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SCATTERED C*-ALGEBRAS

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Introduction.

A topological space X is called scattered, if each subset of X has an isolated point. The compact, scattered spaces are exactly those compact spaces X, for which each Radon measure on X is atomic [9, section 19]. Owing to the Riesz representation theorem the study of positive Radon measures on a compact space X is equivalent with the study of positive functionals on $C(X)$. It is the purpose of this paper, using this equivalence, to extend the notion of scattered, compact spaces to the natural generalization of $C(X)$, that is to C*-algebras, and to start a closer investigation of such algebras.

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1. Atomic functionals on C*-algebras.

The concept of atomic measures on locally compact spaces has been generalized to C*-algebras in [5], but only in the separable case. Since we want to study the general case, we will have to use another approach, which also turns out to be more suitable for our primary purpose, than that in [5].

Let B denote the enveloping von Neumann-algebra of a C*-algebra A. It is well-known, that each positive functional f on A extends in a unique way to a positive, normal functional on B, which is again denoted by f. The extension is pure, if and only if f is pure, and in this case the support of f is a minimal projection in B. We will now use the following definition.

Definition 1.1. Let f denote a positive functional with support e. Then f is called atomic, if for each projection $e_1 \in B$ with $e_1 \leq e$, there exists a minimal projection $e_0 \in B$, such that $e_0 \leq e_1$.

That this definition is reasonable follows from

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THEOREM 1.2. *Let f denote a positive functional on a C*-algebra A. The following two conditions are equivalent.*

$$(1) \qquad f \text{ is atomic}$$

$$(2) \qquad f = \sum f_n, \text{ where each } f_n \text{ is pure .}$$

The sum in (2) may be finite or infinite (but countable). In the last case the convergence is of course pointwise convergence. Before the proof we need a lemma.

LEMMA 1.3. *Let $\pi : A \to L(H)$ be a not degenerate representation of the C*-algebra A, let $\xi \in H$, and let*

$$f(x) = (\pi(x)\xi \mid \xi), \quad x \in A .$$

Suppose, that $\xi = \sum \xi_i$, a finite or countable sum, where each functional $x \to (\pi(x)\xi_i \mid \xi_i)$ is pure. Then $f = \sum f_i$, where each f_i is pure.

PROOF. Let M denote the weak closure of $\pi(A)$, and let p_i be the central support of the vector state ω_{ξ_i} on M. Then p_i is a minimal central projection, because ω_{ξ_i} is pure. If $p = \sum p_i$ where the sum is over a maximal orthogonal subset of the p_i, we have $p\xi = \xi$, and therefore $\omega_\xi(y) = \omega_\xi(py) = \sum \omega_\xi(p_i y)$ for all $y \in M$. Since $p_i M$ is all bounded operators on $p_i(H)$, each functional $f_i(x) = (p_i \pi(x)\xi \mid \xi)$ is pure on A, which completes the proof.

PROOF OF THEOREM 1.2. (1)$\Rightarrow$(2). By Zorn's lemma the support of f in B, denoted by e, is an orthogonal sum of minimal projections e_i in B, and this sum is countable. If $f(x) = (\pi(x)\xi \mid \xi)$, let $\xi_i = e_i \xi$. Then $\xi = \sum \xi_i$, and the functional $x \to (\pi(x)\xi_i \mid \xi_i)$ is pure, since e_i is minimal in B. By lemma 1.3, $f = \sum f_i$ with f_i pure.

(2)$\Rightarrow$(1). Let $f = \sum f_i$, with f_i pure, and let p_i be the central support of f_i in B. If e_1 is a non-zero projection in B with $e_1 \leq e$ (the support of f) there exists a p_i, such that $e_1 p_i \neq 0$. Since $p_i B$ is a type I factor, there is a minimal projection $e_0 \in B$ with $e_0 \leq e_1 p_i \leq e_1$.

The theorem just proved enables us to give a characterization of an atomic functional f in terms of the corresponding representation π_f.

PROPOSITION 1.4. *A positive functional f on a C*-algebra A is atomic if and only if π_f is unitarily equivalent with a subrepresentation of a countable sum of irreducible representations.*

2. Scattered C*-algebras.

These algebras are introduced as follows

DEFINITION 2.1. A C*-algebra A is called scattered, if each positive functional on A is atomic.

Our first structure result is

THEOREM 2.2. *Let A denote a* C*-*algebra, and let B denote the enveloping von Neumann-algebra. The following five conditions are equivalent.*

(1) *A is scattered.*
(2) *Each positive functional f on A is of the form $f=\sum f_n$, where each f_n is pure.*
(3) *Each now-degenerate representation of A is unitarily equivalent with a subrepresentation of a sum of irreducible representations.*
(4) *Each projection in B majorizes a minimal projection in B.*
(5) *The algebra B is isomorphic with an algebra $\prod_{i\in I} L(H_i)$.*

PROOF. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3), (1)$\Leftrightarrow$(4) follows from the preceeding section, and (4) $\Leftrightarrow$(5) is standard.

