MATH. SCAND. 41 (1977), 347–357

SOME REMARKS ON EIGENFUNCTION EXPANSIONS FOR SCHRÖDINGER OPERATORS WITH NON-LOCAL POTENTIALS

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We consider two-body Schrödinger operators $H = -\Delta + V$ in $L^2(\mathbf{R}^3)$ with a not necessarily local potential V that decreases faster than $|x|^{-1-\varepsilon}$ at infinity. We prove the existence of two families $\varphi_\pm(x,\xi)$ of generalized eigenfunctions of H such that the generalized Fourier transforms

$$(\mathscr{F}_\pm f)(\xi) = \lim_{N\to\infty} \int_{|x|<N} f(x)\overline{\varphi_\pm(x,\xi)}\,dx$$

are unitary maps from the subspace of absolute continuity for H onto $L^2(\mathbf{R}^3)$. Let H_{ac} denote the absolutely continuous part of H. Then $H_{\mathrm{ac}} = \mathscr{F}_\pm^* M_{|\cdot|^2}\mathscr{F}_\pm$ where $M_{|\cdot|^2}$ denotes multiplication with $|\xi|^2$. Using the results of Kuroda [4] we establish the connection with scattering theory and prove a rigorous version of the formal connection between the scattering matrix and the generalized eigenfunctions used in physics.

For V multiplication by a real-valued function the above results have been proved by Agmon [2]. We prove the results for a general V by constructing the generalized eigenfunctions using Sobolev's lemma. This method is due to Agmon [1] and has been used by Yamada [8] for constructing eigenfunctions for the Dirac operator with a multiplicative potential. The applicability of the method depends on the fact that we restrict our attention to the physically relevant 3-dimensional case. For a general V the regularity in the x-variable is weaker than the one obtained in [2] for multiplicative V.

1. Definitions, notations and assumptions on V.

We denote by L^2 the space $L^2(\mathbf{R}^3)$, the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$, Ω denotes the unit sphere in $\mathbf{R}^3$ and $L^2(\Omega)$ the space of square integrable functions with respect to the surface measure on Ω. The norm and the inner product are denoted $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$. For $s \in \mathbf{R}$, $L^{2,s} = H^{0,s}$ denotes the weighted L^2 space

Received January 24, 1977.

$$L^{2,s} = \{f \mid (1+|x|^2)^{s/2} f(x) \in L^2\}$$

with the norm

$$\|f\|_{0,s} = \|(1+|x|^2)^{s/2} f\| .$$

$\mathscr{F}$ ($\mathscr{F}^{-1}$) denotes the (inverse) Fourier transform on the temperate distributions in $\mathbf{R}^3$. The Sobolev space $H^m = H^{m,0}$, $m \in \mathbf{R}$, is the Hilbert space of temperate distributions given by

$$H^m = \{f \mid \mathscr{F} f \in L^{2,m}\}$$

with the norm

$$\|f\|_{m,0} = \|\mathscr{F} f\|_{0,m} .$$

The weighted Sobolev space $H^{m,s}$ is defined for $m, s \in \mathbf{R}$ by

$$H^{m,s} = \{f \mid (1+|x|^2)^{s/2} f \in H^m\}$$

with the norm

$$\|f\|_{m,s} = \|(1+|x|^2)^{s/2} f\|_{m,0} .$$

The dual of $H^{m,s}$ is identified with $H^{-m,-s}$ and the duality is written

$$(f, g) = \int_{\mathbf{R}^3} \overline{f(x)}\, g(x)\, dx .$$

For two Hilbert spaces K and L $\mathscr{B}(K, L)$ denotes the bounded linear operators from K to L, equipped with the operator norm. The adjoint of an operator T is denoted T^*. The subspace of compact operators is denoted $\mathscr{C}(K, L)$.

