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s-RINGS AND MODULAR REPRESENTATIONS

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0. Introduction.

If $A = \oplus_{n \geqq 0} A_n$ is a graded ring and N is the set of non-negative integers, then to every element $a = (a_n) \in A$ corresponds a subset of N consisting of all n such that $a_n \neq 0$. This set may be considered as a support of a. There are natural relations between the supports of elements of A, and the supports of their sums and products. A generalization of this situation leads to a kind of graded rings over arbitrary partially ordered sets which are called here s-rings (see Definition 1). We want to apply this generalization to the theory of modular representations of finite groups. We show that if F is a field of characteristic $p > 0$ and G is a finite group, then the defect groups of conjugacy classes of G and the vertices of indecomposable (left) $F(G)$-modules can be considered as supports (in the sense of Definition 1) for elements of well known rings corresponding to the group G (see Examples 3 and 4).

Sections 1 and 2 contain some basic definitions and examples. In Section 3 we discuss the notion of normal supports corresponding to the notion of normal defect groups. In Section 4 we generalize the notion of defect groups of a block and we prove an analogue of Brauer's first main theorem on blocks (Theorem 1). The proof of Brauer's theorem, given at the end of the paper, needs only one very elementary result on groups which is not contained in the general theory of s-rings.

1. Definitions and examples.

Let $(X, \geq)$ be a partially ordered set. If A, $B \subseteqq X$, define

$$[A,B] = \{x \in X : \text{ there are } a \in A \text{ and } b \in B \text{ such that } x \geq a \text{ and } x \geq b\} .$$

Note that if $x \geqq y$ in X means $x = y$, then $[A,B] = A \cap B$.

DEFINITION 1. We say that (A, X) is an s-ring if A is an associative ring with an identity, X is a partially ordered set, and to every element $a \in A$ corresponds a subset $s(a) \subseteqq X$, called the support of a, such that

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(a) $s(a+b) \subseteqq s(a) \cup s(b)$,

(b) $s(ab) \subseteqq [s(a), s(b)]$,

(c) if $a \in A$ and $x \in s(a)$, then there are $b, c \in A$ such that

$$a = b+c, \quad s(b) = \{x\}, \quad x \notin s(c) \subseteqq s(a),$$

(d) $s(a) = \varnothing$ if and only if $a = 0$.

EXAMPLES. 1. Let $A = \bigoplus_{n \geqq 0} A_n$ be a graded ring ($A_i A_j \subseteqq A_{i+j}$), and X the set of non-negative integers ordered by the usual relation $\geqq$. If $a = (a_n) \in A$, define $s(a) = \{n \in X : a_n \neq 0\}$.

2. Let (A_i, X_i) for $i = 1, \ldots, n$ be s-rings, $A = \bigoplus A_i$ the direct sum of the rings A_i, and $X = \coprod X_i$ the disjoint sum of the sets X_i. Define $x \geq x'$ if and only if x, x' belong to the same X_i and $x \geq x'$ in X_i. If $a = (a_i) \in A$, define $s(a) = \bigcup s(a_i)$.

If, for example, (A_i, X_i) is an s-ring such that $X_i = \{x_i\}$ and $s(a_i) = x_i$ for all $a_i \in A_i$, $a_i \neq 0$ (we often write $s(a) = x$ if the support of a has only one element), then $s(a)$ for $a \in \bigoplus A_i$ can be identified with the set of indices of non-zero coordinates of a.

3. (see [2, § 3]). Let G be a finite group, H a subgroup of G, F a field of characteristic $p > 0$ and $F(G)$ the group algebra of G over F. Let

$$ZF(G{:}\ H) = \{x \in F(G) : h^{-1}xh = x \text{ for all } h \in H\}.$$

$ZF(G{:}H)$ is a subalgebra of $F(G)$ generated over F (as a linear space) by the sums of H-conjugacy classes of G, that is, the sums $\sum g$, $g \in C$, where C is a subset of G such that $g, g' \in C$ if and only if $g' = h^{-1}gh$ for some $h \in H$. Let $S(H)$ be the set of conjugacy classes of p-subgroups of H. If x_1 and x_2 are classes of D_1 and D_2, define a partial ordering of $S(H)$ by:

$$x_1 \geq x_2 \Leftrightarrow D_1 \text{ is contained in a subgroup conjugate to } D_2.$$

The element of $S(H)$ corresponding to a p-subgroup D of H will be denoted by x_D.

