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THE AREA OF ANALYTIC VARIETIES IN $\mathbf{C}^n$

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In one complex variable, Jensen's formula gives an upper bound to the frequency with which a holomorphic function (in the disc or the plane) can take on a given value in terms of the growth of the modulus of the function, and in fact this remains valid for holomorphic functions of several complex variables and quite general domains in $\mathbf{C}^n$ (cf. Gruman [6] for a very general setting). This suggested that one might be able to bound above the area of an analytic variety by the growth of the modulii of the functions which define it. Stoll [17] showed this to be possible when the defining functions are polynomials. He showed that the projective area of an analytic variety in $\mathbf{C}^n$ of pure dimension is finite if and only if it is algebraic. This led Griffiths [5] to ask if one could obtain similar bounds for analytic varieties defined by transcendental functions. This became known as the transcendental Bezout problem, in analogy with Bezout's Theorem, which states that if X_1 and X_2 are algebraic varieties of dimension p and q and of degree d_1 and d_2 respectively, then if $X_1 \cap X_2$ is finite, it consists of $d_1 d_2$ points when each is counted with respect to its multiplicity. Cornalba and Shiffman [3] constructed an example which shows that in general no such estimate is possible. They showed that there exist two entire functions in $\mathbf{C}^2$ of order zero such that the intersection of their zero sets can be made to grow arbitrarily fast.

We begin by developing an average Bezout estimate. Let D be a bounded domain given by a $\mathscr{C}^2$ plurisubharmonic function ϱ and let

$$\tilde{D}_r = \{z \in D : \varrho(z) < -1/r\},$$

or let D be a bounded pseudo convex domain with $\mathscr{C}^3$ boundary and

$$\tilde{D}_r = \left\{z \in D : d(z) > \frac{1}{r}\right\} \quad \text{where } d(z) = \inf_{z' \in \complement D} \|z - z'\|$$

(here we use the Euclidean norm). If X is an analytic variety in D of pure complex dimension p, we let $\sigma_X(r)$ be the $2p$-dimensional area of X in $\tilde{D}_r$, and for a holomorphic function f defined on X, we set $M_f(r) = \sup_{\tilde{D}_r \cap X} |f|$. We will

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say that a holomorphic map F of X is *non-degenerate* if the Jacobian of F is of maximal rank almost everywhere on X. If $F=(f_1,\ldots,f_q)$ is a non-degenerate holomorphic map defined on X and $\sigma_X(a,r)$ is the $2(p-q)$ dimensional area of the set $X \cap F^{-1}(a)$, then for $\tau \in \mathbf{R}^q$, $\tau_j>0$, $j=1,\ldots,q$, we estimate

$$\sigma_X(\tau,r) = \int_0^{2\pi} X \ldots X \int_0^{2\pi} \sigma_X(\tau_1 e^{i\varphi_1},\ldots,\tau_q e^{i\varphi_q})\, d\varphi_1 \ldots d\varphi_q$$

in terms of τ, $M_{f_i}(r)$, $\sigma_X(r)$, and r. If $D=\mathbf{C}^n$ and $\tilde{D}_r=\{z : \|z\|<r\}$, we obtain similar estimates.

These estimates resemble those obtained by Carlson [2], who showed that a Bezout type estimate holds except perhaps for an exceptional set in $\mathbf{C}^n$ of measure zero; however, our results differ in that the domains that we consider here are more general, the estimates we obtain are more refined, and in particular, our results imply that the exceptional set for which the estimate does not hold has an intersection of measure zero with the distinguished boundary of every polydisc in $\mathbf{C}^q$. This leads us to conjecture that the exceptional set is polar in $\mathbf{C}^q$ (the set on which a *pluri*subharmonic function is equal to $-\infty$, cf. Lelong [10]). We show this to be the case when $q=1$.

The second phenomenon that we study concerns the geometry of analytic varieties. Let X be an analytic variety of pure dimension p in a pseudoconvex subset D of $\mathbf{C}^n$ and let $G=(g_1,\ldots,g_q)$ be a holomorphic map defined on X. If $\sigma=(j_1,\ldots,j_m)$ is a subset of the numbers $(1,\ldots,q)$ and $|\sigma|=\operatorname{card}\sigma$, we set

$$\tilde{G}_\sigma = \{z \in X : |g_{j_1}(z)|=\ldots=|g_{j_m}(z)|,\ j_k \in \sigma\} \quad \text{and} \quad \tilde{G}^m = \bigcup_{|\sigma|=m} \tilde{G}_\sigma,$$

which is a set of real dimension $(2p-m+1)$.

Of particular interest for us will be the case when $X=\mathbf{C}^n$ and $g_j=(z_j-a_j)/\tau_j$, $\tau_j>0$, $j=1,\ldots,n$, and for this case we will denote $\tilde{G}^m$ by $\Gamma^m(a,\tau)$.

Ronkin [11] showed that if Y is the zero set of an entire function in $\mathscr{C}^n$ (i.e. an analytic variety of co-dimension 1), then Y meets every $\Gamma^n(a,\tau)$ and if $a \notin Y$, the growth of the area of $Y \cap \Gamma^n(a,\tau)$ is comparable to the growth of the area of Y in $\mathbf{C}^n$. We generalize these results to analytic varieties of higher co-dimension. We show that if the analytic polyhedron

$$A = \{z \in X : |g_j|<1,\ j=1,\ldots,q\}$$

is relatively compact in X and Y is an analytic variety of pure dimension s in D such that $Y \cap \{z \in X : g_j(z)=0, j=1,\ldots,q\}$ is empty, then the area of Y in D is bounded above by the area of Y in $\tilde{G}^m$ for $m=s+1$. This will allow us to show that if the distance of Y to $\tilde{G}^m \cap \partial D$ is positive for $m=s+1$, then the area of Y in D is finite. For $X=\mathbf{C}^n$, this will allow us to bound above the area of Y in the ball by the area of $Y \cap \Gamma^{s+1}(a,\tau)$ in the polydisc, and in this case we will also be able to obtain the inverse estimate which will permit us to show that the area of $Y \cap \Gamma^{(s+1)}(a, \tau)$ grows asymptotically like the area of Y in $\mathbf{C}^n$. As an immediate consequence, we will be able to show a result of Carlson [2] that the measure of the intersection of an analytic set Y in $\mathbf{C}^n$ of dimension q with a linear subspace of dimension at least q is bounded above by the area of Y in $\mathbf{C}^n$ for almost all linear subspaces. We will in fact be able to clarify somewhat the nature of the exceptional set.

Our techniques will also permit us to give an alternate proof of part if a theorem of Rudin [13], which says that if X is an analytic variety of pure dimension p in $\mathbf{C}^n$ for which there exist coordinates $z = z' + z''$, z' an $(n-p)$-tuple and z'' a p-tuple, and constants A and B such that $\|z'\| \leqq A(1 + \|z''\|)^B$ whenever $z \in X$, then X is actually an algebraic variety.

Our techniques depend heavily on the theory of positive closed currents (cf. Lelong [9] or Federer [4]), which have become such a powerful tool in complex analysis. Our results reinforce the impression that this theory possesses an inherent geometric subtlety.

1. Plurisubharmonic functions and positive closed currents.

For the basic theory of positive closed currents, we refer the reader to the book of Lelong [9]. We recall here only the basic facts that we shall need.

A real valued function $\varrho(z)$ defined in a domain $D \subset \mathbf{C}^n$ and taking on values in the range $[-\infty, +\infty)$ is plurisubharmonic if it is upper semi-continuous and if for $z \in D$, $w \in \mathbf{C}^n - \{0\}$,

$$\varrho(z) \leqq \frac{1}{2\pi} \int_0^{2\pi} \varrho(z + wre^{i\theta})\, d\theta$$

for $r < \sup\{\tau : z + \lambda e^{i\theta} w \in D \text{ for } |\lambda| < \tau\}$. If ϱ is plurisubharmonic then for every $w \in \mathbf{C}^n$, the distribution

$$\sum_{j,k} \frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k$$

defines a positive measure in D. If ϱ is $\mathscr{C}^2$ in D and there exists a continuous function $C(z) > 0$ such that

$$\sum_{j,k} \frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geqq C(z)\|w\|^2 \quad \text{for all } w \in \mathbf{C}^n,$$

ϱ is said to be strictly plurisubharmonic.

Let $\eta \in \mathscr{C}_0^\infty(B(0,1))$ be such that $0 \leqq \eta \leqq 1$, η depends only on $\|z\|$ and $\int \eta(z)\, d\lambda(z) = 1$. We set

$$\eta_\varepsilon(z) = \frac{1}{\varepsilon^{2n}}\eta\left(\frac{z}{\varepsilon}\right).$$

If t is a distribution in D, we define the distribution

$$t^\varepsilon(\varphi) = t*\eta_\varepsilon(\varphi) = t(\varphi*\eta_\varepsilon).$$

Then, in $D_\varepsilon=\{z : d(z)=\inf_{w\in\complement D}\|z-w\|>\varepsilon\}$, t^ε is a $\mathscr{C}^\infty$ function.

In particular, if ϱ is plurisubharmonic in D, $\varrho_\varepsilon\geqq\varrho$ in D_ε and $\lim_{\varepsilon\to 0}\varrho_\varepsilon=\varrho$, the convergence being uniform on compact sets if ϱ is continuous.

A current θ of pure type (p,p) in D is a linear form on the space of differential forms of degree $(n-p,n-p)$ with coefficients in $\mathscr{C}_0^\infty(D)$, the space of infinitely differentiable functions with compact support in D. For every such θ, there exists a canonical decomposition

$$\theta = \sum_{\substack{|I|=p\\|J|=p}} t_{i_1\ldots i_p,j_1\ldots j_p}dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_p}$$

(the sum is taken over all strictly increasing multi-indices) where the coefficients $t_{i_1\ldots i_p,j_1\ldots j_p}$ are distributions. The exterior differential operators

$$\partial = \sum_{j=1}^n \frac{\partial}{\partial z_j}dz_j, \qquad \bar\partial = \sum_{j=1}^n \frac{\partial}{\partial \bar z_j}d\bar z_j$$

and

$$d = \partial+\bar\partial$$

extend in a natural way to currents via the extension of differential operators to distributions. A current θ is closed if the current $d\theta=0$. A current θ of pure type (p,p) is closed if and only if $\partial\theta=0$ and $\bar\partial\theta=0$, as one sees immediately from degree considerations.

A current θ of pure type (p,p) is said to be positive of degree p if for every system of pure forms with constant coefficients $(\alpha_1,\ldots,\alpha_{n-p})$, $t\wedge i\alpha_1\wedge\bar\alpha_1\wedge\ldots\wedge(i\alpha_{n-p}\wedge\bar\alpha_{n-p})$ defines a positive measure in D. This implies in particular that $t_{i_1\ldots i_p,j_1\ldots j_p}$ is a complex measure for every choice of $i_1\ldots i_p, j_1\ldots j_p$. We shall denote by $\tilde T_+^p$ (respectively Φ_+^p) the positive closed currents of degree p (respectively the positive closed currents of degree p whose coefficients are continuous functions). Then

i) if $\theta\in\tilde T_+^1$ and $\psi\in\tilde\Phi_+^p$, $\theta\wedge\psi\in\tilde T_+^{p+1}$,
ii) if $\theta\in\tilde T_+^p$ and $\psi\in\tilde\Phi_+^1$, $\theta\wedge\psi\in\tilde T_+^{p+1}$.

If θ is a current of degree p with canonical decomposition

$$\theta = \sum_{\substack{|I|=p\\|J|=p}} t_{i_1\ldots i_p,j_1\ldots j_p}dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_p}$$

then, we set

$$\theta^\varepsilon = \sum_{\substack{|I|=p \\ |J|=p}} t^\varepsilon_{i_1 \ldots i_p, j_1 \ldots j_p} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_p},$$

which is a current with $\mathscr{C}^\infty$ coefficients in D_ε.