For any $0<\theta<1$ and $s \in \mathbf{R}$ we denote by $C^{\theta,s}$ the continuous functions on $\mathbf{R}^3$ such that $(1+|x|^2)^{s/2} f \in L^\infty(\mathbf{R}^3)$ and such that $(1+|x|^2)^{s/2} f(x)$ satisfies a uniform Hölder condition of order θ. $C^{\theta,s}$ is a Banach space with the norm

$$|f|_{\theta,s} = \sup_{x \in \mathbf{R}^3} (1+|x|^2)^{s/2} |f(x)| + \sup_{\substack{x,y \\ 0<|x-y|<1}} \left[(1+|x|^2)^{s/2} \frac{|f(x)-f(y)|}{|x-y|^\theta} \right] .$$

H_0 denotes the Laplacian $-\Delta$ in L^2 with domain $D(H_0) = H^{2,0}$.

Assumptions on V. Let V be a closed symmetric operator on L^2 with $D(V) \supseteq D(H_0)$ such that for some $s > \frac{1}{2}$

$$V \in \mathscr{C}(H^{2,0}, H^{0,2s}) , \tag{1.1}$$

and such that V has an extension to $H^{2,-s}$, also denoted V, with the property

$$V \in \mathscr{C}(H^{2,-s}, H^{0,s}) . \tag{1.2}$$

NOTE. In the remainder of this paper s denotes the constant introduced above.

REMARKS (i) When V is multiplication by a real-valued function $v(x)$ condition (1.1) is Agmon's condition (SR), [2]. In this case (1.2) follows from (1.1). A sufficient condition on v is

$$\sup_{x \in \mathbf{R}^3} \left[(1+|x|)^{2+2\varepsilon} \int_{|x-y|<1} |v(y)|^2 |x-y|^{1-2\theta} dy \right] < \infty \tag{1.3}$$

for some $\varepsilon > 0$ and $0 < \theta < \frac{1}{2}$. The results in [2] are proved under this assumption.

(ii) It follows from [3] that (1.3) is also necessary for v to satisfy (1.1). (This remark is due to E. Balslev).

(iii) If $\frac{1}{2} < s' \leqq s$ then (1.1) and (1.2) are satisfied with s' instead of s.

(iv) Under (1.1) and (1.2) the results of [4] hold.

From the assumptions on V and the Kato–Rellich theorem follow that by $H = H_0 + V$, $D(H) = D(H_0)$ is defined a selfadjoint operator on L^2.

We denote by $\gamma(k)$, $k > 0$, the trace operator defined on $C_0^\infty(\mathbf{R}^3)$ by (using polar coordinates $x = k\omega$, $k > 0$, $\omega \in \Omega$) $(\gamma(k)f)(\cdot) = f(k\cdot)$. $\gamma(k)$ has an extension to a bounded linear operator $\gamma(k): H^{t,0} \to L^2(\Omega)$ for any $t > \frac{1}{2}$. There exists a constant C such that

$$\|\gamma(k)\| \leqq C \quad \text{for all } k > 0 , \tag{1.4}$$

and if $t < \frac{3}{2}$ there exists a constant C' such that

$$\|\gamma(k_1) - \gamma(k_2)\| \leqq C' |k_1 - k_2|^{t-\frac{1}{2}} \quad \text{for all } k_1, k_2 > 0 . \tag{1.5}$$

The norms in (1.4) and (1.5) are the operator norm on $\mathscr{B}(H^{t,0}, L^2(\Omega))$. For a proof of these results, see [5 p. 44].

2. Some lemmas.

We state without proof some results which we need in section 3 and 4. The following result is known as Sobolev's lemma. For a proof, see [7].

LEMMA 2.1. *For $t > \frac{3}{2}$, $u \in H^{t,0}$ is a bounded continuous function (after correction on a null set) such that $|u(x)| \leqq C \|u\|_{t,0}$ for some $C > 0$. For any θ, $0 < \theta < t - \frac{3}{2}$, $\theta < 1$, there exists a constant C' such that*

$$|u(x) - u(y)| \leqq C' \|u\|_{t,0} |x-y|^\theta . \tag{2.1}$$

We denote by $e_+(H)$ the discrete set of positive eigenvalues of H. (It is proved in [4] that $e_+(H)$ is discrete under our assumptions on V). The resolvent is denoted $R(z) = (H - z)^{-1}$.

LEMMA 2.2. *For any* $t>\frac{1}{2}$ *and* $\lambda \in \mathbf{R}_+\backslash e_+(H)$ *the following limits exist in the operator norm on* $\mathscr{B}(H^{0,t}, H^{2,-t})$

$$R(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon) . \tag{2.2}$$

The convergence is uniform on compact subsets of $\mathbf{R}_+\backslash e_+(H)$.