When A is scattered, the above theorem especially shows, that the enveloping von Neumann-algebra is of type *I*. Therefore we have

THEOREM 2.3. *Each scattered* C*-*algebra is of type* I.

We recall at this point, that a C*-algebra is of type I, if and only if it is a GCR-algebra ([7] for the non-separable case).

In the non-commutative case the simplest scattered C*-algebra is an algebra of all compact operators on a Hilbert space H, denoted LC(H). Moreover, it will follow below, that each C*-algebra, which is built up, using decomposition series, of scattered C*-algebras, is again scattered.

PROPOSITION 2.4. *Let I denote a closed, two-sided ideal in a* C*-*algebra A. The following two conditions are equivalent.*

(1) *A is scattered.*
(2) *I and A/I are both scattered.*

PROOF. (1)$\Rightarrow$(2). That I is scattered follows from standard-results on extensions, using condition (3) in theorem 2.2. If f is a positive functional on

A/I, then the natural lifting of f to A satisfies conditions (2) in theorem 2.2. Therefore f satisfies this condition, and A/I is scattered.

(2)⇒(1). Let f be a positive functional on A. Then $f = f_1 + f_2$, where f_1 and f_2 are positive functionals, and $\|f_1\| = \|f_1 \mid I\|$, $f_2(I) = 0$. Since, according to condition (2) in theorem 2.2, f_1 and f_2 are both countable sums of pure functionals on I and A/I respectively, the same holds for f, using extensions results. Therefore A is scattered.

PROPOSITION 2.5. *Let (I_α) be a totally ordered family of two-sided closed ideals in a* C**-algebra A, such that the closure of $\bigcup I_\alpha$ is equal to A. If each I_α is scattered, then A is scattered.*

PROOF. Let B_α, resp. B, denote the enveloping von Neumann-algebra of I_α, resp. A. For each α there is a central projection p_α in B, such that $B_\alpha = p_\alpha B$. (See [8, proof of lemma 1.17.3].) From the assumptions it follows, that $p_\alpha \to 1$. Therefore, since each B_α satisfies condition (4) in theorem 2.2, so does B.

From proposition 2.4 and proposition 2.5 we get the following

PROPOSITION 2.6. *Let A be a* C**-algebra with a decomposition series $(I_\alpha)_{0 \leqq \alpha \leqq p}$. If each algebra $I_{\alpha+1}/I_\alpha$ is scattered, then A is scattered.*

This especially shows, that if A has a decompositon series $(I_\alpha)_{0 \leqq \alpha \leqq p}$, such that each algebra $I_{\alpha+1}/I_\alpha$ is isomorphic with an algebra $\mathrm{LC}(H_\alpha)$, then A is scattered.

3. Some separable C*-algebras, which are scattered.

In this section A always is a separable C*-algebra. As usual A^* and $\hat{A}$ denotes the dual space and the spectrum of A respectively. Then we have the following result.

THEOREM 3.1. *The following three conditions are equivalent.*

(1) *A^* is separable in the norm topology.*
(2) *$\hat{A}$ is countable.*
(3) *A has a countable decomposition series (I_p), such that each algebra I_{p+1}/I_p is isomorphic with an algebra* $\mathrm{LC}(H_p)$ *for a separable Hilbert space H_p.*

If A satisfies these equivalent conditions, then A is scattered.

We will prove the implications (1)⇒(2)⇔(3)⇒(1). The last statement of the theorem will then follow from proposition 2.6. First we need a few lemmas.

LEMMA 3.2. *If $\hat{A}$ is countable, the A is of type* I.

PROOF. Let $\pi : A \to L(H_\pi)$ be an irreducible representation, and suppose, that $\pi(A) \cap \mathrm{LC}(H_\pi) = \{0\}$. Then it follows from [3], that there exists an uncountable family of pairwise inequivalent, irreducible representations with the same kernel as π. When $\hat{A}$ is countable, we must therefore have

$$\pi(A) \cap \mathrm{LC}(H_\pi) \neq \{0\},$$

so A is of type I.

LEMMA 3.3. *Suppose, that A has continuous trace, and that $\hat{A}$ is countable. Then A contains a non-zero closed ideal I, which is isomorphic with an algebra* $\mathrm{LC}(H)$ *for a separable Hilbert space H.*

PROOF. Let Prim (A) denote the set of all prime ideals. Since Prim (A) is locally compact and countable, it has an isolated point, say I_1 (see [9, section 8.5]). If I denotes the intersection of the ideals in Prim $(A) \backslash I_1$, it follows, that $I \neq \{0\}$, and that $\hat{I}$ consists of only one point. Therefore I has the desired property (by [2, section 4.7]).