LEMMA 1.1. *If $\theta \in \tilde{T}^p_+$ in D, then $\theta^\varepsilon \in \tilde{T}^p_+$ in D_ε.*

PROOF. Let $(\alpha_1, \ldots, \alpha_{n-p})$ be a system of forms with constant coefficients and $f \in \mathscr{C}_0^\infty(D_\varepsilon)$, $f \geqq 0$. Then

$$t^\varepsilon \wedge f(i\alpha_1 \wedge \bar{\alpha}_1) \wedge \ldots \wedge (i\alpha_{n-p})$$
$$= t \wedge f^\varepsilon(i\alpha_1 \wedge \bar{\alpha}_1) \wedge \ldots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p}) \geqq 0$$

since $f \geqq 0$ and $\eta_\varepsilon \geqq 0$ implies $f * \eta_\varepsilon \geqq 0$. Thus t^ε is positive in D_ε. Let φ be an $(n-p-1, n-p)$ form with coefficients in $\mathscr{C}_0^\infty(D_\varepsilon)$. Then $t^\varepsilon(\partial\varphi) = t((\partial\varphi)^\varepsilon) = t(\partial\varphi^\varepsilon) = 0$ since convolution commutes with differentiation. Similarly, $\bar{\partial} t^\varepsilon = 0$ so $dt^\varepsilon = 0$.

In the applications, there are certain positive closed currents that we shall use. If $\varrho(z)$ is a plurisubharmonic function in D, then $i\partial\bar{\partial}\varrho$ defines a positive closed current of degree 1. We set

$$\beta = \frac{i}{2}\partial\bar{\partial}\|z\|^2 = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k, \quad \beta_q = \frac{\beta^q}{q!}, \quad \alpha = \frac{i}{2}\partial\bar{\partial}\log\|z\|^2 .$$

If t is a positive closed current of degree p, we associate the measure $\sigma_t = t \wedge \beta_{n-p}$. An analytic set of pure dimension p in D is composed of a union of irreducible branches X_j and positive integers m_j giving the multiplicity of X_j (cf. Stoll [18]). Then there exists a positive closed current θ_j of degree $(n-p)$ associated with X_j, the current of integration on X_j, and hence a positive closed current $\theta = \sum m_j \theta_j$ of degree $(n-p)$ associated with X, and σ_θ gives the $2p$ dimensional area (with multiplicity) of the analytic set X.

2. An average Bezout estimate in $\mathbf{C}^n$.

Let X be an analytic variety in $\mathbf{C}^n$ of dimension p and $F = (f_1, \ldots, f_q)$, $q \leqq p$, a non-degenerate holomorphic map of X into $\mathbf{C}^q$. By non-degenerate, we will mean that $(\partial f_1, \ldots, \partial f_q)$ are linearly independent almost everywhere on X. The Bezout estimate consists in majorizing the area of $F^{-1}(a)$ by a function of the area of X and the area of $F_j^{-1}(a_j)$, $j = 1, \ldots, q$, or alternately, the growth of

$\log|f_j|$. It is now known that such an estimate is in general impossible [3]. We shall obtain here an estimate of the average area of $F^{-1}(a)$ over the set

$$\left\{z \in \mathbf{C}^q : \frac{|z_1|}{\tau_1} = \ldots = \frac{|z_q|}{\tau_q}\right\}$$

in terms of τ_j, the growth of the area of X, and the growth of the functions $\log|f_j|$.

Before embarking on the development of this result, we note that there already exist several Bezout Theorems in the literature. Stoll [18] showed that if X is a complex manifold in $\mathbf{C}^n$ or in the unit ball in $\mathbf{C}^n$ and f: $X \to \mathbf{C}^q$ is non-degenerate, then $F_t^{-1}(0)$, where $F_t(z)=F(tz)$, $t=(t_1,\ldots,t_n)$, $0\leqq t_j\leqq 1$, satisfies an average Bezout estimate in t. In [19], Stoll obtains a uniform estimate, but it involves a host of different terms which seem difficult to estimate in practice. The result closest in spirit and content to ours is that of Carlson [2], who obtains a uniform Bezout estimate for almost all $a \in \mathbf{C}^q$ when $X\subset\mathbf{C}^n$. Our estimates are similar in nature, but finer, the domains considered here are more general, and more is obtained on the nature of the exceptional set.

The first case that we shall consider is as follows: D is a bounded domain in $\mathbf{C}^n$ given by a $\mathscr{C}^2$ plurisubharmonic function $\tilde{\varrho}$. That is, there exists a plurisubharmonic function $\tilde{\varrho}$ defined in a neighborhood N of ∂D such that

$$D \cap N = \{z : \tilde{\varrho}(z)<0\} .$$

In fact, we can assume D defined by a function ϱ defined everywhere in D. Let $\mu>0$ be such that

$$\{z : -\mu\leqq\varrho\leqq 0\} \subset\subset N$$

and $\chi(t)$ an increasing $\mathscr{C}^\infty$ convex function of t such that $\chi(0)=0$ and $\chi\equiv-\mu$ for $t\leqq-\mu$ and set $\varrho=\chi(\tilde{\varrho})$, which is plurisubharmonic and defines D. We will always make this assumption. We set

$$D_\delta = \{z : \varrho(z)< -\delta\} .$$

LEMMA 2.1. *Let θ be a closed positive current of degree $(n-p)$ with $\mathscr{C}^\infty$ coefficients, $p<n$. Suppose $\vec{\nabla}\varrho\neq 0$ in a neighborhood of ∂D. If h is a $\mathscr{C}^2$ function in a neighborhood of $\bar{D}$, then*

(2.1)
$$\int_{\partial D} hi\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_D hi\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} - \int_D \varrho i\partial\bar{\partial}h \wedge \theta \wedge \beta_{p-1} .$$

PROOF. Since $\theta \wedge \beta_{p-1}$ is d closed, by Stokes' Theorem

$$\int_{\partial D} hi\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_D dh \wedge i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} + \int_D hi\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1}$$

and

$$0 = \int_{\partial D} \varrho i\partial h \wedge \theta \wedge \beta_{p-1} = \int_D \varrho i\bar{\partial}\partial h \wedge \theta \wedge \beta_{p-1} + \\ + \int_D d\varrho \wedge i\partial h \wedge \theta \wedge \beta_{p-1} .$$

But

$$d\varrho \wedge i\partial h \wedge \theta \wedge \beta_{p-1} = \bar{\partial}\varrho \wedge i\partial h \wedge \theta \wedge \beta_{p-1}$$
$$= \partial\varrho \wedge idh \wedge \theta \wedge \beta_{p-1} = -dh \wedge i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1}$$

and

$$i\bar{\partial}\partial h \wedge \theta \wedge \beta_{p-1} = -i\partial\bar{\partial}h \wedge \theta \wedge \beta_{p-1} .$$

LEMMA 2.2. *Let θ be a positive closed current of degree $(n-p)$ with $\mathscr{C}^\infty$ coefficients and suppose $\vec{\nabla}\varrho \neq 0$ in a neighborhood of ∂D. If $V(z)$ is plurisubharmonic and bounded in $\bar{D}$ and*

$$\lim_{\substack{z \to z_0 \in \partial D \\ z \in D}} V(z) = V(z_0) ,$$

(2.2)

then
$$\int_{\partial D} V(z)i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_D V(z)i\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} - \\ - \int_D \varrho\partial\bar{\partial}V \wedge \theta \wedge \beta_{p-1}$$

PROOF. Let $\delta > 0$ be so small that $\vec{\nabla}\varrho \neq 0$ in a neighborhood of ∂D_δ and let $V^\varepsilon = V * \eta_\varepsilon$. Then V^ε is plurisubharmonic in D_δ for ε sufficiently small, and $V^\varepsilon \downarrow V$ in $\bar{D}_\delta$. Applying (2.1) to V^ε, we obtain

$$\int_{\partial D_\delta} V^\varepsilon(z)i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_{D_\delta} V^\varepsilon(z)i\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} - \\ - \int_{D_\delta} (\varrho + \delta)i\partial\bar{\partial}V^\varepsilon \wedge \theta \wedge \beta_{p-1} .$$

When $\varepsilon \to 0$, $i\partial\bar{\partial}V^{\varepsilon} \wedge \theta \wedge \beta_{p-1}$ converges to the measure $i\partial\bar{\partial}V \wedge \theta \wedge \beta_{p-1}$, so by the Bounded Convergence Theorem

$$\int_{\partial D_{\delta}} Vi\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_{D_{\delta}} V(z)i\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} - \\ -\int_{D_{\delta}} (\varrho+\delta)i\partial\bar{\partial}V \wedge \theta \wedge \beta_{p-1} .$$

As $\delta \to 0$, the right hand side tends to

$$\int_{D} V(z)i\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} - \int_{D} \varrho i\partial\bar{\partial}V \wedge \theta \wedge \beta_{p-1}$$

whereas by the Bounded Convergence Theorem, the left hand side converges to $\int_{\partial D} Vi\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1}$, as one can see easily by choosing ϱ as a local coordinate near ∂D.

LEMMA 2.3. *Suppose ϱ is strictly plurisubharmonic and $\vec{\nabla}\varrho \neq 0$ in a neighborhood of D. If θ is a positive closed current of degree $(n-p)$ with $\mathscr{C}^{\infty}$ coefficients, then $i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1}$ defines a positive measure μ on ∂D,* $\operatorname{supp}\mu \subset \operatorname{supp}\theta \cap \partial D$ *and*

$$\int_{\partial D} d\mu \leqq C_D \int_{D} \theta \wedge \beta_p \quad \textit{where } C_D \leqq K \sup_{D} \sum_{j,k} \left| \frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k} \right| .$$

PROOF. It is clear that $i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1}$ determines a linear functional on the Banach space of continuous functions on ∂D with the supremum norm, so it is a measure on ∂D. Let h be a continuous function on ∂D, $h \geqq 0$. By Bremermann [1], there exists a plurisubharmonic function V_h such that

$$\lim_{\substack{z \to z_0 \in \partial D \\ z \in D}} V(z) = V_h(z_0) = h(z_0) .$$

We can assume without loss of generality that $V_h \geqq 0$, for otherwise, we replace V_h by $\sup(V_h, 0)$ which is also plurisubharmonic in D. By the maximum principle, $0 \leqq V_h \leqq \sup_{\partial D} h$ so by (2.2),

$$\int_{\partial D} hi\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_{D} V_h i\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} - \\ -\int_{D} \varrho i\partial\bar{\partial}V_h \wedge \theta \wedge \beta_{p-1} \geqq 0$$

It is clear that $\operatorname{supp}\mu \subset \operatorname{supp}\theta \cap \partial D$, for if $\operatorname{supp} h \cap \operatorname{supp}\theta \cap \partial D$ is empty then

$$\int_{\partial D} hi\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = 0 .$$

If $h=1$, we obtain

$$\int_{\partial D} i\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} = \int_{D} i\partial\bar{\partial}\varrho \wedge \theta \wedge \beta_{p-1} .$$

By Lelong [9, p. 74], there exists a constant A independent of D such that if $t_{I,J}$ is a coefficient of θ then

$$(2.3) \qquad \int_{D} |t_{I,J}|\, d\lambda \leqq A \int_{D} \theta \wedge \beta_p .$$

Let Y be an analytic variety of pure dimension p in D and θ_Y the closed positive current of degree $(n-p)$ associated with the area of Y. We set

$$\tilde{D}_r = D_{1/r} \quad \text{and} \quad \sigma_Y(r) = \int_{\tilde{D}_r} \theta_Y \wedge \beta_p .$$

If f is holomorphic on Y, we set

$$M_Y(f,r) = \sup_{\tilde{D}_r \cap Y} |f| .$$

For any given $a \in \mathbf{C}$, if f is not identically a on any component of Y, then $Y \cap f^{-1}(a)$ is an analytic variety of pure dimension $(p-1)$ (by Gunning and Rossi [7, p. 245], we can actually suppose f defined in all of D since every holomorphic function on Y is the restriction of a function holomorhic in D). Then

$$\theta_Y(f,a) = \frac{i}{2}\partial\bar{\partial}\left(\frac{1}{2\pi} \log|f-a|\right) \wedge \theta_Y$$

is the current associated with the area of $Y \cap f^{-1}(a)$ (cf. Lelong [9], especially p. 78). We set

$$\sigma_Y(f,a,r) = \int_{\tilde{D}_r} \theta_Y(f,a) \wedge \beta_{p-1} .$$

LEMMA 2.4. *Let f be a non-constant holomorphic function defined on an analytic variety Y of pure dimension p in D. Then for every r and $\tau>0$, $\int_{\tilde{D}_r} \log|f-\tau e^{i\varphi}|\, \theta_Y \wedge \beta_p$ is an integrable function of φ.*

PROOF. The proof will turn on the easily verified identity

$$(2.4) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log|\lambda - e^{i\varphi}|\, d\varphi = \log^+ |\lambda|$$

which is a simple consequence of Jensen's formula.