This result is due to Agmon [2] for multiplicative V. It is proved under our assumptions on V in [4].

We define $\varphi_0(x, k\omega) = \exp(ik\omega\cdot x)$ and for $g \in L^2(\Omega)$ we define

$$\varphi_0^g(x,k) = \int_\Omega \varphi_0(x, k\omega) g(\omega)\, d\omega . \tag{2.3}$$

LEMMA 2.3. *For any* $t>\frac{1}{2}$ *and any integer* $m \geqq 0$

$$\varphi_0^g(\cdot, k) \in H^{m,-t} \cap C^\infty(\mathbf{R}^3) .$$

Let $K \subset \mathbf{R}_+$ *be a compact set. There exists a constant* $C = C(m,t,K)$ *such that*

$$\|\varphi_0^g(\cdot, k)\|_{m,-t} \leqq C\|g\|_\Omega \tag{2.4}$$

for all $g \in L^2(\Omega)$ *and all* $k \in K$. *Moreover,* $k \mapsto \varphi_0^g(\cdot, k)$ *is continuous* $\mathbf{R}_+ \to H^{m,-t}$, *equicontinuous for* g *in a bounded subset of* $L^2(\Omega)$.

For a proof, see [2]. The last statement follows from the proof given in [2]. We note that

$$\varphi_0^g(x,k) = (2\pi)^{\frac{3}{2}}(\mathscr{F}^{-1}\gamma(k)^* g)(x) \tag{2.5}$$

where $\gamma(k)^* \in \mathscr{B}(L^2(\Omega), H^{-t,0})$.

It is a consequence of Lemma 2.2 that H has no singular continuous spectrum. We denote by L_p^2 the closed subspace spanned by the eigenvectors of H and by L_{ac}^2 the subspace of absolute continuity of H. Then $L^2 = L_p^2 \oplus L_{\text{ac}}^2$. The proof is analogous to the proof of theorem 6.1 in [2].

3. Existence of generalized eigenfunctions.

We denote by x and ξ the variables in configuration space and momentum space. We always use polar coordinates in momentum space: $\xi = k\omega$, $\omega \in \Omega$. We use the notation

$$e = \{k \in \mathbf{R}_+ \mid k^2 \in e_+(H)\} .$$

THEOREM 3.1. *There exist two families*

$$\varphi_{\pm}(x,k,\omega), \qquad x \in \mathbf{R}^3,\ k \in \mathbf{R}_+ \backslash e,\ \omega \in \Omega$$

of generalized eigenfunctions for H with the following properties

(a) $\varphi_{\pm} \in L^2_{\mathrm{loc}}(\mathbf{R}^3 \times \mathbf{R}_+ \backslash e \times \Omega;\ dx \times k^2 dk \times d\omega)$

(b) $(x,k) \mapsto \varphi_{\pm}(x,k,\cdot)$ *is continuous* $\mathbf{R}^3 \times \mathbf{R}_+ \backslash e \to L^2(\Omega)$

(c) *for* $g \in L^2(\Omega)$ *we define* $\varphi^g_{\pm}(x,k) = \int_\Omega \varphi_{\pm}(x,k,\omega) g(\omega)\, d\omega$. *Then* $\varphi^g_{\pm}(\cdot,k) \in H^{2,-s} \cap C^{\theta,-s}$, $k \in \mathbf{R}_+ \backslash e$, *where s is the constant from the assumptions on V*, $0 < \theta < \frac{1}{2}$, *and furthermore*

$$(3.1) \qquad (H_0 + V - k^2)\varphi^g_{\pm}(\cdot,k) = 0 \quad \text{for } k \in \mathbf{R}_+ \backslash e,\ g \in L^2(\Omega),$$

$$(3.2) \quad \varphi^g_{\pm}(\cdot,k) = (1 - R(k^2 \mp i0)V)\varphi^g_0(\cdot,k) \quad \text{for } k \in \mathbf{R}_+ \backslash e,\ g \in L^2(\Omega).$$

REMARK. When V is multiplicative the families $\varphi_{\pm}(x,k,\omega)$ are measure-theoretically equivalent to the generalized eigenfunctions constructed in [2]. They are unique in the same sense, see [2].

