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PLURIHARMONICITY IN TERMS OF HARMONIC SLICES

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Abstract.

Let f be a function on the open unit ball in $\mathbf{C}^n$. We prove that if the slices of f through the origin are harmonic and if f is smooth at the origin, then f is pluriharmonic.

1. Introduction.

1.1. We will denote by B the open unit ball in $\mathbf{C}^n$ and by S the unit sphere in $\mathbf{C}^n$. Thus

$$B = \left\{z \;:\; z \in \mathbf{C}^n,\, \sum_1^n z_j\bar{z}_j < 1\right\}$$

and

$$S = \left\{z \;:\; z \in \mathbf{C}^n,\, \sum_1^n z_j\bar{z}_j = 1\right\} = \partial B\,.$$

If f is a complex-valued function on B, then we will denote by $H(f)$ the class of those y in S such that the slice $\mu \to f(\mu y)$ is harmonic on the open unit disc $\mathbf{D}$. Thus $H(f)$ is a circular subset of S. By $C^k(0)$ we mean the class of those $f\colon B \to \mathbf{C}$ which are of differentiablity class C^k in some neighbourhood of 0 (which depends on f). The purpose of this paper is to state and prove the following three theorems on pluriharmonic functions.

1.2. Theorem. *Let $f \in C(B)$ and let $H(f)=S$. If*

$$f \in \bigcap \{C^k(0) \;:\; k=1,2,\ldots\}\,, \tag{1.1}$$

then f is pluriharmonic on B.

1.3. Theorem. *Let $f\colon B \to [0,\infty)$ and let $H(f)=S$. If* (1.1) *holds, then f is pluriharmonic on B.*

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1.4. THEOREM. *Let* $f\colon B \to \mathbf{C}$ *and let* $H(f)=S$. *If* (1.1) *holds, then* f *is pluriharmonic on* B.

1.5. Although Theorem 1.4 contains Theorems 1.2 and 1.3, we prefer to separate them since the proofs of Theorems 1.2 and 1.3 are elementary, whereas the proof of Theorem 1.4 is not.

2. The proofs of Theorems 1.2, 1.3, and 1.4.

2.1. PROPOSITION. *Let* G *be an open subset of* $\mathbf{R}^p$ *and let* $g \in C^k(G)$. *If* $(x,y) \in G \times \mathbf{R}^p$ *and if*

$$h(t) = g(x+ty)\,,$$

then

$$h^{(k)}(0) = g^{(k)}(x,y,\ldots,y)\,.$$

PROOF. We have $[h(t+s)-h(t)]/s=[g(x+ty+sy)-g(x+ty)]/s$, hence

$$h'(t) = g'(x+ty,y)\,.$$

Thus $h''(t)=g''(x+ty,y,y)$, etc.

2.2. We will denote by $\mathbf{N}$ the class of all nonnegative integers and by $\mathbf{N}_+$ the class of all positive integers. If $k \in \mathbf{N}^n$, then by $|k|$ we mean $\sum_1^n k_j$, whereas if $z \in \mathbf{C}^n$, then by $|z|$ we mean

$$|z| = \left(\sum_1^n z_j\bar{z}_j\right)^{\frac{1}{2}}\,.$$

If $s,t \in \mathbf{N}$, then we will denote by H_{st} the class of all polynomials in z and $\bar{z}$ ($z \in \mathbf{C}^n$) that are homogeneous of bidegree (s,t). Thus if $f \in H_{st}$, then

$$f(z) = \sum_{(j,k)\in I(s,t)} c_{jk}z^j\bar{z}^k$$

where $c_{jk} \in \mathbf{C}$ and

$$I(s,t) = \{(j,k) : j,k \in \mathbf{N}^n,\ |j|=s,\ |k|=t\}\,.$$

We let $H_s=H_{s0}$ and $H_{-s}=H_{0s}$. Thus H_s is the class of all polynomials in z that are homogeneous of degree s, and $H_{-s}=\bar{H}_s$. We will denote (as is usual) by $\mathbf{T}$ the class of all μ in $\mathbf{C}$ such that $\mu\bar{\mu}=1$.

We owe the statement of the following proposition to Rudin, who proves it by means of Taylor's theorem. (This proposition was implicit in our first draft of the proof of Theorem 1.2.)

2.3. PROPOSITION. *Let $k \in \mathbf{N}_+$ and let $g\colon B \to \mathbf{C}$ be such that if $\mu \in \mathbf{D}$ and $z \in B$, then*

$$g(\mu z) = \mu^k g(z) . \tag{2.1}$$

If $g \in C^k(0)$, then $g \in H_k$.