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(z)-\tau e^{i\varphi}|\,d\varphi = \sup(\log|f(z)|, log\,\tau)\,.$$

Let $\tilde{f}$ be a homomorphic function in D such that $\tilde{f}|_Y = f$ and let

$$V_\varepsilon^\varphi(z) = (\log|\tilde{f}(z)-\tau e^{i\varphi}|)*\eta_\varepsilon\,,$$

$$V_\varepsilon^\tau(z) = \sup(\log|f(z)|, \log\tau)*\eta_\varepsilon\,.$$

Then for ε sufficiently small V_ε^φ and V_ε^τ are $\mathscr{C}^\infty$ plurisubharmonic functions in a neighborhood of $\bar{D}_r \cap Y$ and are decreasing as $\varepsilon \to 0$. Furthermore, by (2.4),

$$\frac{1}{2\pi}\int_0^{2\pi} V_\varepsilon^\varphi(z)\,d\varphi = V_\varepsilon^\tau(z)\,.$$

Given $\delta > 0$, for ε small enough

$$\sigma_Y(r)[\log^+ M_Y(f,r) + \log^+\tau] + \delta \geqq \int_{\bar{D}_r} V_\varepsilon^\varphi(z)\theta_Y \wedge \beta_p = A_\varepsilon^\varphi$$

and

$$\lim_{\varepsilon\to 0} A_\varepsilon^\varphi = \int_{\bar{D}_r} \log|f(z)-\tau e^{i\varphi}|\theta_Y \wedge \beta_p\,.$$

Then by Fatou's Lemma,

$$\begin{aligned}\int_0^{2\pi}\left(\int_{\bar{D}_r} \log|f-\tau e^{i\varphi}|\theta_Y \wedge \beta_p\right)d\varphi &= \int_0^{2\pi}\left(\lim_{\varepsilon\to 0} A_\varepsilon^\varphi\right)d\varphi \\ &\geqq \lim_{\varepsilon\to 0}\int_0^{2\pi} A_\varepsilon^\varphi\,d\varphi \\ &\geqq \int_{\bar{D}_r} \sup(\log|f|, \log\tau)\theta_Y \wedge \beta_p > -\infty\,.\end{aligned}$$

LEMMA 2.5. *Let f be holomorphic in D and non-constant on Y and let*

$$A_\varepsilon^r(\varphi) = \int_{\bar{D}_r} \frac{i}{2}\partial\bar{\partial}\left(\frac{1}{2\pi}\log|f-\tau e^{i\varphi}|*\eta_\varepsilon\right) \wedge \theta_Y \wedge \beta_{p-1}\,.$$

Then

$$\lim_{\varepsilon\to 0}\int_0^{2\pi} A_\varepsilon^r(\varphi)\,d\varphi = \int_0^{2\pi}\left(\int_{\bar{D}_r} \theta_Y(f,\tau e^{i\varphi}) \wedge \beta_{p-1}\right)d\varphi = \int_0^{2\pi} \sigma_Y(f,\tau e^{i\varphi},r)\,d\varphi$$

and for $\gamma > 1$

$$\int_0^{2\pi} \sigma_Y(f, \tau e^{i\varphi}, r)\, d\varphi \leqq \frac{\gamma r}{(\gamma-1)} C[\log^+ M_Y(f, \gamma r) + \log^- \tau]\sigma_Y(\gamma r) .$$

PROOF. Let

$$V_\varepsilon^\varphi(z) = \frac{1}{2\pi} \log\left|\frac{f}{\tau} - e^{i\varphi}\right| * \eta_\varepsilon ,$$

which is plurisubharmonic in a neighborhood of $\hat{D}_r$ for ε small enough, and let $\theta^{\varepsilon'} = \theta_Y * \eta_{\varepsilon'}$. Set

$$\varrho_\xi = \varrho * \eta_\xi + \xi \|z\|^2 ,$$

which is strictly plurisubharmonic in D. Let $\delta > 0$ be given. For ξ sufficiently small

$$\sup_D \sum_{j,k} \left|\frac{\partial^2 \varrho_\xi}{\partial z_j \partial \bar{z}_k}\right| \leqq \sup_D \sum_{j,k} \left|\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}\right| + \delta ,$$

and if

$$\alpha_\xi = \inf_{\partial \hat{D}_r} \varrho_\xi, \quad D_\xi^\lambda = \{z \in D ,\, \varrho_\xi < \alpha_\xi + \lambda\} ,$$

then for ξ, λ and ε sufficiently small,

$$\sup_{D_\xi^\lambda \cap Y} |f| \leqq M_Y(f, r) + \delta .$$

Furthermore, for ε' sufficiently small,

$$\sigma_Y(r) + \delta \geqq \int_{D_\xi^\lambda} \theta^{\varepsilon'} \wedge \beta_p .$$

By Sard's Theorem [14], the set of λ for which $\vec{\nabla} \varrho_\xi = 0$ on ∂D_ξ^λ is a measure zero. For λ nôt in this exceptional set, by Lemma 2.3

$$\int_{\partial D_\xi^\lambda} V_\varepsilon^\varphi i \bar{\partial} \varrho_\xi \wedge \theta^{\varepsilon'} \wedge \beta_{p-1} = \int_{D_\xi^\lambda} V_\varepsilon^\varphi i \partial\bar{\partial} \varrho_\xi \wedge \theta^{\varepsilon'} \wedge \beta_{p-1} - $$
$$- \int_{D_\xi^\lambda} (\varrho_\xi + \alpha_\xi + \lambda) i \partial \bar{\partial} V_\varepsilon^\varphi \wedge \theta^{\varepsilon'} \wedge \beta_{p-1} .$$

We first let $\lambda \to 0$, then $\xi \to 0$, then $\varepsilon' \to 0$ to obtain for any $\mu > 0$ and ε sufficiently small (depending on μ)

$$(2.5)\qquad \mu + C[\log^+ M_Y(f,r) + \log^- \tau]\sigma_Y(r) \geqq \int_{\tilde{D}_r} V_\varepsilon^\varphi i\partial\bar{\partial}\varrho \wedge \theta_Y \wedge \beta_{p-1} - \\ - \int_{\tilde{D}_r} \left(\varrho + \frac{1}{r}\right) i\partial\bar{\partial} V_\varepsilon^\varphi \wedge \theta_Y \wedge \beta_{p-1} .$$

Thus, since

$$V_\varepsilon^\varphi(z) \geqq \frac{1}{2\pi}\log\left|\frac{f}{\tau} - e^{i\varphi}\right| ,$$

if we apply (2.4) for γr, we obtain

$$\mu + C\sigma_Y(\gamma r)[\log^+ M_Y(f,\gamma r) + \log^- \tau] - \\ - \int_{\tilde{D}_{\gamma r}} \frac{1}{2\pi} \log\left|\frac{f}{\tau} - e^{i\varphi}\right| i\partial\bar{\partial}\varrho \wedge \theta_Y \wedge \beta_{p-1} \\ \geqq \frac{(\gamma-1)}{\gamma r} \int_{\tilde{D}_r} i\partial\bar{\partial} V_\varepsilon^\varphi \wedge \theta_Y \wedge \beta_{p-1} \geqq 0$$

so if we can show that $\lim_{\varepsilon\to 0} A_\varepsilon^r(\varphi) = \sigma_Y(f, \tau e^{i\varphi}, r)$ for almost φ, the first part of the lemma will follow from Lemma 2.4 and the Lebesgue Dominated Convergence Theorem, and the second part will follow from (2.4).

The set of points $Y' \subset Y$ for which Y is not a manifold is an analytic variety of dimension at most $(p-1)$ so there are at most a finite number of φ for which $f^{-1}(\tau e^{i\varphi}) \cap Y \subset Y'$, hence for all but a finite number of φ, the set $\partial f = 0, f = \tau e^{i\varphi}$ on Y is an analytic variety of dimension at most $(p-2)$. Thus, if φ is not in this exceptional set, the set on which $f^{-1}(\tau e^{i\varphi}) \cap Y$ is not a complex manifold is an analytic variety Y_φ of dimension at most $(p-2)$. Let $\omega \in \mathscr{C}_0^\infty(\tilde{D}_r)$ and $\psi \in \mathscr{C}_0^\infty(D)$ such that $0 \leqq \psi \leqq 1$ and $\psi \equiv 1$ on Y_φ. Then

$$\lim_{\varepsilon\to 0} \int_{\tilde{D}_r} (1-\psi)\omega i\partial\bar{\partial} V_\varepsilon^\varphi \wedge \theta_Y \wedge \beta_{p-1} \\ = \int_{\tilde{D}_r} \omega(1-\psi) i\partial\bar{\partial} \frac{1}{2\pi} \log|f - \tau e^{i\varphi}| \wedge \theta_Y \wedge \beta_{p-1} \\ = \int_{\tilde{D}_r} (1-\psi)\omega\theta_Y(f, \tau e^{i\varphi}) \wedge \beta_{p-1}$$

since we can use f as a local coordinate on $Y \cap \complement(\operatorname{supp}\psi)$. If we can find a sequence $\psi_n \to \psi_{Y_\varphi}$, the characteristic function of Y_φ, such that

$$\int_{\tilde{D}_r} |i\partial\bar{\partial}(\psi_n\omega) \wedge \theta_Y \wedge \beta_{p-1}| \to 0$$

then, by (2.4),

$$f_n(\varphi) = \int_{\tilde{D}_r} \log\left|\frac{f}{\tau} - e^{i\varphi}\right| |i\partial\bar{\partial}(\psi_n\omega) \wedge \theta_Y \wedge \beta_{p-1}|$$

converges in measure to zero, so there exists a subsequence, which we will again denote by $f_n(\varphi)$, which converges almost everywhere to zero, and since

$$C(r) \geqq V_\varepsilon^\varphi \geqq \log\left|\frac{f}{\tau} - e^{i\varphi}\right| ,$$

$\int_{\tilde{D}_r} V_\varepsilon^\varphi |i\partial\bar{\partial}(\psi_n\omega) \wedge \theta_Y \wedge \beta_{p-1}|$ will also converge to zero for almost all φ uniformly in ε, so that for almost all φ

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\tilde{D}_r} \omega i\partial\bar{\partial} V_\varepsilon^\varphi \wedge \theta_Y \wedge \beta_{p-1} &= \lim_{n\to\infty}\lim_{\varepsilon\to 0} \int_{\tilde{D}_r} \omega(1-\psi_n) i\partial\bar{\partial} V_\varepsilon^\varphi \wedge \theta_Y \wedge \beta_{p-1} \\ &= \lim_{n\to\infty} \int_{\tilde{D}_r} \omega(1-\psi_n)\theta_Y(f, \tau e^{i\varphi}) \wedge \beta_{p-1} \\ &= \int_{\tilde{D}_r} \omega\theta_Y(f, \tau^{i\varphi}) \wedge \beta_{p-1} .\end{aligned}$$

To construct ψ_n, we note that by Lelong [9, p. 77, Proposition 12], there exists a constant K (depending on r) such that

$$\int_{B(z,t)} \theta \wedge \beta_p \leqq Kt^{2p}$$

independent of z (where $B(z,t)$ is the ball of center z and radius t). Since Y_φ is an analytic variety of dimension $(p-2)$ at most, it has finite $2(p-2)$ Hausdorff measure (cf. Federer [4] for a definition of Hausdorff measure), so given an integer n, there exists a finite number of balls $B_i^n = B(z_{i,n}, t_{i,n})$ with

$$\sup_i t_{i,n} < \frac{1}{n} \quad \text{and} \quad \sum_i t_{i,n}^{2(p-2)} \leqq A$$

such that $Y \cap \tilde{D}_r \subset \bigcup B_i^n$. Let $\eta(z)$ be in $\mathscr{C}_0^\infty(B(0,2))$ such that $\eta \equiv 1$ on $B(0,1)$ and $0 \leqq \eta(z) \leqq 1$, and set

$$\eta_i^n(z) = \eta\left(\frac{z - z_{i,n}}{t_{i,n}}\right), \quad \psi_n(z) = 1 - \prod (1 - \eta_i^n(z)) .$$

Then, by (2.3)

$$\int_{\tilde{D}_r} |\partial\bar{\partial}(\psi_n\omega) \wedge \theta_Y \wedge \beta_{p-1}| \leqq C_1 \sum_i \int_{B_i^n} \frac{\theta_Y \wedge \beta_p}{t_{i,n}^2} + \\ + C_2 \sum_{i \neq j} \int_{B_i^n \cap B_j^n} \frac{\theta_Y \wedge \beta_p}{t_{i,n} t_{j,n}}$$

and

$$\sum_{i \neq j} \int_{B_i^n \cap B_j^n} \frac{\theta_Y \wedge \beta_p}{t_{i,n} t_{j,n}} \leqq \sum_i \sum_{\substack{j \neq i \\ t_{j,n} \geqq t_{i,n}}} \int_{B_i^n \cap B_j^n} \frac{\theta_Y \wedge \beta_p}{t_{i,n}^2} \\ \leqq \sum_i \int \frac{\theta_Y \wedge \beta_p}{t_{i,n}^2}$$

so

$$\int_{\tilde{D}_r} |\partial\bar{\partial}(\psi_n\omega) \wedge \theta_Y \wedge \beta_{p-1}| \leqq C_3 \sum_i t_{i,n}^{2p-2} \leqq \frac{C_3 A}{n^2},$$

which concludes the proof.