PROOF. According to lemma 2.3, $\varphi^g_0(\cdot,k) \in H^{2,-s}$ for any $g \in L^2(\Omega)$. Thus for $k \in \mathbf{R}_+ \backslash e$,

$$(1 - R(k^2 \mp i0)V)\varphi^g_0(\cdot,k) \in H^{2,-s}.$$

From lemmas 2.1, 2.2, 2.3 and the definition of $H^{2,-s}$ follow for a fixed $x \in \mathbf{R}^3$ the estimate

$$|((1 - R(k^2 \mp i0)V)\varphi^g_0(\cdot,k))(x)| \leqq C\|g\|_\Omega$$

for all $g \in L^2(\Omega)$ with C independent of g. Hence

$$g \mapsto ((1 - R(k^2 \mp i0)V)\varphi^g_0(\cdot,k))(x)$$

is a bounded linear functional on $L^2(\Omega)$ and thus there exists $\varphi_{\pm}(x,k,\cdot) \in L^2(\Omega)$ such that for all $g \in L^2(\Omega)$

$$\int_\Omega \varphi_{\pm}(x,k,\omega) g(\omega)\, d\omega = ((1 - R(k^2 \mp i0)V)\varphi^g_0(\cdot,k))(x).$$

By lemmas 2.1, 2.2, 2.3 and some elementary estimates together with

$$\|\varphi_{\pm}(x,k,\cdot) - \varphi_{\pm}(x',k',\cdot)\|_\Omega = \sup_{\substack{g \in L^2(\Omega) \\ \|g\|_\Omega = 1}} \left| \int_\Omega (\varphi_{\pm}(x,k,\omega) - \varphi_{\pm}(x',k',\omega)) g(\omega)\, d\omega \right|$$

the two maps $(x,k) \mapsto \varphi_{\pm}(x,k,\cdot)$ are continuous $\mathbf{R}^3 \times \mathbf{R}_+ \backslash e \to L^2(\Omega)$. Thus (b) is proved. (a) follows easily from (b). Now it is a consequence of our definitions that for $k \in \mathbf{R}_+ \backslash e$

$$\varphi_\pm^g(x,k) = ((1-R(k^2\mp i0)V)\varphi_0^g(\cdot,k))(x) .$$

Thus $(1+|\cdot|^2)^{-s/2}\varphi_\pm^g(\cdot,k) \in H^{2,0}$ and from Lemma 2.1 we get $\varphi_\pm^g(\cdot,\mathrm{k}) \in C^{\theta,-s}$ for any $0<\theta<\frac{1}{2}$. Equation (3.1) is proved as follows. For $g \in L^2(\Omega)$, $k \in \mathbf{R}_+\backslash e$ we have

$$(H_0+V-k^2)(1-R(k^2\mp i\varepsilon)V)\varphi_0^g(\cdot,k) = \pm i\varepsilon R(k^2\mp i\varepsilon)V\varphi_0^g(\cdot,k)$$

since H_0+V-k^2 is a bounded linear operator from $H^{2,-s}$ to $H^{0,-s}$. Using lemmas 2.3 and 2.2 we can let ε tend to zero and thus prove (3.1).

REMARK. The proof depends on the dimension of the space being 3. For the case $L^2(\mathbf{R}^n)$ one must require instead of (1.2) that

$$V \in \mathscr{C}(H^{m,-s}(\mathbf{R}^n), H^{m-2,s}(\mathbf{R}^n)) \tag{3.3}$$

for some $m>n/2$. The above results can be established under this assumption. For V multiplication by $v(x)$ (3.3) implies $v \in H^{m-2}_{\mathrm{loc}}(\mathbf{R}^n)$ and is thus stronger than Agmon's assumption for $n>3$.

THEOREM 3.2. *Let $\varphi_\pm$ be the two families of generalized eigenfunctions from Theorem* 3.1.

