MATH. SCAND. 42 (1978), 53–63.

A NOTE ON TANGENT BUNDLES IN A CATEGORY WITH A RING OBJECT

G. E. REYES* and G. C. WRAITH

The aim of this note is to point out that certain infinitesimal structures appearing in Algebraic Geometry (e.g. tangent bundle of a scheme, Lie algebra of a group scheme cf. [5]) may be defined, and their basic properties proved in the context of a finitely complete category with a distinguished ring object. We assume that the category has some exponentials.

This possibility was first pointed out by F. W. Lawvere [10] in an unpublished lecture. It was taken up by the second author who sketched a proof that the tangent vectors over the neutral element of a monoid object form a Lie algebra. The requirements on the category needed for the proof were, however, unnecessarily strong.

We present an improved version of that result based on work done by the first author.

We are grateful to A. Kock for telling us about Lawvere's work and for valuable discussions. In particular, he explained to us his work on "differential calculus" in this categorical context (Kock [7], [8]).

Throughout, we shall use the set-theoretical notation, well established for toposes and related categories (see e.g. Osius [11], Boileau [1], Coste [2], etc.) leaving to the sceptical reader the uninviting task of manipulating diagrams.

1. Infinitesimally linear objects and their tangent bundles.

Let $\boldsymbol{E}$ be a category with finite limits and some exponentials (namely those which we shall have occasion to use) and let A be a commutative ring object with unit element in $\boldsymbol{E}$. We define

$$D_0 = \mathbf{1} \rightarrowtail A$$

$$D_1 = [a \in A \mid a^2=0] \rightarrowtail A$$

$$D_2 = [(a,b) \in A^2 \mid a^2=b^2=ab=0] \rightarrowtail A^2$$

$$D_3 = [(a,b,c) \in A^3 \mid a^2=b^2=c^2=ab=bc=ac=0] \rightarrowtail A^3$$

* Partially supported by a Canada Council Leave Fellowship.

Received January 7, 1977.

and we have the diagrams

$$(*)\qquad \begin{cases} D_0 \xrightarrow{0} D_1 \overset{i_0, i_1}{\rightrightarrows} D_2 \\ D_0 \xrightarrow{0} D_1 \overset{j_0, j_1, j_2}{\rightrightarrows} D_3 \end{cases}$$

where

$$i_0(d) = (d,0), \qquad i_1(d) = (0,d)$$

$$j_0(d) = (d,0,0), \quad j_1(d) = (0,d,0), \quad j_2(d) = (0,0,d) .$$

Intuitively we think of A as a "line" and of D_1 as a "disembodied tangent vector" or as "point with an infinitesimal linear neighbourhood". Under this interpretation it is natural to define the tangent space of "variety" M at a "point" x as the "set" of maps from D_1 to M which take the base-point 0 of D_1 to x, that is,

$$T_x(M) = [f \in M^{D_1} \mid f(0)=x] .$$

The *tangent bundle* of M becomes M^{D_1} together with the canonical projection

$$\pi\colon M^{D_1} \to M$$

sending f to its "base-point" $f(0)$. Of course, we want $T_x(M)$ to be a "vector space" (or rather an A-module in our context), i.e. an abelian group with a scalar multiplication

$$A \times T_x(M) \to T_x(M)$$

satisfying the usual axioms. Notice that this operation can be defined in the obvious way by

$$(a \circ f)(d) = f(ad) .$$

As for the group structure, we first assume that M is a *Euclidean* A-module, that is to say an A-module such that all "functions" in M^{D_1}, M^{D_2}, $M^{D_1 \times D_1}$ are "analytic". This should mean, since $d^2=0$, that the maps

$$\alpha\colon M \times M \to M^{D_1}$$

$$\beta\colon M \times M \times M \to M^{D_2}$$

$$\gamma\colon M \times M \times M \times M \to M^{D_1 \times D_1}$$

defined by $\alpha(x,y)=\lambda d(x+yd)$, that is,

$$\alpha(x,y)(d) = x+yd$$

$$\beta(x,y,z) = \lambda d_1 d_2(x+yd_1+zd_2)$$

$$\gamma(x,y,z,u) = \lambda d_1 d_2(x+yd_1+zd_2+ud_1d_2)$$

are isomorphisms (Compare Kock [8]). In Kock [0], A is said to be of *line type* if