REMARK. If $\varphi_1, \varphi_2 \in \mathscr{C}_0^\infty$, then

$$\iint \varphi_1(u)\varphi_2(u-v)\eta_\varepsilon(v)\,dv\,du = \int \varphi_2(w)\varphi_1(w+v)\eta_\varepsilon(v)\,dw\,dv \\ = \int \varphi_2(w)\varphi_1(w-v)\eta_\varepsilon(v)\,dw\,dv$$

since $\eta_\varepsilon(-v) = \eta_\varepsilon(v)$. If $\omega \in \mathscr{C}_0^\infty(\tilde{D}_r)$ and ε sufficiently small

$$\int_{\tilde{D}_r} \omega\,\partial\bar{\partial}\log|f-\tau e^{i\varphi}| \wedge \theta_Y^\varepsilon \wedge \beta_{p-1} \\ = \int_{\tilde{D}_r} (\omega i\partial\bar{\partial}\log|f-\tau e^{i\theta})^\varepsilon \wedge \theta_Y \wedge \beta_{p-1}$$

and so as in Lemma 2.5, one has for almost all φ

$$\lim_{\varepsilon\to 0} \int_{\tilde{D}_r} i\,\partial\bar{\partial}\log|f-\tau e^{i\varphi}| \wedge \theta_Y^\varepsilon \wedge \beta_{p-1} = \int_{\tilde{D}_r} \sigma_Y(f, \tau e^{i\varphi}, r)\,.$$

THEOREM 2.6. *Let D be a bounded domain in $\mathbf{C}^n$ given by a $\mathscr{C}^2$ plurisubharmonic function ϱ and let Y be a pure p dimensional analytic variety in D. Suppose $F\colon Y \to \mathbf{C}^q$ is a non-degenerate holomorphic map defined on Y and let $M_i(r) = \sup_{\tilde{D}_r \cap Y} |f_i|$. If $\tau \in \mathbf{R}^q$, $\tau_j > 0$, we set $\theta_\tau(\varphi_1, \ldots, \varphi_q)$ to be the closed positive current associated with the area of $Y \cap F^{-1}(\tau_1 e^{i\varphi_1}, \ldots, \tau_q e^{i\varphi_q})$ and*

$$\sigma_\tau(\Phi,r) = \int_{\tilde{D}_r} \theta_\tau(\varphi_1,\ldots,\varphi_q) \wedge \beta_{p-q} .$$

Let T^q be the q-dimensional torus and $d\Phi$ the product measure on T^q. Then for every $\gamma>1$,

$$\int_{T^q} \sigma_\tau(\Phi,r)\,d\Phi \leqq \left[\frac{\gamma r C'}{(\gamma-1)}\right]^q \sigma_Y(\gamma^q r) \prod_{j=1}^{q} [\log^+ M_j(\gamma^{q-j+1}r)+\log^- \tau_j] .$$

PROOF. We establish the result by induction on q. For $q=1$, this is just Lemma 2.5. Assume the result for $(q-1)$. By Lemma 2.5

$$\int_0^{2\pi} \sigma_Y(f_1,\tau_1 e^{i\varphi_1},\gamma^{q-1}r)\,d\varphi_1 \leqq \left[\frac{\gamma r C'}{(\gamma-1)}\right][\log^+ M_1(\gamma^q r)+\log^- \tau_1]\sigma_Y(\gamma^q r) .$$

Let $\tilde{F}=(f_2,\ldots,f_q)$. The set on which $\tilde{F}$ does not have rank $(q-1)$ is an analytic variety on Y of dimension at most $(p-1)$, so for all but a finite number of φ_1, $\tilde{F}$ is non degenerate on $Y\cap f_1^{-1}(\tau_1 e^{i\varphi})$. The estimate now follows from the induction hypothesis and Fubini's Theorem.

COROLLARY 2.7. *Let Y and F be as in Theorem 2.6. Then given $\gamma>1$, $\alpha>0$, and $\varepsilon>0$, there exist constants C_1 and C_1' depending only on $\alpha,\gamma,\varepsilon,\tau$ and D such that for almost all Φ*

i) $\sigma_\tau(\Phi,r) \leqq C_1 r^{q+1+\alpha}\sigma_Y(\gamma^q(r+\varepsilon)) \prod_{j=1}^{q} [\log^+ M_j(\gamma^q(r+\varepsilon))]$

or

ii) $\sigma_\tau(\Phi,r) \leqq C_1' r^q(\log^+ r)^{1+\alpha}\sigma_Y(\gamma^{q+1}r) \prod_{j=1}^{q} [\log^+ M_j(\gamma^{q+1}r)]$

for $r\geqq r_\Phi$.

PROOF. The proof of (i) is contained essentially in [2], so we consider only the second assertion. Let $r_n=\gamma_n$. Then if $T(\gamma,r)$ is the function given by Theorem 2.6, and if

$$\Sigma_n = \{\Phi : \sigma_\tau(\Phi,\gamma^n)>(n\log\gamma)^{1+\alpha}T(\gamma,\gamma^n)\} ,$$

then meas $\Sigma_n\leqq K(\gamma)/n^{1+\alpha}$ and since $\sum_{n=1}^{\infty} 1/n^{1+\alpha}<+\infty$, if $\Omega_n=\bigcup_{l\geqq n}\Sigma_l$, then for n sufficiently large, $m(\Omega_n)<\varepsilon$. If $\Phi\notin\bigcap_{n=1}^{\infty}\Omega_n$, which has measure zero, then for n sufficiently large (depending on Φ)

$$\sigma_\tau(\Phi,\gamma^n) \leqq C(\log\gamma^n)^{1+\alpha}\sigma_Y(\gamma^{q+n})\left[\prod_{j=1}^{q} \log^+ M(\gamma^{q+n})\right](\gamma^n)^q$$

and by the increasing nature of $\sigma_\tau(\Phi, r)$, this holds for all τ such that $\gamma^{n-1} \leqq r \leqq \gamma^n$ so

$$\sigma_\tau(\Phi, r) \leqq C' r^q (\log^+ r)^{1+\alpha} \sigma_Y(\gamma^{q+1} r) \left[\prod_{j=1}^{q} \log^+ M(\gamma^{q+1} r) \right].$$

We now consider a slight variation which in some sense has a wider application as we do not require a plurisubharmonic function defined in a neighborhood of D. Let D be a bounded pseudoconvex domain (domain of holomorphy, cf. [7]) with $\mathscr{C}^3$ boundary. By this, we will mean that there exists a function $\tau(z)$ defined in a neighborhood N of ∂D such that $\vec{\nabla}\tau \neq 0$ near ∂D, $\tau \in \mathscr{C}^3$, and $N \cap D = \{z : \tau(z) < 0\}$. Then if

$$d(z) = \inf_{w \in \complement D} \|z - w\|,$$

$d(z)$ is a $\mathscr{C}^2$ function near ∂D and since D is a pseudoconvex domain $\varrho = -\log d(z)$ is plurisubharmonic and

$$\left| \frac{\partial^2 \varrho(z)}{\partial z_j \partial \bar{z}_k} \right| \leqq \frac{C}{d(z)^2}.$$

Thus, exactly as for Theorem 2.6, one shows the following result.

THEOREM 2.8. *Let D be a bounded domain of holomorphy in $\mathbf{C}^n$ with $\mathscr{C}^3$ boundary, and let $\tilde{D}_r = \{z : d(z) > 1/r\}$. Let Y be an analytic variety of pure dimension p and F a non-degenerate holomorphic map of Y into $\mathbf{C}^q$. Then for every $\gamma > 1$,*

$$\int_{T^q} \sigma_\tau(\Phi, r)\, d\Phi \leqq C \left[\frac{r^2}{\log \gamma} \right]^q \sigma_Y(\gamma^q r) \prod_{j=1}^{q} [\log^+ M_j(\gamma^{q-j+1} r) + \log^- \tau_j]$$

which implies that for $\gamma > 1$, $\alpha > 0$, $\varepsilon > 0$ and almost all Φ.

i) $\sigma_\tau(\Phi, r) \leqq C r^{2q+1+\alpha} \sigma_Y(\gamma^q (r + \varepsilon)) \prod_{j=1}^{q} [\log^+ M_j(\gamma^q (r + \varepsilon))]$

ii) $\sigma_\tau(\Phi, r) \leqq C^1 r^{2q} (\log^+ r)^{1+\alpha} \sigma_Y(\gamma^{q+1} r) \left[\prod_{j=1}^{q} \log^+ M_j(\gamma^{q+1} r) \right]$

for $r \geqq r_\Phi$.

In the case $D = \mathbf{C}^n$, we obtain different estimates.

THEOREM 2.9. *Let Y be an analytic set of pure dimension p in $\mathbf{C}^n$ and let F be a non-degenerate holomorphic map of Y into $\mathbf{C}^q$. Let $M_j(r) = \sup_{Y \cap B(0,r)} |f_j|$. For*

$\tau \in \mathbf{R}^q$, $\tau_j > 0$, *let* $\theta_\tau(\Phi)$ *be the positive closed current associated with the area of the analytic variety* $Y \cap F^{-1}(\tau_1 e^{i\varphi_1}, \ldots, \tau_q e^{i\varphi_q})$ *and*

$$\sigma_\tau(\Phi, r) = \int_{B(0,r)} \theta_\tau(\Phi) \wedge \beta_{p-q}.$$

Then for every $\gamma > 1$,

$$\int_{T^q} \sigma_\tau(\Phi, r)\, d\Phi \leqq \frac{1}{[(\gamma^2-1)r^2]^q} \sigma_Y(\gamma^q r) \prod_{j=1}^q [\log^+ M_j(\gamma^{q-j+1} r) + \log^- \tau_j].$$

Thus for $\alpha > 0$, $\varepsilon > 0$ *and almost every* Φ, *either*

i) $$\sigma_\tau(\Phi, r) \leqq \frac{Cr^{1+\alpha}}{[(\gamma^2-1)r^2]^q} \sigma_Y(\gamma^q(r+\varepsilon)) \prod_{j=1}^q [\log^+ M(\gamma^q(r+\varepsilon))]$$

or

ii) $$\sigma_\tau(\Phi, r) \leqq \frac{C'}{[(\gamma^2-1)r^2]^q} (\log^+ r)^{1+\alpha} \sigma_Y(\gamma^{q+1} r) \left[\prod_{j=1}^q \log^+ M_j(\gamma^{q+1} r)\right]$$

for $r \geqq r_\Phi$.

PROOF. We consider the ball of radius γr and $\tilde{Y}$ an analytic variety of pure dimension s. Then, since $\|z\|^2$ is strictly plurisubharmonic, if f is holomorphic in $\tilde{Y}$, one shows exactly as in Lemma 2.5 that

$$\begin{aligned} &\sigma_{\tilde{Y}}(\gamma r)[\log^+ M_{\tilde{Y}}(f, \gamma r) + \log^- \tau] \\ &\geqq \int_0^{2\pi} \left(\int_{B(0,\gamma r)} ((\gamma r)^2 - \|z\|^2) \theta_{\tilde{Y}}(f, \tau e^{i\varphi}) \wedge \beta_{s-1} \right) d\varphi \\ &\geqq r^2(\gamma^2 - 1) \int_0^{2\pi} \sigma_{\tilde{Y}}(f, \tau e^{i\varphi}, r)\, d\varphi. \end{aligned}$$

It is now a simple matter to apply induction as in Theorem 2.6 to arrive at the desired conclusion.