(a) *If $s>\frac{3}{2}$ then $\varphi_\pm(x,k,\omega)$ are jointly continuous in (x,k,ω). For fixed $k \in \mathbf{R}_+\backslash e$, $\omega \in \Omega$,*

$$\varphi_\pm(\cdot,k,\omega) \in H^{2,-s} \cap C^{\theta,-s}, \quad 0<\theta<\tfrac{1}{2} ,$$

and

$$(H_0+V-k^2)\varphi_\pm(\cdot,k,\omega) = 0 . \tag{3.4}$$

(b) *If $s>1$ and V has an extension to $H^{2,-s-\frac{1}{2}}$ such that*

$$V \in \mathscr{C}(H^{2,-s-\frac{1}{2}}, H^{0,s-\frac{1}{2}}) \tag{3.5}$$

then $\varphi_\pm(x,k,\omega)$ are jointly continuous in (x,k,ω). For fixed $k \in \mathbf{R}_+\backslash e$, $\omega \in \Omega$,

$$\varphi_\pm(\cdot,k,\omega) \in H^{2,-s-\frac{1}{2}} \cap C^{\theta,-s-\frac{1}{2}}, \quad 0<\theta<\tfrac{1}{2} ,$$

and

$$(H_0+V-k^2)\varphi_\pm(\cdot,k,\omega) = 0 .$$

REMARK. These results are well known for multiplicative V. See [2, p. 170] and the references given there. We note that for a multiplicative V (3.5) is implied by assumption (1.1).

PROOF. We prove (a), the proof of (b) is analogous. $s>\frac{3}{2}$ implies that

$\varphi_0(\cdot, k\omega) \in H^{2,-s}$ and that $(k,\omega) \to \varphi_0(\cdot, k\omega)$ is a continuous map from $\mathbf{R}_+\backslash e \times \Omega$ to $H^{2,-s}$. It is easy to see that (after correcting $\varphi_\pm$ on a null set) we have

$$\varphi_\pm(x,k,\omega) = \big((1-R(k^2 \mp i0)V)\varphi_0(\cdot,k,\omega)\big)(x) .$$

Now the regularity results follow from lemma 2.1, 2.2 and the above mentioned continuity property of $\varphi_0(\cdot, k\omega)$. Equation (3.4) can be proved in the same manner as (3.1) was proved.

4. The expansion theorem.

Instead of giving a selfcontained proof of the expansion theorem we use the results from [4].

THEOREM 4.1. *Let $\varphi_\pm$ denote the two families of generalized eigenfunctions from theorem 3.1. There exist two bounded linear operators $\mathscr{F}_\pm$ on L^2 with the following properties:*

(a) *$\mathscr{F}_\pm$ are partial isometries with initial space L^2_{ac} and final space L^2.*

(b) *For any $f \in L^2$*

$$(\mathscr{F}_\pm f)(k\omega) = \lim_{N\to\infty} (2\pi)^{-\frac{3}{2}} \int_{|x|<N} f(x)\overline{\varphi_\pm(x,k,\omega)}\,dx \tag{4.1}$$

and

$$(\mathscr{F}_\pm^* f)(x) = \lim_{j\to\infty} (2\pi)^{-\frac{3}{2}} \int_{I_j}\int_{\Omega} f(k\omega)\varphi_\pm(x,k,\omega)\,d\omega k^2\,dk \tag{4.2}$$

where I_j is an increasing sequence of compact sets such that $\bigcup_j I_j = \mathbf{R}_+\backslash e$.

(c) *Let P_{ac} denote the projection of L^2 onto L^2_{ac}. Then*

$$(P_{\mathrm{ac}}H)f = (\mathscr{F}_\pm^* M_{|\cdot|^2}\mathscr{F}_\pm)f \quad \text{for all } f \in D(H) ,$$

where $M_{|\cdot|^2}$ denotes multiplication by k^2.