PROOF. If $-1<t<1$, then

$$t^k g(z) = g(tz) ,$$

hence by Proposition 2.1

$$k!g(z) = g^{(k)}(0, z, \ldots, z) . \tag{2.2}$$

Furthermore $g^{(k)}(0, v_1, \ldots, v_k)$ is a complex-valued k-linear symmetric form on $\times_1^k \mathbf{C}^n$ over $\mathbf{R}$, hence by (2.2) g is a polynomial in z and $\bar{z}$ which is homogeneous of degree k. That is to say

$$g \in \sum_{s+t=k} H_{st} .$$

We have

$$g = \sum_{s+t=k} g_{st} \qquad \text{where } g_{st} \in H_{st} .$$

If $\mu \in \mathbf{T}$, then by (2.1)

$$\mu^k g = \sum_{s+t=k} \mu^s \bar{\mu}^t g_{st} = \sum_{s+t=k} \mu^{s-t} g_{st} ,$$

hence $g_{st}=0$ if $(s,t) \neq (k,0)$ which completes the proof of Proposition 2.3.

2.4. PROPOSITION. *Let $f\colon B \to \mathbf{C}$ and let $H(f)=S$. If $k \in \mathbf{Z}$, then we define $f_k\colon B \to \mathbf{C}$ by*

$$f_k(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z) e^{-ik\theta} \, d\theta . \tag{2.3}$$

2.4.1. *If $z \in B$, then $\sum_{-\infty}^{\infty} |f_k(z)| < \infty$ and $f(z) = \sum_{-\infty}^{\infty} f_k(z)$.*

2.4.2. *If $z \in B$, then $f_0(z) = f(0)$.*

2.4.3. *If $k>0$ and $f \in C^k(0)$, then $f_k \in H_k$ and $f_{-k} \in H_{-k}$.*

PROOF. If $z \in \mathbf{C}^n$, if $\mu \in \mathbf{C}$, and if $|\mu| < 1/|z|$, then

$$f_k(\mu z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} \mu z) e^{-ik\theta} \, d\theta . \tag{2.4}$$

Furthermore $\mu \to f(\mu z)$ is harmonic if $|\mu| < 1/|z|$, hence

$$f(\mu z) = \sum_1^\infty c_{-j}(z)\bar{\mu}^j + c_0 + \sum_1^\infty c_j(z)\mu^j .$$

If $k \in \mathsf{N}$, then by (2.4)

$$f_k(\mu z) = c_k(z)\mu^k \tag{2.5}$$

and

$$f_{-k}(\mu z) = c_{-k}(z)\bar{\mu}^k .$$

Thus if $k \in \mathsf{Z}$ and $z \in B$, then

$$f_k(z) = c_k(z) , \tag{2.6}$$

hence 2.4.1 and 2.4.2 hold.

If $k \in \mathsf{N}_+$, if $z \in B$, and if $|\mu| < 1/|z|$, then by (2.5) and (2.6)

$$f_k(\mu z) = \mu^k f_k(z) .$$

Furthermore if $f \in C^k(0)$, then by (2.3) $f_k \in C^k(0)$, hence by Proposition 2.3 $f_k \in H_k$. Likewise $f_{-k} \in H_{-k}$ which completes the proof of Proposition 2.4.

2.5. The proof of Theorem 1.2. Let f_k be defined by (2.3). Since $f \in C(B)$, f is bounded on compact subsets of B, hence by (2.3) the f_k are uniformly bounded on compact subsets of B. Thus by Proposition 2.4 f is pluriharmonic on B.

2.6. The proof of Theorem 1.3. Let f_k be defined by (2.3). If $z \in B$, then by (2.3)

$$|f_k(z)| \leqq \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}z)\,d\theta = f(0) ,$$

hence by Proposition 2.4 f is pluriharmonic on B.

2.7. For the purpose of the proof of Theorem 1.4 we recall the following proposition of Hartogs and Hörmander (Theorem 1.6.13 of [2]).

2.8. Proposition. *Let G be an open subset of* R^p *and let* $\{g_k : k \in \mathsf{N}\}$ *be a sequence of subharmonic functions on* G *such that if* $k \in \mathsf{N}$ *and* $x \in G$, *then*

$$g_k(x) \leqq \alpha \quad \textit{and} \quad \limsup_{j \to \infty} g_j(x) \leqq a$$

where $\alpha, a \in \mathbf{R}$. If F is a compact subset of G and if $b>a$, then there is an index $N=N(F,b)$ such that if $k \geqq N$ and $x \in F$, then

$$g_k(x) < b .$$

2.9. THE PROOF OF THEOREM 1.4. Let f_k be defined by (2.3). It is to be proved that the f_k are uniformly bounded on compact subsets of B. The following proof of this fact is a paraphrase of the proof (of another fact) on pp. 9–11 of the classic paper [1] of Hartogs.