LEMMA 3.4. *Suppose, that A is scattered and $\hat{A}$ is countable. Then A^* is separable.*

PROOF. We may assume, that A has a unit. Let $\{g_n\}$ be a sequence of pure states, dense among all pure states, and let

$$K = \{\textstyle\sum \mu_n g_n \mid \mu_n \geqq 0,\ \mu_n \text{ rational}, \textstyle\sum \mu_n = 1\}.$$

Then K is countable. Let f be a state. Since A is scattered, $f = \sum \lambda_i f_i$ where each f_i is a pure state, and $\lambda_i > 0$, $\sum \lambda_i = 1$. If $\varepsilon > 0$ is given, we can for each i choose $g_{n_i} \in \{g_n\}$, and μ_i rational, $\mu_i > 0$, such that

$$\|\lambda_i f_i - \mu_i g_{n_i}\| < \varepsilon \cdot 2^{-i}.$$

Since there is no restriction to assume $\sum \mu_i = 1$, we have $\sum \mu_i g_{n_i} \in K$, and $\|f - \sum \mu_i g_{n_i}\| < \varepsilon$. Therefore K is in the state space, so A^* is separable.

PROOF OF THEOREM 3.1. The implication (1)$\Rightarrow$(2) is well-known (and easy, using [4]). Let us next suppose, that $\hat{A}$ is countable. Then A is of type I (lemma 3.2), so A contains a non-zero ideal I, which has continuous trace. According to lemma 3.3 I has a closed ideal I_1 isomorphic with $\mathrm{LC}(H_1)$, where H_1 is separable. By repeated use of this argument it follows by transfinite induction, that A has a decomposition series with the property in (3), and this series is countable, since A is separable. This proves (2)$\Rightarrow$(3).

Let us finally suppose, that (3) is satisfied. We then have

$$\hat{A} = \bigcup_p (I_{p+1}/I_p)\hat{}\,.$$

Since each I_{p+1}/I_p is isomorphic with an algebra $LC(H_p)$, it follows, that $\hat{A}$ is countable. Moreover, A is scattered (proposition 2.6), and therefore A^* is separable (lemma 3.4).

The equivalence (1)$\Leftrightarrow$(2) in theorem 3.1 is a generalization of the well-known fact [9, section 19], that for a locally compact space X, the dual space $C_0(X)^*$ is separable in the norm topology, if and only if X is countable. Moreover, from theorem 3.1 we get the following result, which was earlier obtained by Tomiyama [10] in a different formulation.

COROLLARY 3.5. *Let A denote a* C*-*algebra. Then A^* is separable, if and only if A has a countable decomposition series (I_p), such that each algebra I_{p+1}/I_p is isomorphic with an algebra $LC(H_p)$ for a separable Hilbert space H_p.*

PROOF. If A^* is separable, then A is separable (by [6, section 3]), and theorem 3.1 gives the desired conclusion. On the other hand, if A has a decomposition series as mentioned above, then, since $LC(H_n)$ is separable, one concludes by induction, that A is separable, and theorem 3.1 can therefore be applied.

It should be remarked, that arguing as in the first half of the proof of theorem 3.1, one could use Tomiyama's result to prove the first part of this theorem, and then lemma 3.4 could be omitted. But that the stated conditions imply, that the algebra is scattered, seems not to follow directly from corollary 3.5. Concerning this point, it is the authors conjecture, that the converse of theorem 3.1 is also true, and more generally, that each scattered C*-algebra has a decomposition series as mentioned after proposition 2.6.

REFERENCES

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien* (Cahier Scientifiques 25), Gauthier-Villars, Paris, 1969.
2. J. Dixmier, *Les C*-algèbres et leur représantations*, (Cahier Scientifiques 24), Gauthier-Villars, Paris, 1964.
3. J. Dixmier, *Utilisation des facteur hyperfinis dans la theorie des C*-algèbres*, C.R. Acad. Sci. Paris Sér. A. 258 (1964), 4184.

4. J. Glimm and R. V. Kadison, *Unitary operators in* C*-*algebras*, Pacific J. Math. 10 (1960), 547–556.
5. G. K. Pedersen, *Atomic and diffuse functionals on a* C*-*algebra*, Pacific J. Math. 37 (1971), 795–800.
6. M. Reed and B. Simon, *Methods of Modern Mathematical Physics* I, *Functional Analysis*, Academic Press, New York · London, 1972.
7. S. Sakai, *On a characterization of type* I C*-*algebras*. Bull. Amer. Math. Soc. 72 (1966), 508–512.
8. S. Sakai, C*-*algebras and* W*-*algebras* (Ergebnisse Math. 60), Springer-Verlag, Berlin · Heidelberg · New York, 1971.
9. Z. Semadini, *Banach spaces of continuous functions*, (Monografie Matematyczne 55), Polish Scientific Publishers, Warszawa, 1971.
10. J. Tomiyama, *A characterization of* C*-*algebras, whose conjugate spaces are separable*, Tôhuku Math. J., 15 (1963), 96–102.

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