$$\alpha\colon A \times A \to A^{D_1}$$

is an isomorphism. Assuming, then, that M is Euclidean, there is an obvious group structure on $T_x(M)$:

$$\lambda d(x+yd)+\lambda d(x+y'd) = \lambda d(x+(y+y')d) .$$

Although this assumption that M is Euclidean is too restrictive for the examples we have in mind, the following corollary of it will do as well to define our group structure:

1) the map

$$[h \in M^{D_2} \mid h(0,0)=x] \to T_x(M) \times T_x(M)$$

given by

$$h \mapsto (\lambda dh(d,0), \lambda dh(0,d))$$

is an isomorphism whose inverse is given by

$$(\lambda d(x+yd), \lambda d(x+y'd)) \mapsto \lambda d_1 d_2(x+yd_1+y'd_2) .$$

In other words, this corollary says that a couple of functions in $T_x(M)$ may be coded up by a single function in $[h \in M^{D_2} \mid h(0,0)=x]$. Notice that if h codes up the couple (f,g) in $T_x(M)$, then $f+g$ in the sense defined above is just $\lambda dh(d,d)$. Of course, this will be our new definition of addition in $T_x(M)$ whenever M satisfies condition 1).

For our main result, the following corollary of M being Euclidean is critical:

2) If $k \in [k \in M^{D_1 \times D_1} \mid k(0,0)=x]$ satisfies the condition

$$\lambda d_1 d_2 k(d_1,0) = \lambda d_1 d_2 k(0,d_2)$$

then there is a unique f in $T_x(M)$ such that

$$k = \lambda d_1 d_2 f(d_1 d_2) .$$

To see this, assume that

$$k = \lambda d_1 d_2(x+yd_1+zd_2+ud_1d_2)$$

is such that $\lambda d_1 d_2 k(d_1,0)=\lambda d_1 d_2 k(0,d_2)$. Then $y=z=0$ so we may take $f=\lambda d(x+ud)$.

We hope that the preceding discussion motivates the following axioms on an object M of $\boldsymbol{E}$.

AXIOM 1. The diagrams

$$M = M^{D_0} \xleftarrow{\pi} M^{D_1} \overset{\pi_0,\pi_0}{\leftleftarrows} M^{D_2}$$

$$M = M^{D_0} \xleftarrow{\pi} M^{D_1} \overset{\varrho_0,\varrho_1,\varrho_2}{\Lleftarrow} M^{D_3}$$

obtained from (*) by applying the functor $M^{(\cdot)}$ are exact.

Axiom 1 implies that π is a group object in $\mathbf{E}/M$. The group operation is

$$\partial_1 = M^{\Delta},$$

where $\Delta\colon D_1 \to D_2$ is given by $\Delta(d)=(d,d)$, so the group structure is clearly abelian. Furthermore, if $\pi_1\colon A\times M \to M$ denotes projection to the second factor, it should be evident from our discussion of 1) that we have:

THEOREM. *If M satisfies axiom 1), then the tangent bundle*

$$M^{D_1} \xrightarrow{\pi} M$$

is a $(A\times M \xrightarrow{\pi_1} M)$-module in $\mathbf{E}/M$.

To formulate our next axiom on an object M of $\mathbf{E}$, consider the diagram

$$D_1\times D_1 \overset{p_0,p_1}{\rightrightarrows} D_1\times D_1 \xrightarrow{\cdot} D_1$$

where $p_0(d_1,d_2)=(d_1,0)$, $p_1(d_1,d_2)=(0,d_2)$ and $\cdot$ is multiplication.

AXIOM 2. The diagram

$$M^{D_1} \xrightarrow{M^{\cdot}} M^{D_1\times D_1} \overset{M^{p_0},M^{p_1}}{\rightrightarrows} M^{D_1\times D_1}$$

obtained by applying the functor $M^{(-)}$ to the diagram above, is exact.