(d) *For $f \in H^{0,s}$, $k \in \mathbf{R}_+\backslash e$*

$$(\mathscr{F}_\pm f)(k\omega) = \big(\gamma(k)(\mathscr{F}(1-VR(k^2 \pm i0))f)\big)(\omega) .$$

PROOF. For $f \in H^{0,s}$ we define

$$(F_\pm f)(k\omega) = \gamma(k)(\mathscr{F}(1-VR(k^2 \pm i0))f)(\omega) .$$

It is proved in [4, section 5.5] that $F_\pm$ can be extended to a bounded linear operator on L^2 with the properties (a) and (c). We denote by L^2_c the L^2-functions with compact support. For $f \in L^2_c$, $h \in L^2(\Omega)$ we have

$$\int_{\Omega} \overline{h(\omega)}(\mathscr{F}(1-VR(k^2\pm i0))f)(k\omega)\,d\omega$$

$$= \int_{\Omega} \overline{h(\omega)} \lim_{N\to\infty} (2\pi)^{-\frac{3}{2}} \int_{|x|<N} ((1-VR(k^2\pm i0))f)(x)\varphi_0(x,-k\omega)\,dx d\omega$$

$$= \lim_{N\to\infty} (2\pi)^{-\frac{3}{2}} \int \overline{h(\omega)} \int_{|x|<N} ((1-VR(k^2\pm i0))f)(x)\varphi_0(x,-k\omega)\,dx d\omega$$

$$= \lim_{N\to\infty} (2\pi)^{-\frac{3}{2}} \int_{|x|<N} ((1-VR(k^2\pm i0))f)(x) \overline{\int_{\Omega} h(\omega)\varphi_0(x,k\omega)\,d\omega}\,dx$$

$$= (2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} f(x)\overline{(1-R(k^2\mp i0)V)\varphi_0^h(\cdot,k)}\,dx$$

$$= (2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} f(x) \overline{\int_{\Omega} \varphi_{\pm}(x,k,\omega)h(\omega)\,d\omega}\,dx$$

$$= \int_{\Omega} \overline{h(\omega)}(2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} f(x)\overline{\varphi_{\pm}(x,k,\omega)}\,dx d\omega\,.$$

The exchange of the limit and the integration above is legitimate, because

$$(\mathscr{F}(1-VR(k^2\pm i0))f)(k\omega)$$

$$= (2\pi)^{-\frac{3}{2}} \lim_{N\to\infty} \int_{|x|<N} ((1-VR(k^2\pm i0))f)(x)e^{-ix\cdot k\omega}\,dx$$

the convergence takes place in $H^{s,0}$, as $\mathscr{F}$ maps $H^{0,s}$ onto $H^{s,0}$. We have also used the continuity of the trace operator and the inner product on $L^2(\Omega)$. Furthermore we have used the fact that

$$\int_{\mathbf{R}^3} f(x)\overline{(R(k^2\mp i0)Vg)(x)}\,dx = \int_{\mathbf{R}^3} (VR(k^2\pm i0)f)(x)\overline{g(x)}\,dx$$

for $f\in H^{0,s}$, $g\in H^{2,-s}$, which is a consequence of the assumptions on V and lemma 2.2. We have also used Fubini's theorem twice. The first use is easy to justify, the second use is justified by the following estimate

$$\int_{\mathbf{R}^3}\int_{\Omega} |\varphi_{\pm}(x,k,\omega)h(\omega)f(x)|\,d\omega dx$$

$$\leqq \|h\|_{\Omega}\|f\|_{L^1} \sup_{x\in\operatorname{supp}(f)} \|\varphi_{\pm}(x,k\,\cdot)\|_{\Omega} < \infty\,.$$

It follows from theorem 3.1(a) that for $f\in L_c^2$ we can define

$$(\mathscr{F}_{\pm}f)(k\omega) = (2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} f(x)\overline{\varphi_{\pm}(x,k,\omega)}\,dx$$

and that $\mathscr{F}_{\pm} f$ belongs to $L^2_{\mathrm{loc}}(\mathbf{R}_+\backslash e \times \Omega;\ k^2 dk \times d\omega)$. We see that for each $k \in \mathbf{R}_+\backslash e$

$$(\mathscr{F}_{\pm} f)(k\cdot) = (F_{\pm} f)(k\cdot) \quad \text{in } L^2(\Omega)\,.$$

