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THE NUMBER OF REPRESENTATIONS OF A GROUP INDUCED FROM THE IRREDUCIBLE REPRESENTATIONS OF A NORMAL SUBGROUP

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Let G be a finite group and let K be an arbitrary field of characteristic $p \geqq 0$. The well-known result of Berman and Witt [1] states that the number of irreducible representations of G over K is equal to the number of K-conjugacy classes of p'-elements in G.

The aim of this paper is to prove the following application of the mentioned result.

THEOREM. *Let H be a normal subgroup of the finite group G, and let K be an arbitrary field of characteristic $p \geqq 0$. Then the number of non-isomorphic KG-modules induced from the irreducible KH-modules is equal to the number of K-conjugacy classes of p'-elements of G which are in H.*

1. Preliminaries.

All groups in this paper are assumed to be finite.

NOTATIONS AND DEFINITIONS. K is any field of characteristic $p \geqq 0$.

A K-character is a character of a linear representation of a group G over K.

Let L be a left KH-module where H is a normal subgroup of G. For a fixed $g \in G$, let $g \otimes L = L^g$ be the left KH-module whose underlying vector space is L and on which H acts according to the rule $h * l = g^{-1}hgl$, $l \in L$, where $h * l$ denotes the module operation in L^g and hl the operation in L. Two KH-modules L_1 and L_2 are called G-conjugate if $L_2 \cong L_1^g$ for some $g \in G$. Similarly two K-characters α and β of the group H are called G-conjugate if $\beta = \alpha^g$ for some $g \in G$ where $\alpha^g(h) = \alpha(g^{-1}hg)$ for all $h \in H$.

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Let n be the least common multiple of the orders of the p'-elements in G and let ε be a primitive nth root of unity over K. Let further I_n be the multiplicative group consisting of those integers r, taken modulo n, for which $\varepsilon \to \varepsilon^r$ defines an automorphism of $K(\varepsilon)$ over K. Two p'-elements $a, b \in G$ are called K-conjugate if $x^{-1}bx = a^r$ for some $x \in G$ and some $r \in I_n$. Thus a K-conjugacy class is a union of ordinary conjugacy classes.

Lemma 1. *Let $\chi_1, \ldots, \chi_s$ be all irreducible K-characters of G. Then*

1) *$\chi_1, \ldots, \chi_s$ are linearly independent over K;*
2) *Two irreducible KG-modules are isomorphic if and only if they have the same characters.*

Proof. Straightforward by applying (30.12), (30.15) and (29.7) of [3].

Lemma 2. *Let $H \triangle G$ and let $T_h(K_h)$ be the K-conjugacy class of p'-elements of the group $G(H)$ with representative $h \in H$. Then*

$$T_h = \bigcup_{g \in G} K_{g^{-1}hg} .$$

Proof. Let m be the least common multiple of the orders of the p'-elements in H, so that $n = mk$ for some natural number k and $\delta = \varepsilon^k$ is the primitive mth root of unity over K. Suppose $s \in T_h$. Then $s = a^{-1}h^{\mu}a$ for some $a \in G$ and some $\mu \in I_n$. If $\mu \equiv r \pmod m$, $0 \leqq r \leqq m-1$, then $s = a^{-1}h^r a$. The automorphism $\varepsilon \to \varepsilon^{\mu}$ of $K(\varepsilon)$ over K induces the automorphism $\delta \to \delta^{\mu} = \delta^r$ of $K(\delta)$ over K. Hence $r \in I_m$ and $s \in K_{a^{-1}ha}$. As

$$\mathrm{Gal}\,(K(\varepsilon)/K) \geqq \mathrm{Gal}\,(K(\delta)/K)$$

it follows that if two elements of H are K-conjugate in H, then they are K-conjugate in G, from which the other inclusion follows.