Let C be an H-conjugacy class of G, $c = \sum g$, $g \in C$ its sum, and D a defect group of C, that is, D is a Sylow p-subgroup of the centralizer $C_H(g)$, where g is an element of C. Since all the defect groups of C are conjugate, they define an element $x_c \in S(H)$. Let $s(c) = \{x_c\}$ and if $a = \sum_{i=1}^{k} r_i c_i$, $r_i \neq 0$, $r_i \in F$, then

$$s(a) = \{x_{c1}, \ldots, x_{ck}\}.$$

$(ZF(G{:}H), S(H))$ is an s-ring. Conditions (a), (c) and (d) are evident. To prove (b), it suffices to prove that

$$s(cc') \subseteqq [s(c), s(c')],$$

for arbitrary H-conjugacy classes C and C' of G. This inclusion follows directly from [2, Lemma 3.2a] or [1, Lemma 54.2].

REMARK. If $H=G$, then $ZF(G:H)$ is the center $ZF(G)$ of the group algebra $F(G)$ and we get the s-ring $(ZF(G), S(G))$.

For later use, let us note a fact, whose proof is trivial:

LEMMA 1. *If C is a conjugacy class of G and D a p-subgroup of G, then*

$$x_c \leq x_D \text{ if and only if } C \cap C_G(D) \neq \varnothing ,$$

where $C_G(D)$ is the centralizer of D in G.

4. (See [3]). Let G be a finite group, and F a field of characteristic $p>0$. Let $\mathcal{M}(F(G))$ be the abelian group with generators $[M]$, where M is a finitely generated left $F(G)$-module, and relations $[M]=[M']+[M'']$ if $M \cong M' \oplus M''$.

Since the Krull–Schmidt theorem is valid for fintely generated $F(G)$-modules, $\mathcal{M}(F(G))$ is a free abelian group with base $[M]$, where M represent the isomorphism classes of indecomposable $F(G)$-modules. The G-action on $M \otimes_F M'$ given by $g(m \otimes m') = gm \otimes gm'$ defines a ring structure on $\mathcal{M}(F(G))$ if

$$[M][M'] = [M \otimes_F M'] .$$

If M is an indecomposable $F(G)$-module, then a vertex of M (see [1, § 53]) is a p-subgroup D of G such that M is D-projective, and if M is H-projective for some subgroup H of G, then H contains a p-subgroup G-conjugate to D. This shows that the vertices of M, which are determined up to conjugacy in G, define an element of $S(G)$. We shall denote this element by $\mathrm{vx}(M)$. If $X \in \mathcal{M}(F(G))$, $X = \sum_{i=1}^{k} r_i [M_i]$, where r_i are non-zero integers and M_i are non-isomorphic indecomposables, define

$$s(X) = \{\mathrm{vx}(M_1), \ldots, \mathrm{vx}(M_k)\} .$$

$(\mathcal{M}(F(G)), S(G))$ is an s-ring. Conditions (a), (c) and (d) of Definition 1 are evident. Condition (b) is satisfied for arbitrary elements if

$$s([M][M']) \subseteq [s([M]), s([M'])]$$

for any indecomposable $F(G)$-modules M and M'. Let

$$M \otimes_F M' \cong \bigoplus M_i ,$$

where M_i are indecomposable $F(G)$-modules. Since $M \otimes_F M'$ is $\mathrm{vx}(M)$-projective (we mean an arbitrary representant of $\mathrm{vx}(M)$ — see [1, Lemma 60.2]), all the direct summands M_i are also $\mathrm{vx}(M)$-projective. Hence $\mathrm{vx}(M_i) \geq \mathrm{vx}(M)$. Similarly, $\mathrm{vx}(M_i) \geq \mathrm{vx}(M')$, so we get $\mathrm{vx}(M_i) \in [s([M]), s([M'])]$.

