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A NOTE ON FREE DIRECT SUMMANDS

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1. Introduction.

In the paper [1] Bass demonstrated a useful technique for dealing with big projective modules (i.e. non-finitely generated projective modules). By the so-called "Eilenberg-swindle" he proved: Let P be a big projective module which is a quotient of a big free module F; if P has a direct summand isomorphic to F then P is isomorphic to F. With this result at hand the problem of showing that a given projective module is free naturally leads on to look for free direct summands in modules.

In the paper [2] we proved a theorem of this kind. Let R be a ring with Jacobson-radical N, and let F be a free module. If F' is a submodule of F such that $F=F'+NF$ then F' has a direct summand isomorphic to F. Theorem 2 of the present paper is an improvement of this earlier result. In theorem 3 we give a necessary and sufficient condition for a module M to have a direct summand isomorphic to $R^{(I)}$, where I is an infinite set, and we give some applications.

In the following rings are associative with identity and modules are left unitary modules. For a free R-module F with basis $\{e_i\}_{i\in I}$ we set the support of an element $x \in F$, $x=\sum r_ie_i$, to be

$$\operatorname{supp}(x) = \{i \in I \mid r_i \neq 0\} .$$

2. Free direct summands.

To simplify the proof of Theorem 2 we first prove a set-theoretic lemma.

LEMMA 1. *Let A be a set and let X be a subset of the cartesian product $A \times A$. There exists a subset B of A satisfying:*

(i) *B has a (non-reflexive) well-ordering such that if $b_1,b_2 \in B$ and $b_1<b_2$ then $(b_1,b_2) \in X$.*
(ii) *If $a \in A$ and $(b,a) \in X$ for all $b \in B$, then $a \in B$.*

PROOF. Let A be non-empty and let Ω be the set of all pairs $(C, <_c)$ where $C \subseteq A$ and $<_c$ is a well-ordering of C such that if $c_1,c_2 \in C$ and $c_1 <_c c_2$ then

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$(c_1, c_2) \in X$. Ω is partially ordered by $(C, <_C) \leq (D, <_D)$ if $C \subseteqq D$ and the restriction of $<_D$ to C is $<_C$ and $c <_D d$ whenever $c \in C$ and $d \in D \backslash C$. The set Ω is not empty and inductively ordered. It is then straightforward to verify that a maximal member $(B, <)$ of Ω satisfies conditions (i) and (ii) of Lemma 1.

For a free R-module F with basis $\{e_i\}_{i \in I}$ let $\pi_i(x) = r_i$ $(i \in I)$ when $x = \sum r_i e_i$ is an element of F. These coordinate-projections are surjective R-homomorphisms from F to R.

Theorem 2. *Let F be a free R-module with basis $\{e_i\}_{i \in I}$ where I is an infinite index-set. If $\{x_i\}_{i \in I}$ is a set of elements of F such that $\pi_i(x_i) = 1$ for every $i \in I$, then there exists a set $J \subseteqq I$ of the same cardinality as I such that $\{x_j\}_{j \in J} \cup \{e_i\}_{i \in I \backslash J}$ is a basis for F.*

Proof. Let X be the following subset of $I \times I$:

$$X = \{(i, j) \in I \times I \mid j \notin \operatorname{Supp}(x_i)\} .$$

According to Lemma 1 there exists a set $J \subseteqq I$, well-ordered by $<$, such that

(i) If $j_1, j_2 \in J$ and $j_1 < j_2$ then $j_2 \notin \operatorname{Supp}(x_{j_1})$
(ii) $(\bigcup_J \operatorname{Supp}(x_j)) \cup J = I$.

Since $i \in \operatorname{Supp}(x_i)$ we have $\bigcup_J \operatorname{Supp}(x_j) = I$ and as $\operatorname{Supp}(x_j)$ is a finite set for every $j \in J$ we obtain that $\operatorname{card}(J) = \operatorname{card}(I)$. It remains to prove that $\{x_j\}_{j \in J} \cup \{e_i\}_{i \in I \backslash J}$ is a basis for F. Let $\varphi \colon F \to \sum_J Re_j$ be the projection, that is, $\varphi(e_j) = e_j$ for $j \in J$ and $\varphi(e_i) = 0$ for $i \in I \backslash J$. Set $y_j = \varphi(x_j)$ for $j \in J$. Due to (i) we have

$$x_j = e_j + \sum_{k \in J, k < j} r_k e_k + \sum_{t \notin J} s_t e_t$$

hence

$$y_j = e_j + \sum_{k \in J, k < j} r_k e_k .$$

Assume that

$$u_i y_{j_1} + \ldots + u_n y_{j_n} = 0, \quad \text{where } u_i \in R, \text{ and } j_1 < \ldots < j_n .$$

Since e_{j_n} occurs only in the representation of y_{j_n} and with coefficient 1 it follows that $u_n = 0$. By induction we get all the u_i's to be zero. Thus: if $u_1 x_{j_1} + \ldots + u_n x_{j_n} \in \operatorname{Ker} \varphi$ then $u_1 = \ldots = u_n = 0$, implying that $\{x_j\}_{j \in J}$ is a basis for the free module $\sum_J Rx_j$.and

$$\sum_J Rx_j \cap \operatorname{Ker} \varphi = \{0\} .$$

To prove that $\sum_J Rx_j + \operatorname{Ker} \varphi = F$ it suffices to show that $\sum_J Ry_j = \sum_J Re_j$.