Lemma 3. *Let H be a normal subgroup of G and let $L(M)$ be the irreducible KH-module with the character $\alpha(\beta)$. Then the induced KG-modules L^G and M^G are isomorphic if and only if α and β are G-conjugate.*

Proof. Let $G = g_1H + \ldots + g_sH$ be the left coset decomposition of G with respect to H. Then

$$L_H^G = g_1 \otimes L + \ldots + g_s \otimes L \quad (M_H^G = g_1 \otimes M + \ldots + g_s \otimes M)$$

is the decomposition of the KH-module L_H^G (M_H^G) into the direct sum of the irreducible submodules. If ψ: $L^G \to M^G$ is a KG-isomorphism then

$$M_H^G = \psi(g_1 \otimes L) + \ldots + \psi(g_s \otimes L)$$

is another decomposition of M_H^G into the direct sum of the irreducible submodules and it follows from the Jordan-Hölder theorem that L and M are G-conjugate. Conversely, let $\psi\colon L \to g \otimes M$ be the KH-isomorphism, $g \in G$. If

$$x = g_1 \otimes l_1 + \ldots g_s \otimes l_s \qquad (l_i \in L;\ 1 \leqq i \leqq s)$$

is an arbitrary element in L^G then

$$\psi(x) = g_1[\psi(l_1)] + \ldots + g_s[\psi(l_s)]$$

is the KG-isomorphism between L^G and M^G. Hence $L^G \cong M^G$ if and only if L and M are G-conjugate. Now apply the second part of the lemma 1. This proves the lemma.

LEMMA 4. {2, p. 934]. *Let M be a nonsingular matrix of degree m, and let A and B be two permutation groups of degree m which are both homomorphic to the same group G. If for every $g \in G$, the corresponding element a_g of A, applied to the rows of M, and the corresponding element b_g of B, applied to the columns of M, both carry M into the same matrix, then the number of systems of transitivity is the same for A and for B.*

2. Proof of the theorem.

Let $T = \{\chi_1, \ldots, \chi_t\}$ $(S = \{K_1, \ldots, K_t\})$ be the set of irreducible K-characters of H (the set of K-conjugacy classes of p'-elements in H, $l \leqq i \leqq t$). Consider the matrix $M = \|\chi_i(K_j)\|$ $(l \leqq i, j \leqq t)$. By Lemma 1 $\chi_1, \ldots, \chi_t$ are linearly independent over K and hence the matrix M is invertible. The formulae

$$A(g) = \begin{pmatrix} \chi_i \\ \chi_{i(g)} \end{pmatrix} \quad \text{and} \quad B(g) = \begin{pmatrix} K_j \\ K_{j(g)} \end{pmatrix}$$

$$(l \leqq i, j, i(g), j(g) \leqq t;\ \chi_{i(g)} = \chi_i^g,\ K_{j(g)} = g^{-1} K_j g)$$

are the representations of G by the permutations of the sets T and S respectively if the product $\alpha_1\alpha_2$ $(\beta_1\beta_2)$ of a pair of permutations in $A(G)$ $(B(G))$ is defined by $(\alpha_1\alpha_2)x = \alpha_1(\alpha_2 x)$, $x \in T$ $((\beta_1\beta_2)y = \beta_2(\beta_1 y)$, $y \in S)$. Moreover, for every $g \in G$ the permutation $A(g)$ applied to the rows of M and the permutation $B(g)$ applied to the columns of M both carry M into the same matrix $\|\chi_i^g(K_j)\|$ $(l \leqq i, j \leqq t)$. Thus the matrix M and the groups $A = A(G)$, $B = B(G)$, G satisfy the conditions of lemma 4. It follows from lemma 4 that the groups A and B have the same number of systems of transitivity. Now applying lemma 3 we obtain that the number of systems of transitivity in A is the same as the number of the nonisomorphic KG-modules induced from the irreducible

KH-modules. On the other hand it follows from lemma 2 that the number of systems of transitivity in B is the same as the number of K-conjugacy classes of p'-elements of G which are in H. This completes the proof of the theorem.

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REFERENCES

1. S. D. Berman, *The number of irreducible representations of a finite group over an arbitrary field*, Dokl. Akad. Nauk. SSSR(N.S.) 106 (1956), 767–769.
2. R. Brauer, *On the connection between the ordinary and the modular characters of groups of finite order*, Ann. of Math. 142 (1941), 926–935.
3. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience Publ., New York, London, 1962.

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