We shall prove some easy consequences of Definition 1.

Proposition 1. *If* (A, X) *is an s-ring, then*

(a) $s(a)=s(-a)$,
(b) *if* $a=b+c$ *and* $s(b)\subseteqq s(a)$, *then* $s(c)\subseteqq s(a)$,
(c) *if* $s(a)\cap s(b)=\varnothing$, *then* $s(a+b)=s(a)\cup s(b)$.

Proof. (a) If $s(a)=x$, then $x\in s(a)\subseteqq[s(-1), s(-a)]$, so there is $y\in s(-a)$ such that $x\geq y$. Since $y\in s(-a)\subseteqq[s(-1), s(a)]$, we get $y\geq x$. Hence $y=x$. Thus $s(a)=x$ implies $x\in s(-a)$. Now if $s(a)$ is arbitrary and $x\in s(a)$, then $a=a'+a_1$, where $s(a')=x$ and $x\notin s(a_1)\subseteqq s(a)$. Hence

$$s(-a') = s(-a+a_1) \subseteqq s(-a)\cup s(a_1)\,.$$

Since $x\in s(-a')$ and $x\notin s(a_1)$, so $x\in s(-a)$. Thus $s(a)\subseteqq s(-a)$.
(b) and (c) are trivial.

Proposition 2. *If* (A, X) *is an s-ring and* $s(a)=\{x_1,\ldots,x_r\}$, *then* $a=a_1+\ldots+a_r$, *where* $s(a_i)=x_i$ *and* a_i *are unique.*

Proof. *Existence.* We have $a=a_1+a'$, where $s(a_1)=x_1$ and $x_1\notin s(a')\subseteqq s(a)$. By Proposition 1 (c), $s(a')=\{x_2,\ldots,x_r\}$, so we can proceed by induction.
Uniqueness. Let $a=a'+a_1=a''+a_2$, where $s(a')=s(a'')=x$ and $x\notin s(a_i)\subseteqq s(a)$ for $i=1,2$. We have

$$s(a'-a'') \subseteqq s(a')\cup s(a'') = \{x\}\,.$$

But $s(a'-a'')=s(a_2-a_1)\subseteqq s(a_1)\cup s(a_2)$, so $x\notin s(a_i)$ gives $s(a'-a'')=\varnothing$, that is, $a'=a''$.

2. Homomorphisms of s-rings.

Let (A, X) be an s-ring and $x_0\in X$. Define:

$$A_{x0} = \{a\in A : x\in s(a) \Rightarrow x\leq x_0\}\,,$$
$$A^0_{x0} = \{a\in A : s(a)\subseteqq\{x_0\}\}\,,$$
$$I_{x0} = \{a\in A : x\in s(a) \Rightarrow x\geq x_0\}\,,$$
$$J_{x0} = \{a\in A : x\in s(a) \Rightarrow x\nleq x_0\}\,.$$

A_{x_0} and $A^0_{x_0}$ are subgroups of the additive group of A, while I_{x_0} and J_{x_0} are two-sided ideals of A. Call (A, X) *s-finite* if $s(a)$ is finite for every $a\in A$. In this case,

$$A = \bigoplus_{x\in X} A^0_x \quad \text{and} \quad A = A_{x0}\oplus J_{x0}$$

(direct sums of abelian groups).

PROPOSITION 3. *If (A, X) is an s-ring, then $(A/J_{x_0}, X)$ is an s-ring if*

$$s^*(a^*) = \{x \in s(a) : x \leq x_0\} .$$

PROOF. Obvious.

DEFINITION 2. We say that (f, f^s): $(A, X) \to (B, Y)$ is a homomorphism of s-rings if f: $A \to B$ is a homomorphism of rings, f^s: $X \to Y$ is a morphism of partially ordered sets, that is, $x_1 \geq x_2$ implies $f^s(x_1) \geq f^s(x_2)$, and

$$s(f(a)) \subseteqq f^s(s(a)) \quad \text{if } a \neq 0 .$$

If there exists a homomorphism of s-rings (g, g^s): $(B, Y) \to (A, X)$ such that $g = f^{-1}$, then (f, f^s) is an isomorphism of s-rings.