We note that in Theorem 2.6, and 2.8 and 2.9, (i) gives the better estimate for functions of finite order whereas (ii) gives the better estimate for functions of more rapid growth.

In [2], Carlson asks if it is possible to give some kind of an analytic characterization of the sets on which a Bezout estimate of the kind i) or ii) does not hold. It is clear from Theorems 2.6, 2.8, and 2.9 that the intersection of such a set with the distinguished boundary of any polydisc is a set of measure zero.

This is a property shared by $\mathbf{C}^q$-polar sets (sets on which a plurisubharmonic function takes on the value $-\infty$, cf. Lelong [10]). This leads us to conjecture that the set of exceptional values is indeed a $\mathbf{C}^q$-polar set. We show this for the case $q=1$.

THEOREM 2.10. *The set of* $\lambda \in \mathbf{C}$ *for which*

i) $$\overline{\lim_{r\to\infty}} \frac{\sigma_Y(f,\lambda,r)}{r^{2+\alpha}\sigma_Y(\gamma(r+\varepsilon))\log^+ M_Y(f,\gamma(r+\varepsilon))} = +\infty$$

or

$$\overline{\lim_{r\to\infty}} \frac{\sigma_Y(f,\lambda,r)}{(\log^+ r)^{1+\alpha}\sigma_Y(\gamma^2 r)\log^+ M_Y(f,\gamma^2 r)} = +\infty$$

in Theorem 2.6,

ii) $$\overline{\lim_{r\to\infty}} \frac{\sigma_Y(f,\lambda,r)}{r^{3+\alpha}\sigma_Y(\gamma(r+\varepsilon))\log^+ M_Y(f,\gamma(r+\varepsilon))} = +\infty$$

or

$$\overline{\lim_{r\to\infty}} \frac{\sigma_Y(f,\lambda,r)}{r^2(\log^+ r)^{1+\alpha}\sigma_Y(\gamma^2 r)\log^+ M_Y(f,\gamma^2 r)} = +\infty$$

in Theorem 2.8,

iii) $$\overline{\lim_{r\to\infty}} \frac{r^{1-\alpha}\sigma_Y(f,\lambda,r)}{\sigma_Y(\gamma(r+\varepsilon))\log^+ M_Y(f,\gamma(r+\varepsilon))} = +\infty$$

or

$$\overline{\lim_{r\to\infty}} \frac{r^2\sigma_Y(f,\lambda,r)}{(\log^+ r)^{1+\alpha}\sigma_Y(\gamma^2 r)\log^+ M_Y(f,\gamma^2 r)} = +\infty$$

in Theorem 2.9,

for every $\alpha>0$, $\varepsilon>0$, $\gamma>1$ *is an* $\mathbf{R}^2$*-polar set.*

PROOF. We shall prove only the first part of i) since the others are proved in a similar fashion.
Let

$$V_r(\lambda) = \int_{\bar{D}_r} \frac{\log|f-\lambda| i\partial\bar{\partial}\varrho \wedge \theta_Y \wedge \beta_{p-1}}{r\sigma_Y(\gamma r)\log^+ M_Y(f,\gamma r)}$$

which is a subharmonic function of λ for every r. By Lemmas 2.4 and 2.5, we have

$$1+\log^+|\lambda| \geqq V_r(\lambda)+\frac{\sigma_Y(f,\lambda,r)}{[r\sigma_Y(\gamma r)\log^+ M_Y(f,\gamma r)]}.$$

Let

$$V(\lambda) = \sum_k \frac{1}{k^{1+\alpha/2}}[V_{k\varepsilon/2}(\lambda)-k^{\alpha/4}].$$

Since for fixed λ, $V_{k\varepsilon/2}(\lambda)-k^{\alpha/4}$ is negative for k large enough, either $V(\lambda)$ defines a subharmonic function or converges to $-\infty$ uniformly on compact sets. Since by (2.4),

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f-\lambda^{ei\varphi}|\,d\varphi = \sup(\log|f|,\log|\lambda|)$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} V_r(\lambda e^{i\varphi})\,d\varphi \geqq \log|\lambda|$$

and $V(\lambda)$ cannnot converge uniformly to $-\infty$, hence $V(\lambda)$ is subharmonic. Suppose for λ there exists a sequence $r_n \to \infty$ such that

$$\frac{\sigma_Y(f,\lambda,r_n)}{r_n^{2+\alpha}\sigma_Y(\gamma(r_n+\varepsilon))\log^+ M_Y(f,\gamma(r_n+\varepsilon))} \to \infty, \qquad k_{n-1}\leqq r_n\leqq k_n .$$

Then

$$\frac{\sigma_Y(f,\lambda,k_n)}{k_n^{2+\alpha}\sigma_Y(\gamma k_n)\log^+ M_Y(f,k_n)} \to \infty \quad \text{when } n\to\infty$$

and hence $V(\lambda)=-\infty$, so for fixed $\alpha>0$, $\varepsilon>0$, $\gamma>1$, the set on which an estimate of the form i) does not hold is a polar set. We now choose sequences $\alpha_n \searrow 0$, $\varepsilon_n \searrow 0$, $\gamma_n \searrow 1$. Since a countable union of polar sets is polar, the set of λ for which we do not have an estimate of the form i) for every $\alpha>0$, $\varepsilon>0$, $\gamma>1$ is polar.

An $\mathbf{R}^2$-polar set is also a set of logarithmic capacity zero (cf. Landkof [8]).

3. The geometry of analytic sets.

In [11], Ronkin showed the following surprising result. Let Y be an analytic variety of co-dimension 1 in $\mathbf{C}^n$ and let

$$\Sigma = \left\{z\in\mathbf{C}^n : \frac{|z_1|}{\tau_1} = \ldots = \frac{|z_n|}{\tau_n}\right\},$$

which is a set of real dimension $(n+1)$. Then if $0 \notin Y$, the measure of $Y \cap \Sigma \cap B(0,r)$ grows asymptotically like the measure of $Y \cap B(0,r)$ (see below where these measures are made more precise). This imposes some kind of a geometric restriction on Y. The proof depends heavily on the fact that there exists an entire function f in $\mathbf{C}^n$ with Y as its zero set, and since $\log|f|$ is plurisubharmonic,

$$\int_{T^n} \log|f(\tau_1 e^{i\varphi_1},\ldots,\tau_n e^{i\varphi_n})|\, d\varphi_1,\ldots d\varphi_n$$

is an increasing function of $\tau_1,\ldots \tau_n$. If θ is a positive closed current of degree 1 in $\mathbf{C}^n$, then there exists a plurisubharmonic function V in $\mathbf{C}^n$ such that $\theta = i\partial\bar{\partial}V$ (cf. Skoda [15]), so by the same reasoning as that of Ronkin, one can conclude, for instance, — if the coefficients of θ are continuous functions, that the growth of the mass of θ in Σ is asymptotically the same as the growth of the mass of θ in $\mathbf{C}^n$. We shall generalize these results to analytic varieties of pure dimension p defined in $\mathbf{C}^n$, or more generally, to closed positive currents.

Let X be an analytic variety of pure dimension p in a pseudoconvex domain $D \subset \mathbf{C}^n$ and let

$$A = \{z \in X \mid |g_j(z)| < 1,\ j=1,\ldots,q\}$$

be compact in X. We will say that $G = (g_1,\ldots,g_q)$ is non-degenerate if for every subset $g_{j_1},\ldots,g_{j_p}$ of p elements, $\partial g_{j_1},\ldots,\partial g_{j_p}$ are linearly independent almost everywhere on X. If $\sigma = (j_1,\ldots,j_m)$ is a subset of the numbers $(1,\ldots,q)$ and $|\sigma| = \operatorname{card}\sigma$, we set

$$\tilde{G}_\sigma = \{z \in X : |g_{j_1}(z)| = \ldots = |g_{j_m}(z)|,\ j_k \in \sigma\} \quad \text{and} \quad \tilde{G}^m = \bigcup_{|\sigma|=m} \tilde{G}_\sigma .$$

We let $\tilde{G}^1 = A$.

LEMMA 3.1. *Let $G = (g_1,\ldots,g_q)$ be non-degenerate in an open set $D \subset \mathbf{C}^n$ and let*

$$\varrho = \sup_{j=1,\ldots,q} (\log|g_j|), \qquad \varrho_{kj} = \sup(\log|g_k|, \log|g_j|)$$

and $\tilde{\varrho} = \sum_{k<j} \varrho_{kj}$, which are plurisubharmonic functions in D.

Let

$$U = \{z \in D : g_j(z) = 0,\ j=1,\ldots,q\} .$$

If θ is a closed positive current of degree $(n-1)$ with $\mathscr{C}^\infty$ coefficients and $U \cap \operatorname{supp}\theta = \{\varnothing\}$ then

$$i\partial\bar{\partial}\tilde{\varrho} \wedge \theta \geqq i\partial\bar{\partial}\varrho \wedge \theta$$

as a measure on D.

PROOF. Let $\omega \in \mathscr{C}_0^\infty(D)$ and let

$$X_{jkl} = \{z \in D : |g_j(z)| = |g_k(z)| = |g_l(z)|\} .$$

Then

$$X = \bigcup_{\substack{j,k,l \\ j<k<l}} X_{jkl}$$

is a set of finite Hausdorff $(2n-2)$ dimensional measure since G is non-degenerate. Thus, given $\varepsilon > 0$, there exists a finite number of balls $B_i = B(z_i, t_i)$ which cover $\operatorname{supp}\theta \cap \operatorname{supp}\omega \cap X$ such that $\sum_i t_i^{2n-2} \leqq C$ and $\sup t_i \leqq \varepsilon$. Furthermore, since $U \cap \operatorname{supp}\theta = \{\emptyset\}$, we can suppose that $d(B_i, U) \geqq C > 0$ independent of i and ε. Let $\eta(z) \in \mathscr{C}_0^\infty(B(0,2))$ such that $\eta \equiv 1$ on $B(0,1)$, $0 \leqq \eta(z) \leqq 1$ and set

$$\eta_i(z) = \eta\left(\frac{z-z_i}{t_i}\right), \qquad \beta_\varepsilon(z) = 1 - \prod_i (1-\eta_i(z)) .$$

Suppose $\omega \geqq 0$. Then

$$\int_D \omega i\partial\bar\partial\tilde\varrho \wedge \theta \geqq \int_D (1-\beta_\varepsilon)\omega i\partial\bar\partial\tilde\varrho \wedge \theta \geqq \int_D (1-\beta_\varepsilon)\omega i\partial\bar\partial\varrho \wedge \theta$$

for in the support of $(1-\beta_\varepsilon)$, either $i\partial\bar\partial\varrho = i\partial\bar\partial\tilde\varrho$ or ϱ is pluriharmonic. But

$$\int \beta_\varepsilon \omega i\partial\bar\partial\varrho \wedge \theta = -\int \partial(\beta_\varepsilon\omega) \wedge i\bar\partial\varrho \wedge \theta ,$$

and since ϱ is a Lipschitz continuous function with exponent 1 outside a fixed neighborhood of U, $\bar\partial\varrho$ as a distribution is a bounded function on $\operatorname{supp}\theta$ so

$$\int_D |\partial(\beta_\varepsilon\omega) \wedge i\bar\partial\varrho \wedge \theta| \leqq K \sum_i t_i^{2n-1} \leqq CK\varepsilon .$$

Since ε was arbitrary, this completes the proof.

LEMMA 3.2. *Let D, G, U and ϱ be as in Lemma 3.1 and let*

$$A = \{z \in D : |g_j(z)| < 1,\ j = 1, \ldots, q\}$$

with $\bar A$ compact in D. Let θ be a positive closed current of degree $(n-1)$ with $\mathscr{C}^\infty$ coefficients defined in a neighborhood of $\bar A$ such that $\operatorname{supp}\theta \cap U = \{\emptyset\}$. *Then*

$$t \int_A i\partial\bar\partial\varrho \wedge \theta \geqq -\int \varrho\theta \wedge \beta ,$$

where $t = \sup_{A \cap \operatorname{supp}\theta} \|z\|^2$.

