as defined in 3.1 has a continuous square root $c^{\frac{1}{2}}$ there exists a strongly continuous mapping $u\colon G \to \mathscr{U}$ such that $\sigma = \operatorname{Ad} u$ and for all g and h*

$$u_{g+h} = c(h,g)^{\frac{1}{2}} u_g \alpha_g(u_h) .$$

PROOF. When $\mathscr{M}$ is a von Neumann algebra with separable predual, $\mathscr{U}$ is a Polish group in the strong topology, thus $\mathscr{U}/\mathscr{Z}$ is a standard Borel space in the Borel structure coming from the quotient topology. It is also wellknown that $\operatorname{Aut}\mathscr{M}$ is Polish in the topology of pointwise normconvergence on the predual.

Let $\iota\colon \operatorname{Int}\mathscr{M} \to \operatorname{Aut}\mathscr{M}$ denote the injection map, and let $\varphi\colon \mathscr{U}/\mathscr{Z} \to \operatorname{Int}\mathscr{M}$ take $\dot{u} = u\mathscr{Z}$ into $u \cdot u^* = \operatorname{Ad} u$.

$$\begin{tikzcd} \mathscr{U}/\mathscr{Z} \arrow[r, "\varphi"] & \operatorname{Int}\mathscr{M} \arrow[r, "\iota"] & \operatorname{Aut}\mathscr{M} \\ & G \arrow[ul, "\pi"] \arrow[u, "\sigma"'] & \end{tikzcd}$$

$\iota \circ \varphi\colon \mathscr{U}/\mathscr{Z} \to \operatorname{Aut}\mathscr{M}$ is an injective Borel map from a standard space into a Polish space, thus the image $\operatorname{Int}\mathscr{M}$ is a Borel subset of $\operatorname{Aut}\mathscr{M}$ and φ a Borel isomorphism. It follows that if G is l.c.a. separable and $\pi\colon G \to \mathscr{U}/\mathscr{Z}$ a 1-cocycle, then π is continuous if and only if $\sigma = \varphi \circ \pi\colon G \to \operatorname{Int}\mathscr{M}$ is continuous.

Let $\mathscr{U} \underset{\alpha}{\overset{c}{\times}} G$ denote the "c-semi-direct" product of $\mathscr{U}$ and G, i.e. define

$$(u,g) \cdot (v,h) = (c(h,g)^{\frac{1}{2}} u \alpha_g(v), g+h) .$$

It is easily checked that $\mathscr{U} \underset{\alpha}{\overset{c}{\times}} G$ is a group. Let k denote the quotient map from $\mathscr{U}$ onto $\mathscr{U}/\mathscr{Z}$, and define the subgroup E of $\mathscr{U} \underset{\alpha}{\overset{c}{\times}} G$ to be

$$E = \left\{ (u,g) \in \mathscr{U} \underset{\alpha}{\overset{c}{\times}} G \mid k(u) = \pi(g) \right\} .$$

We claim that E is abelian. To see this, note that when $\sigma_g = \operatorname{Ad} u$ and $\sigma_h = \operatorname{Ad} v$ we have

$$u\alpha_g(v) = c(g,h) v \alpha_h(u) ,$$

thus

$$\begin{aligned} (u,g) \cdot (v,h) &= (c(h,g)^{\frac{1}{2}} u \alpha_g(v), g+h) \\ &= ((c(g,h)^{\frac{1}{2}})^* c(g,h) v \alpha_h(u), g+h) \\ &= (c(g,h)^{\frac{1}{2}} v \alpha_h(u), g+h) \\ &= (v,h) \cdot (u,g) . \end{aligned}$$

This shows that E in the factor case is an abelian extension of the l.c.a. group G over $\mathcal{U} \cap \mathcal{Z} = \mathsf{T}$, and it now follows from [9, section 2.2, page 52] that E must be l.c.a. Thus the exact sequence

$$\mathsf{T} \to E \to G$$

splits (see e.g. [3] or [11, comment before theorem 4]), i.e. there exists a continuous homomorphism from G into E, $g \to (u_g, g)$. But this is exactly what we wanted, since $g \to u_g$ is then continuous and satisfies that $\sigma_g = \operatorname{Ad} u_g$ and

$$u_{g+h} = c(h,g)^{\frac{1}{2}} u_g \alpha_g(u_h) .$$

COROLLARY 4.3. *Let $(\mathcal{M}, G, \alpha)$ and σ be as in 4.2. When $G = \mathsf{R}$, a separable compact connected abelian group K, a product of the form $\mathsf{R} \times K$ or the integers, σ is implementable.*

PROOF. We want to see that for G as above, $c \equiv 1$. By the proof of lemma 4.1, it suffices to show this for $G = \mathsf{R}$. This is true due to the fact that the vector-space of skew-symmetric bilinearforms from $\mathsf{R} \times \mathsf{R}$ into R is 0-dimensional.