$(F_{\pm} f)(k\cdot)$ is continuous in k with values in $L^2(\Omega)$, so we get

$$\mathscr{F} f_{\pm} = F_{\pm} f, \quad f \in L^2_c\,. \tag{4.3}$$

From the properties of $F_{\pm}$ stated above follow that $\mathscr{F}_{\pm}$ have unique extensions, also denoted $\mathscr{F}_{\pm}$, to bounded operators on L^2 with the properties (a) and (c). (d) follows from (4.3) and the density of L^2_c in $H^{0,s}$. Let χ_N denote the characteristic function of $\{x \mid |x| < N\}$. Then for any $f \in L^2$

$$\begin{aligned}(\mathscr{F}_{\pm} f)(k\omega) &= \lim_{N\to\infty} \mathscr{F}_{\pm}(\chi_N f)(k\omega) \\ &= \lim_{N\to\infty} (2\pi)^{-\frac{3}{2}} \int_{|x|<N} f(x)\overline{\varphi_{\pm}(x,k,\omega)}\, dx\end{aligned}$$

(convergence in L^2) and (4.1) is proved. It follows from theorem 3.1(a) and Fubini's theorem that for $f \in L^2_c$, $\operatorname{supp}(f) \subseteqq \{k\omega \mid k \in \mathbf{R}_+\backslash e, \omega \in \Omega\}$

$$(\mathscr{F}^*_{\pm} f)(x) = (2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}_+\backslash e} \int_{\Omega} f(k\omega)\varphi_{\pm}(x,k,\omega)\, d\omega k^2 dk\,.$$

Now the proof of (4.2) is analogous to the proof of (4.1).

5. Representation of the scattering matrix.

We refer to [4] for a discussion of scattering theory. We summarize some results from [4] in the following theorem:

THEOREM 5.1. (Kuroda). *Define* $W_{\pm} = \mathscr{F}^*_{\pm}\mathscr{F}$. *Then*

$$W_{\pm} = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$$

and range $(W_{\pm}) = L^2_{\mathrm{ac}}$, *that is the wave operators exist and are complete. The unitary scattering operator* $S = W^*_+ W_-$ *has a representation*

$$(\mathscr{F} S f)(k\omega) = S(k)(\mathscr{F} f)(k\cdot)(\omega)$$

where $S(k)$, $k \in \mathbf{R}_+\backslash e$ *is a family of unitary operators on* $L^2(\Omega)$, *depending continuously on* k *in the operator norm on* $L^2(\Omega)$. $S(k)$ *has the following representation*

$$S(k) = 1 - \pi i k \gamma(k) \mathscr{F}[V - VR(k^2 + i0)V]\mathscr{F}^{-1}\gamma(k)^*\,. \tag{5.1}$$

REMARK. It follows from the remark after lemma 2.3 that $\mathscr{F}^{-1}\gamma(k)^*$ maps $L^2(\Omega)$ into $H^{2,-s}$ boundedly.

THEOREM 5.2. *$S(k)-1$ has the kernel $t(k;\,\omega,\omega')$, symbolically given by*

$$t(k;\,\omega,\omega') \sim -\frac{i}{8\pi^2}k\int_{\mathbf{R}^3}\varphi_0(x,-k\omega)V\varphi_-(x,k,\omega')\,dx\,. \tag{5.2}$$

This is to be interpreted as follows. For $f,g\in L^2(\Omega)$ and $k\in \mathbf{R}_+\backslash e$

$$(f,(S(k)-1)g)_\Omega = -\frac{i}{8\pi^2}k\int_{\mathbf{R}^3}\overline{\varphi_0^f(x,k)}\,V\varphi_-^g(x,k)\,dx\,. \tag{5.3}$$

REMARK. (5.2) is the representation of the scattering amplitude in terms of generalized eigenfunctions formally derived in physics textbooks, see for instance [6].

PROOF. From (5.1) follows for $k\in\mathbf{R}_+\backslash e$, $f,g\in L^2(\Omega)$

$$\begin{aligned}
&(f,(S(k)-1)g)_\Omega\\
&= -\pi ik(f,\gamma(k)\mathscr{F}[V-VR(k^2+i0)V]\mathscr{F}^{-1}\gamma(k)^*g)_\Omega\\
&= -\pi ik(\mathscr{F}^{-1}\gamma(k)^*f,\,V(1-R(k^2+i0)V)\mathscr{F}^{-1}\gamma(k)^*g)\\
&= -\frac{ik}{8\pi^2}(\varphi_0^f(\cdot,k),\,V\varphi_-^g(\cdot,k))
\end{aligned}$$

where we have used the remark after lemma 2.3 and (3.2).

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