PROOF. Let $\gamma = d(U, \operatorname{supp}\theta)$. We form

$$\varrho_\varepsilon = \varrho * \eta_\varepsilon + \varepsilon\|z\|^2 ,$$

which is strictly plurisubharmonic in a neighborhood of A. By Sard's Theorem [14], the set of ξ for which $\vec{\nabla}\varrho_\varepsilon = 0$ on $\{z : \varrho_\varepsilon(z) = \xi\}$ is of measure 0. Let

$$D_\varepsilon^\xi = \{z : \varrho_\varepsilon < \xi\} .$$

Suppose $\delta > 0$ is given. Then for ξ not in the exceptional set and ε sufficiently small, by Stokes' Theorem

$$(t+\delta)\int_{D_\varepsilon^\xi} i\partial\bar{\partial}\varrho_\varepsilon \wedge \theta = (t+\delta)\int_{\partial D_\varepsilon^\xi} i\bar{\partial}\varrho_\varepsilon \wedge \theta$$

and by Lemmas 2.2 and 2.3

$$(t+\delta)\int_{\partial D_\varepsilon^\xi} i\bar{\partial}\varrho_\varepsilon \wedge \theta \geqq \int_{\partial D_\varepsilon^\xi} \|z\|^2 i\bar{\partial}\varrho_\varepsilon \wedge \theta$$

$$\geqq \int_{D_\varepsilon^\xi} \|z\|^2 i\partial\bar{\partial}\varrho_\varepsilon \wedge \theta - \int_{D_\varepsilon^\xi} \varrho_\varepsilon \theta \wedge \beta$$

so if we let $\xi \to 0$, $\varepsilon \to 0$, then $\delta \to 0$, the Lemma is proved.

If X is an analytic variety contained in a pseudoconvex domain D in $\mathbf{C}^n$ and g is holomorphic on X, then g is the restriction to X of a function holomorphic in D (cf. Gunning and Rossi [7, p. 245]), so we can assume without loss of generality that g is defined in D. If Y is a subvariety of X of dimension s, then Y is a subvariety of $\mathbf{C}^n$ of dimension s. Let

$$Y_\sigma(\varphi_1, \ldots, \varphi_{m-1}) = \{z : g_{j_1} - g_{j_m} e^{i\varphi_1} = \ldots = g_{j_{m-1}} - g_{j_m} e^{i\varphi_{m-1}} = 0,\ j_k \in \sigma\} .$$

Let $\theta_\sigma(\Phi)$ be the positive closed current of degree $(n-s+m-1)$ associated with the area of the analytic variety $Y \cap Y_\sigma(\Phi)$. Let

$$A_r = \{z \in D : |g_j(z)| < r,\ j = 1, \ldots, q\} \quad \text{for } r \leqq 1 ,$$

and set

$$n_\sigma^r(\Phi) = \int_{A_r} \theta_\sigma(\Phi) \wedge \beta_{(s-m+1)}$$

(with the convention that this is zero if $Y \cap Y_\sigma(\Phi)$ is not of dimension $(s-m+1)$) and

$$n_\sigma(r) = \int_{T^{m-1}} n_\sigma^r(\Phi)\, d\Phi .$$

THEOREM 3.3. *Let X be an analytic variety contained in a pseudoconvex domain D in $\mathbf{C}^n$ and let $G=(g_1,\ldots,g_q)$ be a non-degenerate map of D into $\mathbf{C}^q$ such that*

$$A = \{z \in X : |g_j(z)|<1,\ j=1,\ldots,q\}$$

is relatively compact in D.

Suppose that Y is a subvariety of X of pure dimension s such that

$$Y \cap U = \{z \in X : g_1(z)=\ldots=g_q(z)=0\}$$

is empty. Let θ be the positive closed current associated with the area of Y. Then for every $\gamma>1$, there exists a constant c (depending only on γ) such that

$$\int_{A_r} \theta \wedge \beta_s \leqq (ct)^s \sum_{\substack{\sigma \\ |\sigma|=s+1}} n_\sigma(r')$$

where $r'=r+\gamma^{-1}(1-r)$ and $t=\sup_{A\cap Y}\|z\|^2$.

PROOF. We shall proceed by induction on s. For $s=0$, the result is trivial since $\tilde{G}^1=A$. It is sufficient to treat the case Y irreducible since every variety can be decomposed into a countable union of irreducible branches.

Suppose first that $Y\subset Z=\{z : g_j(z)-g_k(z)e^{i\varphi}=0\}$ for some j,k and φ. Then we can replace G by $G'=(g'_1,\ldots,g'_{q-1})$ where we omit g_j, and X by $Z\cap X$. Thus, we can assume that this does not occur.

Since X is an analytic variety in $\mathbf{C}^n$, there exist $(n+1)$ functions $F=(f_1,\ldots,f_{n+1})$ such that X is just the set of common zeros of the f_j (cf. Skoda [16]). Then the set

$$\tilde{A} = \left\{z \in D : |g_j|<1,\ \frac{|f_k|}{\delta}<1\right\}$$

is relatively compact in D for δ sufficiently small. If we set

$$\tilde{U} = \{z : g_1=\ldots=g_j=f_1=\ldots=f_{n+1}=0\},$$

then by hypothesis $Y\cap\tilde{U}$ is empty.

Let θ^ε be the regularization of θ, which for ε small enough, gives a positive closed current of degree $(n-s)$ in a neighborhood of $\tilde{A}_r$, and let $\psi^\varepsilon=\theta^\varepsilon\wedge\beta_{s-1}$.

Let

$$\varrho = \sup_{j,k}\left(\log\frac{|g_j|}{r},\ \log\frac{|f_k|}{\delta r}\right).$$

Then by Lemma 3.2, for ε sufficiently small,

$$t\int_{\tilde{A}_r} i\partial\bar{\partial}\varrho \wedge \psi^\varepsilon \geqq -\int_{\tilde{A}_r} \varrho\psi^\varepsilon \wedge \beta .$$

Since $\operatorname{supp}\partial\bar{\partial}\varrho \subset A_1 \cup A_2 \cup A_3$, where

$$A_1 = \bigcup_{j,k}\left\{z : \frac{|g_j|}{r} = \frac{|g_k|}{r} > \sup\left(\frac{|g_l|}{r}, \frac{|f_m|}{\delta r}\right),\ l \neq j,\ l \neq k\right\}$$

$$A_2 = \bigcup_{j,k}\left\{z : \frac{|f_k|}{\delta r} = \frac{|f_j|}{\delta r} > \sup\left(\frac{|g_m|}{r}, \frac{|f_l|}{\delta r}\right),\ l \neq j,\ l \neq k\right\}$$

$$A_3 = \bigcup_{j,k}\left\{z : \frac{|g_j|}{r} = \frac{|f_k|}{\delta r} > \sup\left(\frac{|g_m|}{r}, \frac{|f_l|}{\delta r}\right),\ m \neq j,\ l \neq k\right\}$$

$\operatorname{supp}\psi \cap (A_2 \cup A_3) = \{\varnothing\}$ since $Y \subset \{z : f_j(z) = 0,\ j = 1, \ldots, n+1\}$. Thus, for ε sufficiently small,

$$\operatorname{supp}\psi^\varepsilon \cap (A_3 \cup A_2) = \{\varnothing\}$$

so if $\varrho' = \sup_j \log|g_j|/r$,

$$t\int_{\tilde{A}_r} i\partial\bar{\partial}\varrho' \wedge \psi^\varepsilon \geqq -\int_{\tilde{A}_r} \varrho\psi^\varepsilon \wedge \beta .$$

Let

$$D_\xi = \{z \in D : |g_j| > \xi,\ j = 1, \ldots, q\} .$$

Then D_ξ is again a pseudoconvex domain and for ξ sufficiently small, $\operatorname{supp} i\partial\bar{\partial}\varrho' \wedge \psi^\varepsilon \subset D_\xi$ for all $\varepsilon < \varepsilon_0$ for some ε_0. Thus

$$t\int_{\tilde{A}_r \cap D_\xi} i\partial\bar{\partial}\varrho' \wedge \psi^\varepsilon \geqq -\int_{\tilde{A}_r} \varrho\psi^\varepsilon \wedge \beta$$

so by Lemma 3.1, if $\varrho_{jk} = \sup(\log|g_j|, \log|g_k|)$

$$t\sum_{j,k}\int_{\tilde{A}_r \cap D_\xi} i\partial\bar{\partial}\varrho_{jk} \wedge \psi^\varepsilon \geqq -\int_{\tilde{A}_r} \varrho\psi^\varepsilon \wedge \beta .$$

By (2.3),

$$i\partial\bar{\partial}\varrho_{jk} = \frac{1}{2\pi}\int_0^{2\pi} i\partial\bar{\partial}\log|g_j - g_k e^{i\varphi}|\, d\varphi$$

so

$$t\sum_{j,k}\int_0^{2\pi}\left(\int_{\tilde{A}_r \cap D_\xi} \frac{1}{2\pi} i\partial\bar{\partial}\log|g_j - g_k e^{i\varphi}| \wedge \theta^\varepsilon \wedge \beta_{s-1}\right)d\varphi$$
$$\geqq -\int_{\tilde{A}_r} \varrho\theta^\varepsilon \wedge \beta_s .$$

Clearly we can find a strictly pseudoconvex domain $\tilde{D}$ such that $\tilde{A}_r \cap D_\xi \cap X \subset \tilde{D} \cap X \subset A$ and since

$$\sup(\log|g_j|, \log|g_k|) > \xi > 0$$

in D_ξ, we can apply the reasoning of Lemma 2.5 and let $\varepsilon \to 0$ to obtain

$$t \sum_{j,k} \int_0^{2\pi} \left(\int_{\tilde{A}_r \cap D_\xi} \frac{1}{2\pi} i\partial\bar{\partial} \log|g_j - g_k e^{i\varphi}| \wedge \theta \wedge \beta_{s-1} \right) d\varphi$$

$$\geqq -\int_{\tilde{A}_r} \varrho\theta \wedge \beta_s$$

so clearly

$$t \sum_{j,k} \int_0^{2\pi} \left(\int_{\tilde{A}_r} \frac{1}{2\pi} \partial\bar{\partial} \log|g_j - g_k e^{i\varphi}| \wedge \theta \wedge \beta_{s-1} \right) d\varphi \geqq -\int_{\tilde{A}_r} \varrho\theta \wedge \beta_s .$$

We replace r by $\eta^{-1/s} r$ and obtain by the induction hypothesis

$$\int_{\tilde{A}_r} \theta \wedge \beta_s \leqq (ct)^s \sum_{|\sigma| = s+1} n_\sigma(r) .$$

COROLLARY 3.4. *Suppose that D, X and G are as in Theorem* 3.3 *and that Y is a subvariety of X of pure dimension s such that* $d(Y, \tilde{G}^{s+1} \cap \partial A) > 0$. *Then* $\int_A \theta \wedge \beta_s < +\infty$ *where* θ *is the positive closed current associated with the area of Y, and Y consists of only a finite number of irreducible branches.*

PROOF. We choose a value $b \in C^q$ such that $Y \cap G^{-1}(b)$ is empty. This is possible if we choose b close to the distinguished boundary of the polydisc in C^q. We let

$$G' = (g'_1, \ldots, g'_q) \quad \text{where } g'_j = \frac{g_j - b_j}{1 - g_j \bar{b}_j} ;$$

we can find b for which G' remains non-degenerate. Then if $d(Y, \tilde{G}^{s+1} \cap \partial A) > 0$,

$$d(Y, \tilde{G}'^{(s+1)} \cap \partial A) > 0$$

also, so Theorem 3.3 implies that the area of Y is finite in A.

Suppose Y consists of an infinite number of branches. Let $R = \max r$ such that $Y \cap A_r \cap \tilde{G}^{s+1}$ is non-empty. Then $R < 1$ and there are at most a finite number of branches Y_j of Y which intersect A_R. Every branch of Y must intersect A_R, so Y has only a finite number of branches.

We refer the reader to Rudin [12] for a non-trivial example of an analytic variety which satisfies the hypotheses of Corollary 3.4.