REMARK 4.4. Let $G = \mathsf{R}^n$, then the vectorspace of bilinearforms from $\mathsf{R}^n \times \mathsf{R}^n$ into R can be identified with the space of real $n \times n$-matrices. The skew-symmetric forms correspond to the skew-symmetric $n \times n$-matrices, and so form a real vector-space of dimension $\frac{1}{2}n(n-1)$. This yields a rather precise description of the implementation possibilities in theorem 4.2.

E.g. for $n = 2$ we get that either

$$d((s,t),(r,p)) = 0 \text{ or } d((s,t),(r,p)) = a \cdot (sp - rt)$$

with $a \in \mathsf{R}$. Thus the family $u_{(s,t)}$ is either a 1-cocycle or satisfies that

$$u_{(s+r,t+p)} = e^{\frac{ia}{2}(sp-rt)} u_{(s,t)} \alpha_{(s,t)}(u_{(r,p)}) .$$

LEMMA 4.5. *Let $(\mathcal{M}, G, \alpha)$ be a dynamical system, σ a uniformly continuous 1-cocycle with respect to* $\operatorname{Ad}\alpha$. *Assume $\mathcal{M}$ to be a factor. When G is connected or the integers there exists a continuous bihomomorphism $d \colon G \times G \to \mathsf{R}$ such that for all g and h in G*

$$c(g,h) = \exp(id(g,h)) .$$

When $G = \mathsf{R}$, a compact connected abelian group K, a product of the form $\mathsf{R} \times K$ or the integers, d is identically 0.

PROOF. That c is definable and continuous follows from lemma 3.5. The rest of the argument goes as in lemma 4.1 (where the separability of G was only assumed in order to ensure continuity of c).

The above lemma ensures that both conclusions of our next theorem are non-void.

THEOREM 4.6. *Let $(\mathcal{M}, G, \alpha)$ be a dynamical system, σ a uniformly continuous* 1-*cocycle with respect to* $\mathrm{Ad}\,\alpha$. *Assume $\mathcal{M}$ to be a factor. When c as defined in* 3.1 *has a continuous square root, σ is quasi-implementable, and a norm-continuous implementing quasi-*1-*cocycle u: $G \to \mathcal{U}$ may be chosen. When $c \equiv 1$, u may be chosen a* 1-*cocycle.*

PROOF. Note that the group $\mathrm{Int}\,\mathcal{M}$ in its uniform topology is homeomorphic to $\mathcal{U}/\mathsf{T}$ in the quotient norm-topology ([1, Proposition 2.3]). We are thus able to "identify" σ: $G \to \mathrm{Int}\,\mathcal{M}$ with its corresponding projective 1-cocycle π: $G \to \mathcal{U}/\mathsf{T}$. From this point we may proceed word-to-word as in the proof of theorem 4.2, the only difference being that we now regard $\mathcal{U}$ in its norm-topology instead of its strong topology.

Recall that a C*-algebra A is said to be simple if it contains no non-trivial closed two-sided ideals. If A is a simple C*-algebra with unit 1, its centre consists of the "scalars" $\mathsf{C}\cdot 1$. For simple C*-algebras we have a result analogous to theorem 4.6.

PROPOSITION 4.7. *Let A be a simple C*-algebra with unit, α and β strongly continuous representations of the* l.c.a. *group G as automorphisms on A. Assume that $\sigma_g = \beta_g \alpha_{-g}$ defines a uniformly continuous map from G into* $\mathrm{Aut}\,A$. *Then σ is pointwise inner, and when the map c as defined in* 3.1 *has a continuous square-root, σ is quasi-implementable.*

Especially we note that when G is connected, σ is quasi-implementable, and that when $G = \mathsf{R}$, a compact (connected abelian) group K or a product of the form $\mathsf{R} \times K$, σ is implementable.

PROOF. It follows from [12, Corollary 2.5, Remark 2.3] that σ is pointwise inner, and that the implementing unitary family u may be chosen norm-continuous at 0. Thus we deduce that c is a bicontinuous map into T.