REMARK. Let $X(A)$ be the set of $x \in X$ such that $x \in s(a)$ for some $a \in A$. Define similarly $Y(B)$ for (Y, B). It is easy to prove that if (f, f^s) is an isomorphism, then f^s is a one-to-one mapping of $X(A)$ onto $Y(B)$.

EXAMPLES. 5. If (A, X) is an s-ring and $x_0 \in X$, then

$$(n, n^s): (A, X) \to (A/J_{x_0}, X) ,$$

where n is the natural surjection, and n^s the identity, is a homomorphism of s-rings.

6. Let G be a finite group, D_0 an arbitrary normal subgroup of G, and F a field of characteristic $p > 0$. The natural surjection of G onto G/D_0 induces a homomorphism of rings

$$\tau: ZF(G) \to ZF(\bar{G}) ,$$

where $\bar{G} = G/D_0$. If C is a conjugacy class of G and $g \in C$, then $\tau(c) = n\bar{c}$, where $\bar{C}$ is the conjugacy class of $\bar{g} = \tau(g)$, and

$$n = (G : C_G(g))/(\bar{G} : C_{\bar{G}}(\bar{g})) .$$

Define τ^s: $S(G) \to S(\bar{G})$ by $\tau^s(x_D) = x_{\bar{D}}$, where D is a p-subgroup of G and $\bar{D} = DD_0/D_0$. It is easy to check that (τ, τ^s) is an s-homomorphism.

7. Let G be a finite group, F a field of characteristic $p > 0$, and $(ZF(G), S(G))$ the corresponding s-ring defined in Example 3. If D_0 is a p-subgroup of G and $H = N_G(D_0)$ the normalizer of D_0 in G, then the Brauer homomorphism

$$\sigma: ZF(G) \to ZF(H)$$

is defined by (see [1, § 56] or [4, p. 211]): $\sigma(c) = \sum g$, $g \in C \cap C_G(D_0)$ and $\sigma(c) = 0$ if $C \cap C_G(D_0) = \emptyset$. Note that by Lemma 1, $\sigma(c) \neq 0$ if and only if $x_c \leq x_{D_0}$, so

$$\text{(1)} \qquad \operatorname{Ker} \sigma = J_{x_{D_0}} .$$

In the general case, it is not possible to define a morphism $\sigma^s\colon S(G) \to S(H)$ such that (σ, σ^s) is a homomorphism of s-rings (e.g. $G = S_4$ and D_0 generated by $(1,2)(3,4)$)). If, however, D_0 is a normal p-subgroup of G, then, of course, σ^s can be defined as the identity on $S(G) = S(H)$.

Brauer's first main theorem on blocks and some generalizations of this theorem can be considered as theorems on such subrings or quotient rings of $ZF(G)$ that for the homomorphism induced by σ a good mapping of supports can be defined. We shall discuss this problem later in Theorem 1 and in the proof of Brauer's theorem.

Let us assemble some simple observations.

PROPOSITION 4. *Let* $(f, f^s)\colon (A, X) \to (B, Y)$ *be a homomorphism of s-rings and* (A, X) *be s-finite. Then*

(a) $(\operatorname{Im} f, Y)$ *is an s-subring of* (B, Y),
(b) *if* $\operatorname{Ker} f = J_{x0}$, *then*

$$(f^*, f^s)\colon (A/J_{x0}, X) \to (B, Y)\,,$$

where $f^*(a^*) = f(a)$, *is a homomorphism of s-rings.*

PROPOSITION 5. *If* (A, X) *is s-finite and* $x_0 \in X$, *then the s-homomorphism* $(n, n^s)\colon (A, X) \to (A/J_{x0}, X)$ *induces an isomorphism of abelian groups* A^0_{x0} *and* $(A/J_{x0})^0_{x0}$.