Rudin [13] has shown that an analytic variety Y of pure dimension p is algebraic (i.e. defined by polynomials) if and only if after a non-singular linear change of variables, we have $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_n)$ and constants A and B such that for every $z \in Y$

$$\|z''\| \leqq A(1 + \|z'\|)^B \tag{3.1}$$

since this property is invariant with respect to translations, the sufficiency follows from the following more general result.

COROLLARY 3.5. *Suppose that Y is an analytic variety of pure dimension p such that after a non-singular linear change of variables, $0 \notin Y$ and for positive integers m_j, $j=1,\ldots,n$ and $z \in Y$, there exist $j \leqq p$ and $k \geqq p+1$ such that $|z_j|^{m_j} \neq |z_k|^{m_k}$ for $\|z\|$ sufficiently large. Then Y is an algebraic variety.*

PROOF. The mapping F: $(z_1, \ldots, z_n) \to (z_1^{m_1}, \ldots, z_n^{m_n})$ is a proper mapping so $\tilde{Y} = F(Y)$ is a variety of pure dimension p in $\mathbf{C}^n$ by the proper mapping theorem. Let w be the variable in the image space. Then by Theorem 3.3, if $\tilde{\theta}$ is the current of integration on Y,

$$\int_{B(0,r)} \tilde{\theta} \wedge \beta_p \leqq Cr^{2p}$$

since for $\|w\|$ large, $\tilde{Y} \cap \tilde{G}^{p+1}$ is empty, $\tilde{G} = \{w_1, \ldots, w_n\}$. Then $\int \tilde{\theta} \wedge \alpha_p \leqq C$ (cf. Lelong [9, p. 73]) and so Y is an algebraic variety by Stoll [17]; thus, there exist polynomials $P_1, \ldots, P_{n+1}$ such that $w \in \tilde{Y}$ if and only if $P_j(w) = 0$. But then Y is contained in an algebraic variety $P_j(F(z)) = 0$ and since Y has at most a finite number of irreducible branches (since $\tilde{Y}$ has at most a finite number and F is proper), if θ is the current associated with the area of Y, $\int \theta \wedge \alpha_p < \infty$ and hence Y is also algebraic.

Theorem 3.3 is in reality a statement about positive closed currents which will hold whenever the coefficients are sufficiently regular. Thus, we reformulate it under a slightly different form. Suppose θ is a closed positive current of degree $n-p$ and that θ has continuous coefficients. If $\tilde{\theta}_{Y_\sigma}(\varphi)$ is the positive closed current of degree $|\sigma| - 1$ associated with $Y_\sigma(\varphi)$, we let

$$\theta_\sigma(\varphi) = \tilde{\theta}_{Y_\sigma}(\varphi) \wedge \theta\,,$$

$$n_\sigma^r(\varphi) = \int_{A_r} \theta_\sigma(\varphi) \wedge \beta_{p-m+1}, \quad n_\sigma(r) = \int_{T^{m-1}} n_\sigma^r(\varphi)\, d\varphi\,.$$

THEOREM 3.5. *Let $D \subset \mathbf{C}^n$ be pseudoconvex and $A = \{z \in D : |g_j(z)| < 1, j = 1, \ldots q\}$ compact in D for a non-degenerate mapping $G = (g_1, \ldots, g_q)$. Then if θ is a positive closed current of degree $(n-s)$ with continuous coefficients, for every $\eta > 0$, there exists a constant C such that if $t = \sup_{A_r} \|z\|^2$,*

$$\int_{A_r} \theta \wedge \beta_s \leqq (Ct)^s \sum_{|\sigma| = s+1} n_\sigma(r')$$

for $r' = r + \eta^{-1}(1-r)$.

COROLLARY 3.6. *Let D, G and A be as above. Then if θ is a positive closed current of degree $(n-s)$, $\operatorname{supp} \theta \cap \tilde{G}^{s+1}$ is non-empty in A.*

PROOF. We choose $r < 1$ so large that $\operatorname{supp} \theta \cap A_r \neq \{\emptyset\}$. Then θ^ε for ε sufficiently small is $\mathscr{C}^\infty$ in A_r so by Theorem 3.5, $\operatorname{supp} \theta^\varepsilon \cap \tilde{G}^{s+1} \neq \{\emptyset\}$ and thus $d(\operatorname{supp} \theta, \tilde{G}^{s+1}) < \varepsilon$. Since this is true for all $\varepsilon > 0$, the corollary is proved.

Let Y be an analytic variety of pure dimension s in $\mathbf{C}^n$ and let

$$G = \left(\frac{z_1 - a_1}{\tau_1}, \ldots, \frac{z_n - a_n}{\tau_n}\right), \qquad \tau_j > 0 .$$

For this case, we will denote $\tilde{G}^m$ by $\Gamma^m(a, \tau)$. If $a \notin Y$, then Theorem 3.3 gives a lower estimate for the area of $Y \cap \Gamma^{(s+1)}(a, \tau)$ in terms of the area of Y in $\mathbf{C}^n$. We will now obtain an upper estimate for $Y \cap \Gamma^{(s+1)}(a, \tau)$. Let $T_\sigma(a, \tau, \Phi)$ be the linear subspace of all z such that

$$z_{j_k} - z_{j_m} e^{i\varphi_k} = 0, \qquad k = 1, \ldots, m-1, \ j_k, j_m \in \sigma .$$

If $|\sigma| = s+1$, we let $n_\sigma^r(a, \tau, \Phi)$ be the number of points (counted with multiplicity) in $Y \cap \Gamma_\sigma(a, \tau, \Phi) \cap B(0, r)$ (perhaps infinite) and set

$$n_\sigma(a, \tau, r) = \int_{T^s} n_\sigma^r(a, \tau, \Phi)\, d\Phi, \qquad N_\sigma(a, \tau, r) = \int_0^r \frac{n_\sigma(a, \tau, t)}{t} dt .$$

If θ is the positive closed current associated with the area of Y, we set

$$n_Y(r) = \int_{B(0, r)} \theta \wedge \alpha^s \quad \text{and} \quad N_Y(r) = \int_0^r \frac{n_Y(t)}{t} dt .$$

THEOREM 3.7. *Let Y be an analytic variety of pure dimension s in $\mathbf{C}^n$. Then there exist constant c_1, c_2, k_1, and k_2 depending only on τ such that if $a \notin Y$*

$$c_1 \sum_{\substack{\sigma \\ |\sigma| = s+1}} N_\sigma(a, \tau, k_1 r) \leqq N_Y(r) \leqq c_2 \sum_{\substack{\sigma \\ |\sigma| = s+1}} N_\sigma(a, \tau, k_2 r) .$$

PROOF. We assume without loss of generality that $a=0$ and $\tau_j=1$ for all j. We note that the right hand inequality follows immediately from Theorem 3.3 since $B(0,r)$ is contained in the polydisc $D_r=\{z : |z|_j|<r\}$ and $\sup_{D_r} \|z\|^2=nr^2$. Thus, we show only the left hand inequality.

Before embarking on the proof, we note that it is easy to see that the set of Φ for which $n_\sigma^r(\Phi)$ is not a finite set of points is of measure zero. This is easily established from the fact that if $\tilde{Y}$ is an analytic variety of pure dimension q with irreducible branches $\tilde{Y}_j$, then $\{z : z_l - z_k e^{i\psi}\} \cap \tilde{Y}_j$ is either $\tilde{Y}_j$ or a variety of pure dimension $(q-1)$ (cf. Gunning and Rossi [7]), so there are at most a countable number of ψ for which $\tilde{Y}_j \subset z_l - z_k e^{i\psi}$ for some j. Thus, by induction, there exists a countable number of sets Ω_j each of measure zero such that $Y \cap \Gamma_\sigma(a,\tau,\Phi)$ is zero dimensional for $\Phi \notin \bigcup_j \Omega_j$.

Let θ^{ε_0} be the regularization of the current θ. Set

$$\varrho_1 = \sup_{j>2}\left[\log\frac{|z_1|}{r}, \log\frac{|z_2|}{r}, \log\frac{\mu|z_j|}{r}\right] \quad \text{where } \tfrac{1}{2}<\mu<1\ ,$$

$$\varrho_j = \sup\left(\log\frac{|z_j|}{r}, \log\frac{|z_{j+1}|}{r}\right), \quad j=2,\ldots,s\ ,$$

$$\varrho_j^{\varepsilon_j} = \varrho_j * \eta_{\varepsilon_j} \quad \text{and} \quad \tilde{\varrho}_j^{\varepsilon_j} = \exp \varrho_j^{\varepsilon_j}\ .$$

Let

$$\psi_j^\varepsilon = \theta^{\varepsilon_0} \wedge i\partial\bar{\partial}\varrho_1^{\varepsilon_1} \wedge \ldots \wedge i\partial\bar{\partial}\varrho_j^{\varepsilon_j}, \qquad \varepsilon = (\varepsilon_0,\ldots,\varepsilon_j)$$

and $\tau = \|z\|^2 - r^2$.

On $\partial B(0,r)$, $\varrho_j \leqq 0$ so for small ε_j, $\tilde{\varrho}_j^{\varepsilon_j} \leqq 2$. Hence by Lemmas 2.2 and 2.3,

$$0 > \int_{\partial B(0,r)} (\tilde{\varrho}_j^{\varepsilon_j}-2) i\bar{\partial}\tau \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)} \geqq \int_{B(0,r)} (\tilde{\varrho}_j^{\varepsilon_j}-2)\psi_{j-1}^\varepsilon \wedge$$
$$\wedge\ \beta^{(s-j+1)} + \int_{B(0,r)} (r^2-\|z\|^2) i\partial\bar{\partial}\tilde{\varrho}_j^{\varepsilon_j} \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)}$$

and

$$i\partial\bar{\partial}\tilde{\varrho}_j^{\varepsilon_j} \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)} = i\partial\bar{\partial}(\exp \varrho_j^{\varepsilon_j}) \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)}$$
$$= \exp \varrho_j^{\varepsilon_j}[i\partial\varrho_j^{\varepsilon_j} \wedge \bar{\partial}\varrho_j^{\varepsilon_j} \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)} + i\partial\bar{\partial}\varrho_j^{\varepsilon_j} \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)}]$$
$$\geqq \exp \varrho_j^{\varepsilon_j} i\partial\bar{\partial}\varrho_j^{\varepsilon_j} \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)}.$$

Thus

$$2\int_{B(0,r)} \psi_{j-1}^\varepsilon \wedge \beta^{(s-j+1)} \geqq$$
$$\geqq \int_{B(0,r)} (r^2-\|z\|^2)\exp \varrho_j^{\varepsilon_j} i\partial\bar{\partial}\varrho_j^{\varepsilon_j} \wedge \psi_{j-1}^\varepsilon \wedge \beta^{(s-j)}\ .$$

Let $k<1$ be given. Since $0\notin \operatorname{supp}\theta^{\varepsilon_0}$ for ε_0 small enough and since

$$\operatorname{supp} i\partial\bar{\partial}\varrho_1 \cap \operatorname{supp}\theta \cap \{z\ :\ \varrho_j(z)=-\infty\}$$

is empty for ε small enough,

$$\exp\varrho_j^{\varepsilon_j}\to\exp\varrho_j\ =\ \sup\left(\frac{|z_j|}{r},\frac{|z_{j+1}|}{r}\right)$$

uniformly on $\operatorname{supp}\psi_{j-1}^{\varepsilon}$ when $\varepsilon_j\to 0$. Thus, for ε_j small enough.