It should be clear that axiom 2) formalizes corollary 2) of our discussion. Our discussion above, and the arguments which led us to adopt axioms 1 and 2, suggest the following idea: "manifolds" are objects which are "locally euclidean" and thus have a "locally linear" structure. We can formalize this notion more precisely as follows:

Let $D(n)=[a\in A \mid a^{n+1}=0] \rightarrowtail A$. We may call $D(n)$ the n-th infinitesimal neighbourhood of 0 in A. Let us define a *unifold* to be the limit of a finite diagram of infinitesimal neighbourhoods of 0 in A. For example, D_1 and D_2 are unifolds.

We may say that an object M of $\mathbf{E}$ is *infinitesimally linear* if the objects M^U, for U a unifold, satisfy all the formal consequences of supposing that M is an A-module and that the elements of M^U are analytic, i.e. defined by polynomials. This concept requires further elaboration to be properly defined, and we do not wish to go into the matter further at this point. Suffice it to say that infinitesimally linear objects satisfy axioms 1) and 2).

Let us note that objects which satisfy axiom 1) or 2) have good stability properties.

PROPOSITION.

i) *For any object N, if M satisfies axiom* 1) *or* 2) *so does M^N.*

ii) *If each object of a diagram D in* **E** *satisfies axiom* 1) *or* 2) *then so does* $\varprojlim D$.

The proof is an immediate consequence of the fact that axioms 1) and 2) are defined in terms of limits.

To formulate our main theorem, let G be a monoid object in $\boldsymbol{E}$ satisfying axiom 1) and let

$$1 \xrightarrow{e} G$$

be the neutral element. Define $T_e(G)$, the "set" of tangent vectors at e by the pullback diagram

$$\begin{array}{ccc} T_e(G) & \rightarrowtail & G^{D_1} \\ \downarrow & & \downarrow \pi \\ 1 & \overset{e}{\rightarrowtail} & G \end{array}$$

It is clear that $T_e(G)$ is an A-module object in $\boldsymbol{E}$, obtained by pullback along e from the $(A \times G \to G)$-module $G^{D_1} \xrightarrow{\pi} G$.

THEOREM. *Assume that G is a monoid object in* **E** *which satisfies axioms* 1) *and* 2). *Then $T_e(G)$ has a natural Lie-algebra structure.*

Proof. We define the Lie-bracket operation

$$T_e(G) \times T_e(G) \xrightarrow{[\cdot,\cdot]} T_e(G)$$

as follows: if $f_1, f_2 \in T_e(G)$, let $h \in G^{D_1 \times D_1}$ be defined by

$$h(d_1, d_2) = (-f)(d_1) \underset{G}{\circ} (-f_2)(d_2) \underset{G}{\circ} f_1(d_1) \underset{G}{\circ} f_2(d_2)$$

Since $\lambda d_1 d_2 h(d_1, 0) = \lambda d_1 d_2 h(0, d_2) = \lambda d_1 d_2 e$ there is a unique $\bar{h} \in G^{D_1}$ such that

$$h(d_1, d_2) = \bar{h}(d_1 d_2) .$$

Clearly $\bar{h} \in T_e(G)$, since $\bar{h}(0) = e$.

We define $[f_1, f_2] = \bar{h}$. We need to show that in $T_e(G)$ the following equations hold

i) $[f_1, f_2]+[f_2, f_1] = 0$
ii) $[f, t_1+t_2] = [f, t_1]+[f, t_2]$
iii) Jacobi's idendity

$$[f_1,[f_2,f_3]]+[f_2,[f_3,f_1]]+[f_3,[f_1,f_2]] = 0 .$$

The proof uses the principle of coalescence of commuting operations (cf. Spanier [14, page 43]) to conclude that the monoid structure on $T_e(G)$ induced by G coincides with the abelian group structure given by axiom 1). We note

$$f_1(d)f_2(d) = f_2(d)f_1(d)$$

and

$$(-f)(d) = f(-d) = f(d)^{-1}$$

for $f_1, f_2, f \in T_e(G)$. Let us use the symbol (a, b) for the commutator $a^{-1}b^{-1}ab$. If $f_1, f_2, f_3 \in T_e(G)$ we have