From [1, Lemma 5.2] we know that $\mathcal{U}/\mathsf{T}$ and $\mathrm{Int}\,A$ are homeomorphic when $\mathcal{U}/\mathsf{T}$ is equipped with the quotient norm-topology and $\mathrm{Int}\,A$ with the uniform topology. This allows us to proceed the argument exactly as in 4.6.

For the last part of the statement note that lemma 4.5 carries over to the simple C^*-case.

5. The general von Neumann algebra case.

In this more general setting we deal with implementability only, leaving out the quasi-aspect. In fact, lemma 4.1 carries over only when c is norm-continuous, and a theorem dealing abstractly with a square-root seems out of place. We note that corollary 5.2 is a direct extension of [2, Théorème 1.2.8].

THEOREM 5.1. *Let $(\mathscr{M}, G, \alpha)$ be a centre-fixing dynamical system, σ a pointwise inner 1-cocycle with respect to* $\mathrm{Ad}\,\alpha$. *Assume $\mathscr{M}_*$ to be separable, G to be* R, *a compact connected separable abelian group K, a product of the form* $\mathsf{R} \times K$ *or the integers. Then σ is implementable.*

PROOF. Define $\mathscr{U} \overset{1}{\underset{\alpha}{\times}} G$ and E as in the proof of theorem 4.2. Note that it follows from [11, theorem 1] that E is abelian. (For $G = \mathsf{R}$ we may also conclude this from [3].) Defining $\tilde{\imath}(z) = (z, 0)$ and $\tilde{k}$ to be the projection from E onto G,

$$\mathscr{U} \cap \mathscr{Z} \xrightarrow{\tilde{\imath}} E \xrightarrow{\tilde{k}} G$$

is a short exact sequence, i.e. $\tilde{\imath}$ is an injective homeomorphism from $\mathscr{U} \cap \mathscr{Z}$ onto $\tilde{\imath}(\mathscr{U} \cap \mathscr{Z})$ and $\tilde{k}$ is a continuous and open surjection with kernel equal to $\tilde{\imath}(\mathscr{U} \cap \mathscr{Z})$.

It is well-known that $\mathscr{Z}$ is isomorphic to a space $L^\infty(X, \mu)$ of (equivalence classes of) measurable functions on a standard measure space X with finite measure μ. Thus $\mathscr{U} \cap \mathscr{Z}$ identifies with the measurable functions on X with values in T, $U(X, \mathsf{T})$, equipped with the σ-weak topology (or equivalently the strong operator topology on $L^2(X)$). This topology coincides with the L^1-topology, and so [11, theorem 4] allows us to conclude that the above exact sequence splits.

We obtain the following direct corollary:

COROLLARY 5.2. *Let $\mathscr{M}$ be a von Neumann algebra with separable predual, let φ be a normal semi-finite faithful weight and σ^φ the corresponding modular automorphism group. Assume β to be a σ-weakly continuous one-parameter group of automorphisms of $\mathscr{M}$ such that for every t in* R *there is a unitary u_t which* *satisfies*

$$\beta_t = (\mathrm{Ad}\, u_t) \circ \sigma_t^\varphi .$$

Then β is itself a modular group corresponding to a weight ψ.

6. An implementation result for groups with non-trivial cohomology.

In this section we do not presuppose the group G to be locally compact.

LEMMA 6.1. *Let G be a connected abelian topological group, σ a uniformly continuous representation of G as automorphisms of the von Neumann algebra $\mathcal{M}$. Then c defined as in* 3.1 *with $\alpha = \iota$ is identically* 1.

PROOF. In this setting we have that

$$\|c(g,h)-1\| = \|u_g \sigma_h(u_g^*)-1\| = \|\sigma_h(u_g^*)-u_g^*\| \leqq \|\sigma_h - \iota\| ,$$

i.e. an evaluation of $c(g,h)$ which is independent of g. It easily follows that c is identically 1.

From the above lemma one may then proceed to obtain the implementation result first proved by J. Moffat in [8], that if G is a connected abelian topological group, σ a uniformly continuous representation of G as automorphisms of the von Neumann algebra $\mathcal{M}$, then σ is implemented by a norm-continuous unitary group.

REMARK 6.2. If A is a simple C*-algebra with unit, G a locally compact connected abelian group and σ a uniformly continuous representation of G as automorphisms of A, lemma 6.1 carries over unchanged to show that any implementing unitary family consists of mutually commuting elements. Thus we may reason as in 4.7 to conclude that there exists an implementing norm-continuous unitary representation. A constructive proof of this result can be found in [13].

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