PROOF. If $a \in A^0_{x0}$, $a \neq 0$, then $s^*(n(a)) = x_0$, so $n(a) \in (A/J_{x0})^0_{x0}$ and $n(a) \neq 0$. Thus n induces an injection. If $n(a) \in (A/J_{x0})^0_{x0}$ and $n(a) \neq 0$, then $s^*(n(a)) = x_0$. Hence $a = a_1 + a_2$, where $s(a_1) = x_0$ and $a_2 \in J_{x0}$. Therefore $a_1 \in A^0_{x0}$ and $n(a_1) = n(a)$. Thus n induces a surjection.

3. Normal supports.

If (A, X) is an s-finite ring, then there are some special elements of X which in the case of s-rings $(ZF(G), S(G))$ correspond to the normal p-subgroups of G.

DEFINITION 3. We say that an element $x \in X$ is normal if in the decomposition

$$A = A_x \oplus J_x\,,$$

A_x is a subring of A, and J_x is a nilpotent ideal of A, that is, A is a split extension of its subring A_x with nilpotent kernel J_x.

Note that if x is normal, then A^0_x is a subring of A.

EXAMPLES. 8. If for an s-ring (A, X), there is an element $x_0 \in X$ such that $x \leq x_0$ for every $x \in X(A)$, then x_0 is normal. In this case $A_{x_0} = A$ and $J_{x_0} = 0$. Thus x_0 is normal for $(A/J_{x_0}, X)$.

9. If D is a normal p-subgroup of a finite group G, then $x_D \in S(G)$ is normal for $(ZF(G), S(G))$. To prove that J_{x_D} is nilpotent, let us note (see e.g. [4, Lemma 4.2]) that the kernel of the natural homomorphism

$$\tau\colon ZF(G) \to ZF(G/D)$$

is nilpotent, and if C is a conjugacy class of G such that $C \cap C_G(D) = \emptyset$, then $c \in \operatorname{Ker} \tau$.

If we translate this fact to the language of s-rings, we get by Lemma 1 that $x_c \nleq x_D$ implies $c \in \operatorname{Ker} \tau$. Hence J_{x_D} is nilpotent.

To prove that A_{x_D} is a subring of $A = ZF(G)$, we have to show that if $c, c' \in A_{x_D}$ are sums of conjugacy classes C and C' of G, then $cc' \in A_{x_D}$. Let $cc' = \sum r_i c_i$, $r_i \neq 0$, and let c_i be the sum of the conjugacy class C_i of gg', where $g \in C$ and $g' \in C'$. Since $x_c \leq x_D$ and $x_{c'} \leq x_{D'}$ each of the centralizers $C_G(g)$ and $C_G(g')$ contains a subgroup conjugate to D. But D is normal, so we get $D \subseteqq C_G(g) \cap C_G(g')$. Therefore $gg' \in C_i \cap C_G(D)$. By Lemma 1, $x_{c_i} \leq x_D$, so $c_i \in A_{x_D}$.

In the next section we shall use the following simple fact.

LEMMA 2. *Let $A = A_0 + J$, where A is a commutative ring, A_0 a subring of A, and J a nilpotent ideal of A. If $A_0 \cap J = (0)$, then the idempotents of A are in A_0.*

PROOF. Let $e \in A$ be an idempotent and $e = e_0 + j$, where $e_0 \in A_0$ and $j \in J$. Let r be the least natural number such that $j^r = 0$. Assume $r \geq 2$. $e^2 = e$ gives $e_0^2 = e_0$ and $2e_0 j + j^2 = j$. Multiplying the last equality by j^{r-2} and $e_0 j^{r-2}$, we get $2e_0 j^{r-1} = j^{r-1}$ and $2e_0 j^{r-1} = e_0 j^{r-1}$. Hence $e_0 j^{r-1} = 0$, that is, $j^{r-1} = 0$. This contradiction shows that $r = 1$, so $e = e_0 \in A_0$.

4. Commutative Artinian s-rings.

If A is a commutative Artinian ring with an identity, then

$$1 = e_1 + \ldots + e_n,$$

where e_i are the primitive idempotents of A. A is a direct sum of the local rings $A_{e_i} = Ae_i$. Denote by λ_e the natural homomorphism $A \to A_e \to A_e/m_e$, where e is a primitive idempotent, and m_e is the maximal ideal of the local ring A_e.