$$[f_1,[f_2,f_3]](d_1d_2d_3) = (f_1(d_1), (f_2(d_2), f_3(d_3))) .$$

Since $d^2=0$, if we put $d_1=d_2=d$ we get

$$(f_1(d), (f_2(d), f_3(d_3))) = e$$

so that, using $(f_1(d), f_2(d))=e$, we get

$$(f_2(d), f_3(d_3)) = (f_2(d), f_1(d)^{-1}f_3(d_3)f_1(d)) .$$

This relation, along with the identity

$$(a, bc) = (a, b)(b^{-1}ab, c)$$

and the Magnus identity (cf. Serre [65])

$$(b^{-1}ab, (b, c))(c^{-1}bc, (c, a))(a^{-1}ca, (a, b)) = e$$

give us identities ii) and iii) immediately.

2. Examples.

Axioms 1) and 2) are both of the form "$M^{(-)}$ takes the diagram $\boldsymbol{D}$ to a limit diagram". It will of course follow that every object M will satisfy axioms 1) and 2) if the diagrams $\boldsymbol{D}$ are colimit diagrams in $\boldsymbol{E}$. This will in fact be the case in some of the examples we consider below.

EXAMPLE. Let $\boldsymbol{E}$ be the category of affine schemes, which we may identify with the category of representable set-valued functors on the category of commutative rings with unit. If R is such a ring, we denote the corresponding representable functor, Hom $(R, -)$, by h^R. We take

$$A = h^{\mathbf{Z}[x]}$$

so that

$$D_0 = h^{\mathbf{Z}}$$

$$D_1 = h^{\mathbf{Z}[x]/(x^2)}$$

$$D_2 = h^{\mathbf{Z}[x,y]/(x^2,xy,y^2)}\,.$$

$D_0, D_1, D_1 \times D_1, D_2$ are exponentiable. We have

$$(h^B)^{D_1} = h^{\mathrm{sym}_B(\Omega^1_{B/\mathbf{Z}})}$$

$$(h^B)^{D_2} = h^{\mathrm{sym}_B(\Omega^1_{B/\mathbf{Z}} \oplus \Omega^1_{B/\mathbf{Z}})}\,.$$

All the affine schemes satisfy axioms 1) and 2) because

$$\begin{array}{ccccc} & & \mathbf{Z}[x]/(x^2) & & \\ & \overset{y \mapsto 0}{\nearrow} & & \overset{x \mapsto 0}{\searrow} & \\ \mathbf{Z}[x,y]/(x^2,xy,y^2) & & & & \mathbf{Z}\,. \\ & \underset{x \mapsto 0}{\searrow} & & \underset{y \mapsto 0}{\nearrow} & \\ & & \mathbf{Z}[y]/(y^2) & & \end{array}$$

is a pullback diagram, and so

$$\begin{array}{ccccc} & & D_1 & & \\ & \nearrow & & \searrow & \\ 1 & & & & D_2 \\ & \searrow & & \nearrow & \\ & & D_1 & & \end{array}$$

is a pushout diagram of affine schemes, and because

$$\mathbf{Z}[x]/(x^2) \xrightarrow{x \mapsto x_1x_2} \mathbf{Z}[x_1,x_2]/(x_1^2,x_2^2) \overset{x_2 \mapsto 0,\, x_1 \mapsto 0}{\rightrightarrows} \mathbf{Z}[x_1,x_2]/(x_1^2,x_2^2)$$

is an equalizer diagram, so

$$D_1 \times D_1 \rightrightarrows D_1 \times D_1 \to D_1$$

is a coequalizer diagram of affine schemes. It is an easy exercise to check that if G is a monoid affine scheme, then in $T_e(G)$ the identity $[f, f] = 0$ holds.

EXAMPLE 2. Let $\boldsymbol{E}$ be the Zariski topos and let A be the generic local ring (cf. Hakim [6]). Following Demazure–Gabriel [4], we identify schemes with certain objects in $\boldsymbol{E}$, namely those X which can be written as

$$X = \bigcup_{i=1} U_i$$

where U_i is an affine scheme and $U_i \rightarrowtail X$ is open in the sense of Zariski. We claim that every scheme satisfies axioms 1) and 2).