Obviously $\sum_J Ry_j \subseteqq \sum_J Re_j$. Suppose that $e_j \notin \sum_J Ry_j$ for some $j \in J$. Let j_0 be the smallest $j \in J$ with this property. But as $y_{j_0} = e_{j_0} + \sum r_k e_k$ where the summation is taken over values of $k \in J$ with $k < j_0$ we have a contradiction. As $\operatorname{Ker} \varphi = \sum_{I \setminus J} Re_i$ the proof is complete.

DEFINITION. Let $\{\pi_i\}_{i \in I}$ be a family of R-homomorphisms between R-modules M and N. We shall say that the family is *locally finite* if for each $m \in M$, $\pi_i(m) = 0$ for all but a finite number of i's.

THEOREM 3. *Let M be an R-module and let I be an infinite set. The following conditions are equivalent.*

1° *M has a direct summand isomorphic to $R^{(I)}$*
2° *There exists a locally finite family of surjective R-homomorphisms $\{\pi_i\}_{i \in I}$ from M to R.*

PROOF. It is enough to prove 2° $\Rightarrow$ 1°. Let $\{e_i\}_{i \in I}$ be a basis for $R^{(I)}$ and define $f\colon M \to R^{(I)}$ by

$$f(m) = \sum_I \pi_i(m) e_i .$$

For every $i \in I$ there exists an element $m_i \in M$ with $\pi_i(m_i) = 1$; letting $x_i = f(m_i)$ the family $\{x_i\}_{i \in I}$ satisfies the condition in Theorem 2. Let $J \subseteqq I$ be such that $\operatorname{card} J = \operatorname{card} I$ and $\{x_j\}_{j \in J} \cup \{e_i\}_{i \in I \setminus J}$ is a basis for $R^{(I)}$. Let $F = \sum_J Rx_j$. Then $F \subseteqq f(M)$ and F is a direct summand of $R^{(I)}$. Hence F is also a direct summand of $f(M)$, and consequently M has a surjection onto a module isomorphic to $R^{(I)}$.

Let N denote the Jacobson-radical of a ring R and let F be a free module. If F' is a submodule of F such that $F = F' + NF$, then we proved in [2] that F' has a surjection onto F. This was the keyresult in establishing that $P/NP \cong F/NF$ implies $P \cong F$ whenever P is projective and F is free.

COROLLARY 4. *Let F be a free module and $F' \subseteqq F$ a submodule such that $F' + NF = F$. Then F has a direct summand $F'', F'' \subseteqq F'$ and $F'' \cong F$.*

PROOF. Let $\{e_i\}_{i \in I}$ be a basis for F. If I is finite it follows from Nakayama's lemma that $F' = F$. So assume that I is infinite. Writing $e_i = y_i + z_i$, $y_i \in F'$, $z_i \in NF$, we get that y_i can be written $\sum_I r_{ij} e_j$ where r_{ii} is a unit in R. Letting $x_i = r_{ii}^{-1} y_i$ we get a family $\{x_i\}_{i \in I}$ as in Theorem 2 and we are done.

DEFINITION. Let P be a projective module. A *dual basis* for P is a family

$\{\pi_i, q_i\}_{i \in I}$ where $\pi_i \in \operatorname{Hom}(P, R)$ and $q_i \in P$ such that for all $x \in P$, $x = \sum_I \pi_i(x) q_i$ ($\{\pi_i\}$ locally finite).

THEOREM 5. *Let P be a big projective module of type α (the type is the smallest cardinal of a set of generators). The following conditions are equivalent:*

1° *P is free*
2° *P has a dual basis $\{\pi_i, q_i\}_{i \in I}$ where all the π_i's are surjective.*
3° *P has a dual basis $\{\pi_i, q_i\}_{i \in I}$ such that π_j is surjective for all j in a subset $J \subseteqq I$ with $\operatorname{card} J = \operatorname{card} I = \alpha$.*
4° *There exists a locally finite family $\{\pi_i\}_{i \in I}$, where $\pi_i \in \operatorname{Hom}(P, R)$ is surjective for all $i \in I$ and $\operatorname{card} I = \alpha$.*

PROOF. 1° ⇒ 2° ⇒ 3° ⇒ 4° are all evident.

4° ⇒ 1°. It follows from Theorem 3 that P has a surjection to $R^{(I)}$, and using the result of Bass mentioned in the introduction it follows that P is free.

PROBLEM. Let P be finitely generated projective with dual basis $\{\pi_i, q_i\}$; $i = 1, 2, \ldots, n$ such that all the π_i's are surjective. Is P a free module? This is true if R is commutative or left noetherian.

Elements of the form $x_i = e_i + \sum_{j \neq i} r_{ij} e_j$ often arise as generators of kernels in free modules. Let Q be an R-module with the property that any finitely generated homomorphic image of Q is zero. If R is an integral domain with field of quotients K, $K \neq R$, then K has this property. It is easily seen that a module Q has this property if and only if Q has no maximal submodules. We close this paper with an application of Theorem 2, which shows that the relations of such a module are "big".

THEOREM 6. *Let Q be an R-module with no maximal submodule. Let $\varphi\colon F \to Q$ be surjective with F free. Then $\operatorname{Ker} \varphi$ has a direct summand isomorphic to F.*

PROOF. Let $\{e_i\}_{i \in I}$ be a basis for F and set $q_i = \varphi(e_i)$ (notice that I must be infinite). Then for every $i \in I$, $q_i \in \sum_{j \neq i} R q_j$. It follows that for every $i \in I$ there exists $x_i \in \operatorname{Ker} \varphi$ such that

$$x_i - e_i \in \sum_{j \neq i} R e_j .$$

Now the proof is easily completed by Theorem 2.

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