By (2.3) there is an $r>0$ and a $\beta>0$ such that if $k \in \mathbf{Z}$ and $|z| \leqq r$, then $|f_k(z)| \leqq \beta$. Thus if $k \in \mathbf{N}_+$ and $z \in B$, then

$$|f_k(z)|^{1/k} = \frac{1}{r}|f_k(rz)|^{1/k} \leqq \alpha .$$

Furthermore if $z \in B$ and $\mu \in \mathbf{D}$, then the series $\sum_1^\infty f_j(z)\mu^j$ converges, hence

$$\limsup_{j \to \infty} | f_j(z)|^{1/j} \leqq 1 .$$

Thus if $0 \leqq t < 1$ and if $b>1$, then by Proposition 2.8 there is an index $N=N(t,b)$ such that if $k \geqq N$ and $|z| \leqq t$, then $|f_k(z)|^{1/k} < b$, which is to say that if $k \geqq N$ and $|z| \leqq t/b$, then

$$|f_k(z)| = b^{-k}|f_k(bz)| < 1 .$$

Thus if $0 \leqq t < 1$ and if $b>1$, then by Proposition 2.8 there is an index $N=N(t,b)$ such that if $k \leqq N$ and $|z| \leqq t$, then $|f_k(z)|^{1/k} < b$, which is to say that if $k \geqq N$ and $|z| \leqq t/b$, then

3. An application and a remark.

3.1. We begin with an application of Theorem 1.2 to a recent theorem of Rudin. We will denote by Δ' the Laplace–Beltrami operator with respect to the Bergman metric on B. Thus if $f \in C^2(B)$, then

$$\Delta' f = 4(1-|z|^2)\left(\sum_{j=1}^{n} \partial^2 f/\partial \bar{z}_j \partial z_j - \sum_{j,k=1}^{n} z_j \bar{z}_k \partial^2 f/\partial \bar{z}_k \partial z_j\right).$$

We let

$$\Delta'' = 4 \sum_{j,k=1}^{n} z_j \bar{z}_k \partial^2/\partial \bar{z}_k \partial z_j .$$

Thus

$$\Delta' = (1-|z|^2)(\Delta - \Delta'') . \tag{3.1}$$

3.2. PROPOSITION. *Let $y \in S$, let $\mu \in \mathsf{D}$, and let $z=\mu y$. If $f \in C^2(B)$, then*

$$(\Delta'' f)(z) = 4\mu\bar{\mu}\frac{\partial^2}{\partial\bar{\mu}\partial\mu} f(\mu y) . \tag{3.2}$$

PROOF. We have $z_j=\mu y_j$, $\bar{z}_j=\bar{\mu}\bar{y}_j$, and

$$f(\mu y) = f(\mu y_1,\ldots,\mu y_n) .$$

Thus

$$\frac{\partial}{\partial\mu} f(\mu y) = \sum_1^n \left(\frac{\partial f}{\partial z_j}\frac{\partial z_j}{\partial\mu} + \frac{\partial f}{\partial\bar{z}_j}\frac{\partial\bar{z}_j}{\partial\mu}\right) = \sum_1^n y_j\frac{\partial f}{\partial z_j},$$

hence

$$\begin{aligned}\frac{\partial^2}{\partial\bar{\mu}\partial\mu} f(\mu y) &= \sum_1^n y_j\frac{\partial}{\partial\bar{\mu}}\left(\frac{\partial f}{\partial z_j}\right)\\ &= \sum_{j=1}^n y_j \sum_{k=1}^n \left(\frac{\partial^2 f}{\partial z_k\partial z_j}\frac{\partial z_k}{\partial\bar{\mu}} + \frac{\partial^2 f}{\partial\bar{z}_k\partial z_j}\frac{\partial\bar{z}_k}{\partial\bar{\mu}}\right)\\ &= \sum_{j,k=1}^n y_j\bar{y}_k\partial^2 f/\partial\bar{z}_k\partial z_j .\end{aligned}$$

Thus (3.2) holds.

3.3. THEOREM (Rudin [3]). *Let $f\colon B \to \mathsf{C}$. If f is harmonic with respect to both the Laplace–Beltrami operator Δ' and the Laplace operator Δ, then f is pluriharmonic.*

PROOF. We have

$$\Delta' f = \Delta f = 0 ,$$

hence by (3.1)

$$\Delta'' f = 0 .$$

Thus by Proposition 3.2 $H(f)=S$, hence by Theorem 1.2, f is pluriharmonic.

3.4. Let $j,k \in \mathsf{N}_+$, let $g \in H_{j+k}$, $g \neq 0$, and let $h \in H_j$, $h \neq 0$. Furthermore let

$$f(z) = g(z)\bar{h}(z)/|z|^{2j}$$

if $z \neq 0$, and let $f(0)=0$. Thus if $y \in S$ and $\mu \in \mathsf{C}$, then

$$f(\mu y) = \mu^k g(y)\bar{h}(y) , \tag{3.3}$$

hence $H(f)=S$.

If f is pluriharmonic, then by (3.3) f is holomorphic. Hence $\bar{h}(z)/|z|^{2j}$ is holomorphic if $z \neq 0$, hence $\bar{h}(z)/|z|^{2j}$ is bounded on $\{z : z \in B, z \neq 0\}$ if $n \geqq 2$. Thus if $n \geqq 2$, then f is not pluriharmonic, although $f \in C(B)$. This proves that without the hypothesis (1.1), the conclusion of Theorem 1.2 need not hold.

It may be worthwhile to point out that the Riemann removable singularity theorem [4, p. 19] (which was just used) is a corollary of the elementary Theorem 1.2 if in the statement of Theorem 1.2 the hypothesis "f is continuous on B" is replaced by "f is bounded on compact subsets of B".

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