LEMMA 3. *Let I be an ideal of a commutative Artinian ring A, and e a primitive idempotent of A. Then $\lambda_e(I) \neq (0)$ if and only if $e \in I$.*

PROOF. $\lambda_e(I) \neq 0 \Leftrightarrow eI = A_e \Leftrightarrow e \in eI \subseteqq I$.

PROPOSITION 6. *Let (A, X) be an s-finite commutative Artinian ring and $e \in A$ a primitive idempotent. Then there is a uniquely determined element $x_e \in s(e)$ such that $e \in I_{x_e}$.*

PROOF. Let $s(e) = \{x_1, \ldots, x_r\}$ and $e = \sum a_i$, $s(a_i) = x_i$. Since $\lambda_e(e) \neq 0$, there is i such that $\lambda_e(a_i) \neq 0$. Hence $\lambda_e(I_{xi}) \neq 0$, that is, $e \in I_{x_i}$ by Lemma 3. Thus $x \geq x_i$ for every $x \in s(e)$. Denote x_i by x_e. x_e is unique as the least element of $s(e)$.

The element x_e, defined in Proposition 6, will be called the *s-defect* of e.

Now we are ready to prove an analogue of Brauer's first main theorem on blocks.

THEOREM 1. (a) *Let (A, X) be an s-finite commutative Artinian ring and $x_0 \in X$. The s-homomorphism $(n, n^s)\colon (A, X) \to (A/J_{x0}, X)$ induces a one-to-one correspondence between the primitive idempotents $e \in A$ with s-defect $x_e = x_0$, and the primitive idempotents $e^* \in A/J_{x0}$ with s-defect $x_{e^*} = x_0$.*

(b) *If x_0 is normal for (A, X), then all the primitive idempotents $e \in A$ with s-defect $x_e = x_0$ belong to the ring A^0_{x0} and n induces an isomorphism of rings A^0_{x0} and $(A/J_{x0})^0_{x0}$.*

PROOF. (a) Since A is a commutative Artinian ring and J_{x0} is an ideal of A, every primitive idempotent e^* of A/J_{x0} can be uniquely lifted to a primitive idempotent e of A. Therefore, we have to prove that if $e \in A$ and $x_e = x_0$, then $n(e) \neq 0$, and $x_e = x_0$ if and only if $x_{n(e)} = x_0$.

If $x_e = x_0$, then $x \in s(e)$ implies $x \geq x_0$. Hence $s^*(n(e)) = x_0$ by Proposition 3. Thus $n(e) \neq 0$ and $x_{n(e)} = x_0$.

If $x_{n(e)} = x_0$, then $x_0 \in s(e)$, so $x_0 \geq x_e$. Hence $x_e \in s^*(n(e))$, that is, $x_e \geq x_{n(e)} = x_0$. Thus $x_e = x_0$.

(b) Let x_0 be normal and $x_e = x_0$. By Lemma 2, $e \in A_{x0}$. Hence $x \in s(e)$ implies $x \leq x_0$. But $x \geq x_e = x_0$, so we get $x = x_0$. Thus $s(e) = x_0$, that is, $e \in A^0_{x0}$. Since x_0 is normal, A^0_{x0} is a ring, and by Proposition 5, n induces an isomorphism of rings A^0_{x0} and $(A/J_{x0})^0_{x0}$.

If we apply Theorem 1 in the theory of modular representations of finite groups, we get the situation considered in Brauer's first main theorem on blocks. We end the paper by showing what kind of additional facts about groups we need to prove Brauer's theorem.