,

$$\int_{B(0,r)}\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j+1)}$$

$$\geqq\frac{k}{2}\int_{B(0,r)}(r^2-\|z\|^2)\sup\left(\frac{|z_j|}{r},\frac{|z_{j+1}|}{r}\right)i\partial\bar{\partial}\varrho_j^{\varepsilon_j}\wedge\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j)}\,.$$

Furthermore, on $\operatorname{supp} i\partial\bar{\partial}\varrho_1$, $\|z\|/2\leqq\sup(|z_j|,|z_{j+1}|)\leqq\|z\|$, so for ε small enough

$$\int_{B(0,r)}\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j+1)}\geqq C_k r^2\left[\int_{B(0,kr)}i\partial\bar{\partial}\varrho_j^{\varepsilon_j}\wedge\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j)}-\right.$$
$$\left.-\int_{B(0,k^2r)}i\partial\bar{\partial}\varrho_j^{\varepsilon_j}\wedge\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j)}\right].$$

By Lelong [9, p. 73], $r^{-2(s-j)}\int_{B(0,r)}i\partial\bar{\partial}\varrho_j^{\varepsilon_j}\wedge\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j)}$ is an increasing function of r so for $j<s$,

$$\int_{B(0,r)}\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j+1)}\geqq C_k' r^2\int_{B(0,kr)}i\partial\bar{\partial}\varrho_j^{\varepsilon_j}\wedge\psi_{j-1}^{\varepsilon}\wedge\beta^{(s-j)}\,.$$

Thus, by iterating this result, we obtain for $j<s$

$$\int_{B(0,r)}\theta^{\varepsilon_0}\wedge\beta^s\geqq r^{2(s-j)}C_k''\int_{B(0,k^jr)}\psi_j^{\varepsilon}\wedge\beta^{(s-j)}$$

for ε small enough, or alternatively, using Lelong [9, Proposition 10, p. 73],

$$\int_{B(0,r)}\theta^{\varepsilon_0}\wedge\alpha^s\geqq C_k'''\int_{B(0,k^{s-1}r)}\psi_{s-1}^{\varepsilon}\wedge\alpha\,. \tag{3.1}$$

In order to treat the case $j=s$, we will need a slightly different technique. Let R be fixed and let $\tilde{\tau}=\log\|z\|/R$. Then by Lemmas 2.2 and 2.3

$$0>\int_{B(0,R)}\varrho_s^{\varepsilon_s}i\partial\tilde{\tau}\wedge\psi_{s-1}^{\varepsilon}=\int_{B(0,R)}\varrho_s^{\varepsilon_s}\psi_{s-1}^{\varepsilon}\wedge\alpha-\int\tilde{\tau}i\partial\bar{\partial}\varrho_s^{\varepsilon_s}\wedge\psi_{s-1}^{\varepsilon}$$

so if we let $\varepsilon_s\to 0$ we obtain

$$-\int_{B(0,R)} \varrho_s \psi^\varepsilon_{s-1} \wedge \alpha \geqq -\int_{B(0,R)} \log\frac{\|z\|}{R} i\partial\bar\partial\varrho_s \wedge \psi^\varepsilon_{s-1} .$$

Let $\delta > 0$ be given. Then for μ large enough, $\mu < 1$ and ε small enough

$$-\varrho_s \leqq -\log\frac{\|z\|}{R} + \delta \quad \text{on } \operatorname{supp}\psi^\varepsilon_{s-1}$$

so

$$\delta\int_{B(0,R)} \psi^\varepsilon_{s-1} \wedge \alpha - \int_{B(0,R)} \log\frac{\|z\|}{R} \psi^\varepsilon_{s-1} \wedge \alpha$$
$$\geqq -\int_{B(0,R)} \log\frac{\|z\|}{R} i\partial\bar\partial\varrho_s \wedge \psi^\varepsilon_{s-1} .$$

Let

$$A_1(t) = \int_{B(0,t)} \psi^\varepsilon_{s-1} \wedge \alpha \quad \text{and} \quad A_2(t) = \int_{B(0,t)} i\partial\bar\partial\varrho_s \wedge \psi^\varepsilon_{s-1} .$$

Then, since $A_1(0) = A_2(0) = 0$, if we integrate the above expression by parts, we obtain

$$\delta\int_{B(0,R)} \psi^\varepsilon_{s-1} \wedge \alpha + \int_0^R \frac{A_1(t)}{t} dt \geqq \int_0^R \frac{A_2(t)}{t} dt .$$

Combining this with (3.1), we have

$$(3.2) \quad \delta\int_{B(0,r)} \theta^{\varepsilon_0} \wedge \alpha^s + \int_0^r \left(\int_{B(0,t)} \theta^{\varepsilon_0} \wedge \alpha^s\right)\frac{dt}{t}$$
$$\geqq C_k^{(iv)} \int_0^{k^{s-1}r} \left(\int_{B(0,t)} i\partial\bar\partial\varrho_s \wedge \psi^\varepsilon_{s-1}\right)\frac{dt}{t} .$$

If we let

$$\bar\varrho_1 = \sup\left[\log\frac{|z_1|}{r}, \log\frac{|z_2|}{r}\right],$$

then for $\sup\varepsilon_j$ small enough, $\bar\varrho_1 = \varrho_1$ in $\bigcap_{j=2}^s \operatorname{supp}\varrho_j^{\varepsilon_j}$ so by (2.4)

$$\int_{B(0,t)} i\partial\bar\partial\varrho_s \wedge \psi^\varepsilon_{s-1} = \int_{B(0,t)} \left(\int_{T^s} i\partial\bar\partial\frac{1}{2\pi}\log|z_s - z_{s-1}e^{i\varphi_s}|\right.$$
$$\left.\bigwedge_{j=1}^{s-1} i\partial\bar\partial\frac{1}{2\pi}\log|z_j - z_{s+1}e^{i\varphi_j}| * \eta_{\varepsilon_j} \wedge \theta^{\varepsilon_0}\, d\Phi\right).$$

We now let $\varepsilon_0 \to 0$ in (3.2) to obtain

$$C(r)\delta + N_Y(r) \geqq C_k^{(iv)} \int_0^{k^{s-1}r} \frac{dt}{t} \int_{B(0,t)} \left[\int_{T^s} \left(i\partial\bar{\partial}\frac{1}{2\pi}\log|z_s - z_{s+1}e^{i\varphi_s}| \bigwedge_{j=1}^{s-1} i\partial\bar{\partial}\frac{1}{2\pi}\log|z_j - z_{s+1}e^{i\varphi_j}| * \eta_{\varepsilon_j} \wedge \theta \right) d\Phi \right].$$

We now let successively $\varepsilon_j \to 0$, $j=1,2,\ldots,s-1$. Since $\log|z_j|$ is bounded below on $\operatorname{supp}\psi_{s-1}^{\varepsilon}$ for ε sufficiently small, we can repeatedly apply the reasoning of Lemma 2.5 and Fatou's Lemma to pass to the limit under the integral and obtain

$$C(r)\delta + N_Y(r) \geqq N_\sigma(k^{s-1}r).$$

Since δ was arbitrary, the result now follows.

Cornalba and Shiffman [3] have constructed an example of a variety in $\mathbf{C}^3$ such that the intersection with a certain 1-dimensional subspace can be made to grow arbitrarily fast even though its total area is of finite order. Carlson [2] showed that if Y is an analytic variety of pure dimension s and θ is the associated positive closed current of area, then for almost all linear subspaces A of dimension s $\int_{B(0,r)\cap A}\theta_Y$ does not grow more rapidly than $\int_{B(0,r)}\theta \wedge \alpha^s$ (he actually uses the integrated areas). The following result gives some insight into the nature of the exceptional set.

THEOREM 3.8. *Let Y be an analytic variety of pure dimension s in $\mathbf{C}^n$ and suppose $a \notin Y$. Then given $\alpha > 0$, there exist constants c and k depending only on α and τ such that for almost all $\Phi \in T^s$,*

$$N_\sigma^{kr}(a,\tau,\Phi) \leqq C(\log^+ r)^{1+\alpha} N_Y(r) \quad \text{for } r \geqq r\Phi.$$

PROOF. This follows from Theorem 3.7 as in Corollary 2.7 see also [2, p. 136].

In the preceeding presentation, we have singled out the sets $\Gamma^{(s+1)}(a,\tau)$ to measure the global growth of Y, an analytic variety of pure dimension s. These sets are in no way unique, and we sketch here an alternate approach. Suppose we have $\xi_j \in \mathbf{C}^n$, $j=1,\ldots,M$ such that not all the ξ_j lie in the same real hyperplane (hence $M \geqq 2n$). Let

$$G = (g_1,\ldots,g_M), \quad g_j = \exp\langle \xi_j, z\rangle.$$

We will say that Y is non-degenerate for G if Y is not contained in any hyperplane which passes through the origin. We let

$$A_r = \{z \in \mathbf{C}^n : \operatorname{Re}\langle \xi_j, z\rangle < r, \ j=1,\ldots,M\}$$

and we define $n_\sigma^r(\Phi)$ as above, where the integral is to be taken over A_r. The set

$$|\exp\langle \xi_j, z\rangle| = |\exp\langle \xi_k, z\rangle|$$

is just the $(2n-1)$ dimensional hyperplane $\operatorname{Re}\langle \xi_j - \xi_k, z\rangle = 0$.

One establishes just as in Theorems 3.3 and 3.7 that there exist constants c_1, c_2, k_1 and k_2 such that

$$c_1 \sum_{\substack{\sigma \\ |\sigma|=s+1}} n_\sigma(k_1, r) \leqq r^{-2s} \int_{B(0,r)} \theta_Y \wedge \beta_s \leqq c_2 \sum_{\substack{\sigma \\ |\sigma|=s+1}} n_\sigma(k_2 r)$$

for every Y non-degenerate for G. In fact, here there is no need to integrate the above inequalities since

$$\log|\exp\langle \xi_j, z\rangle| = \operatorname{Re}\langle \xi_j, z\rangle$$

is bounded in a neighborhood of the origin. We leave it to the interested reader to verify these inequalities.

NOTE. Since we completed this work, another article by Carlson [20] has appeared which sheds some new light on the problems considered here. In particular, he shows that the set for which a Bezout estimate does not hold is an $\mathbf{R}^{2p}$-polar set, that is the set on which some subharmonic function takes on the value $-\infty$. This reinforces the conjecture that it is actually $\mathbf{C}^p$-polar.

BIBLIOGRAPHY

1. H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains*, Trans. Amer. Math. Soc. 91 (1959), 246–276.
2. J. A. Carlson, *A remark on the transcendental Bezout problem*, *Value-Distribution Theory*, Part A, edited by R. O. Kujala and A. Vitter, pp. 133–142, Marcel Dekker, Inc., New York, 1974.
3. M. Cornalba and B. Shiffman, *A counterexample to the "Transcendental Bezout problem"*, Ann. of Math. 96 (1972), 402–406.
4. H. Federer, *Geometric Measure Theory*, Grundlehren Math. Wiss. 153, Springer-Verlag, Berlin - Heidelberg - New York, 1969.
5. P. Griffiths, *Function theory of finite order on algebraic varieties* I, J. Differential Geometry 6 (1972), 285–306.
6. L. Gruman, *Generalized Hardy and Nevanlinna classes*, Ark. Mat. 14 (1976), 65–78.
7. R. C. Gunning, and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice–Hall, Englewood Cliffs, N.J., 1965.

8. N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss. 180, Springer-Verlag, Berlin - Heidelberg - New York, 1972.

9. P. Lelong, *Fonctions plurisousharmoniques et formes differentielles positives*, Gordon and Breach, Paris, 1968.

10. P. Lelong, *Fonctionelles analytiques et fonctions entières* (*n variables*), Les Presses de l'Université de Montréal, 1968.

11. L. I. Ronkin, *An analog of the canocical product for entire functions of several complex variables*, Trudy Moskov. Mat. Obšč 18 (1968), 105–146 (Russian); English translation Trans. Moscow. Math. Soc. (1968), 117–160.

12. W. Rudin, *Zero-sets in polydiscs*, Bull. Amer. Math. Soc. 73 (1967), 580–583.

13. W. Rudin, *A geometric criterion for algebraic varieties*, Indiana Univ. Math. J. 17 (1968), 671–684.

14. A. Sard, *The measure of critical values of differential maps*, Bull. Amer. Math. Soc. 48 (1942), 883–890.

15. H. Skoda, *Solution à croissance du second problème de Cousin dans* $\mathbf{C}^n$, Ann. Inst. Fourier (Grenoble), 21 (1971), 11–23.

16. H. Skoda, *Sous-ensembles analytiques d'ordre fini ou infini dans* $\mathbf{C}^n$, Bull. Soc. Math. France 100 (1972), 353–408.

17. W. Stoll, *The growth of the area of a transcendental analytic set*, I and II, Math. Ann. 156 (1964), 47–78 and 144–178.

18. W. Stoll, *A Bezout estimate for complete intersections*, Ann. of Math. 96 (1972), 361–401.

19. W. Stoll, *Deficit and Bezout estimates*, Value-Distribution Theory, Part B, edited by R. O. Kujala and A. Vitter, Marcel Dekker, New York, 1974.

20. J. A. Carlson, *A moving Lemma for the transcendental Bezout problem*, Ann. of Math. 103 (1976), 305–330.

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