Recall some notions. If G is a finite group, F a field of characteristic $p > 0$ and $ZF(G)$ the center of the group algebra $F(G)$, then we can consider the commutative Artinian s-ring $(ZF(G), S(G))$. The primitive idempotens e of

$ZF(G)$ define the blocks $F(G)e$ of $F(G)$. Every p-subgroup D of G such that $x_D = x_e$ is called a defect group of the block $F(G)e$ (see [1, § 54] or [4, p. 211]). Let

$$\sigma\colon ZF(G) \to ZF(H) \tag{2}$$

be the Brauer homomorphism defined by a p-subgroup D_0 of G. The first main theorem on blocks says (see [1, Theorem 58.3] or [4, Theorem 5.3]):

THEOREM 2. *Let G be a finite group, D_0 a p-subgroup of G, and $H = N_G(D_0)$ the normalizer of D_0 in G. If F is a field of characteristic $p>0$, then the Brauer homomorphism σ of $ZF(G)$ into $ZF(H)$ induces a one-to-one correspondence between the blocks of $F(G)$ with s-defect $x_{D_0} \in S(G)$, and the blocks of $F(H)$ with s-defect $x_{D_0} \in S(H)$.*

PROOF. $\operatorname{Ker} \sigma = J_{x0}$, where $x_0 = x_{D_0}$ (see (1)). By Theorem 1, there is a one-to-one correspondence between the primitive idempotents (= blocks) e of the ring $ZF(G)$ with s-defect $x_e = x_0$, and the primitive idempotents e^* of the ring $ZF(G)/J_{x0}$ with s-defect $x_{e^*} = x_0$, induced by

$$(n, n^s)\colon (ZF(G), S(G)) \to (ZF(G)/J_{x0}, S(G)) .$$

The Brauer homomorphism (2) induces the monomorphism

$$\sigma^*\colon ZF(G)/J_{x_0} \to ZF(H) .$$

We have to prove that σ^* induces a one-to-one correspondence between the primitive idempotents with s-defect x_0 of the s-ring $(ZF(G)/J_{x0}, S(G))$, and the primitive idempotents with s-defect x_0 of the s-ring $(ZF(H), S(H))$ — we shall write x_0 both in $S(G)$ and $S(H)$.

By Theorem 1, the idempotents $e^* \in ZF(G)/J_{x0}$ such that $x_{e^*} = x_0$ belong to the ring $(ZF(G)/J_{x0})^0_{x0}$, while the idempotents $e' \in ZF(H)$ such that $x_{e'} = x_0$ belong to the ring $(ZF(H))^0_{x0}$. To prove the Theorem, we have to show that σ^* induces an isomorphism of the rings $(ZF(G)/J_{x0})^0_{x0}$ and $(ZF(H))^0_{x0}$. Only this part of the proof needs some additional facts about groups, which are contained in the following well-known lemma.

LEMMA 4. *Let G be a finite group, D a p-subgroup of G, $C_G(D)$ the centralizer, and $N_G(D) = H$ the normalizer of D in G.*

(a) *If C is a conjugacy class of G such that D is a Sylow p-subgroup of $C_G(g)$ for some $g \in C$, then $C \cap C_G(D) = C'$ is a conjugacy class of H and D is a Sylow p-subgroup of $C_H(g)$ for (all) $g \in C'$.*

(b) *If C' is a conjugacy class of H such that D is a Sylow p-sugroup of $C_H(g)$ for (all) $g \in C'$, then there is a conjugacy class of C of G such that $C \cap C_G(D) = C'$ and D is a Sylow p-subgroup of $C_G(g)$ for some $g \in C$.*

PROOF. See [1, Lemma 58.1] or [4, Lemma 3.4].

Now the proof of Brauer's theorem is evident.

If $c^* \in (ZF(G)/J_{x0})^0_{x0}$, where C is a conjugacy class of G, then D_0 is a defect group of C, so by the first part of the Lemma, $\sigma^*(c^*) = \sigma(c)$ is the sum of a conjugacy class of H with D_0 as a defect group. Thus the restriction of σ^* to $(ZF(G)/J_{x0})^0_{x0}$ is an injection into $(ZF(H))^0_{x0}$.

If $c' \in (ZF(H))^0_{x0}$, where C' is a conjugacy class of H, then D_0 is a defect group of C', so by the second part of the Lemma, there is a conjugacy class C of G, with D_0 as a defect group, such that $\sigma^*(c^*) = \sigma(c) = c'$. Hence c^* belongs to the ring $(ZF(G)/J_{x0})^0_{x0}$, and the restriction of σ^* to this ring is a surjection onto $(ZF(H))^0_{x0}$. The proof is completed.

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