For the proof of the claim, we consider a new notion of open subobject: $U \rightarrowtail X$ is D_1-open if

$$\begin{array}{ccc} U^{D_1} & \rightarrowtail & X^{D_1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \rightarrowtail & X \end{array}$$

is a pullback. Intuitively: if a "point" is in U, all "infinitely close points" are also in U.

LEMMA.

1) (Transitivity). *If $V \rightarrowtail U$ and $U \rightarrowtail X$ are D_1-open, so is $V \rightarrowtail X$.*

2) (Stability under pull-back). *For any map $X' \to X$ if $U \rightarrowtail X$ is D_1-open, then $U \times_X X' \rightarrowtail X'$ is D_1-open.*

3) (Topology axioms)

a) *$\varnothing \rightarrowtail X$ and $X \overset{1}{\rightarrowtail} X$ are D_1-open.*

b) *Finite intersections of D_1-open subobjects are D_1-open.*

c) *Arbitrary unions of D_1-open subobjects are D_1-open.*

The proof uses the Box lemma (cf. Kock–Wraith [9]) for 2) and the exactness properties of topos for 3c).

COROLLARY 1. *If $(U_i \rightarrowtail X)_{i\in I}$ are D_1-open, then*

$$\left(\bigcup_{i\in I} U_i\right)^{D_1} \cong \bigcup_{i\in I} (U_i^{D_1}) .$$

COROLLARY 2. *If $X = \bigcup_{i\in I} U_i$ where each $U_i \rightarrowtail X$ is D_1-open and satisfies axiom* 1) (*respectively axiom* 2)) *then X satisfies axiom* 1) (*respectively axiom* 2)).

To finish the proof of our claims, notice that every $U \rightarrowtail X$ which is Zariski open is D_1-open (but the converse is not true).

Finally, notice that the "set of vector fields" on X may be identified with $T_{1_X}(X^X)$, and hence it has a Lie-algebra structure.

We do not know whether the identity $[f, f] = 0$ is true for schemes, although we suspect it is.

EXAMPLE 3. The example of the category of affine schemes can obviously be relativized to affine A-schemes, for any ring A. For $A = \mathsf{R}$ however, we can broaden the scope to include more functions than polynomials. Let C^∞ be the algebraic theory (in the sense of Lawvere) whose n-ary operations are smooth maps $\mathsf{R}^n \to \mathsf{R}$. Since the theory of (commutative) R-algebras is the subtheory of C^∞ whose n-ary operations are the polynomial maps $\mathsf{R}^n \to \mathsf{R}$, a C^∞-model can be thought of as an R-algebra with further structure. If M is a smooth manifold, then $C^\infty(M)$, the R-algebra of smooth R-valued functions on M, has an obvious C^∞-model structure. In particular, $C^\infty(\mathsf{R}^n)$ is the free C^∞-model on n generators (the co-ordinate projections) and $C^\infty(\mathsf{R})$ is just R itself with the obvious C^∞-model structure. Not all C^∞-models are of the form $C^\infty(M)$, since the underlying R-algebra of such a model is reduced and we shall see below how to manufacture C^∞-models with nilpotent elements. However, we can associate to a finitely presented C^∞-model a variety defined by the zeros of a finite number of smooth functions as follows: if

$$C^\infty(\mathsf{R}^n) \rightrightarrows C^\infty(\mathsf{R}^n) \to X$$

is a co-equalizer diagram, we get (by applying the functor $\mathrm{Hom}_{C^\infty}(-,\mathsf{R})$) the equalizer diagram

$$\mathrm{Hom}_{C^\infty}(X,\mathsf{R}) \to \mathsf{R}^n \rightrightarrows \mathsf{R}^m .$$

More generally, we may be interested in sub-theories T of C^∞ containing (R-algebras). For example, we have the Nash theory of functions algebraic over polynomials (cf. Palais [12]) or we may consider the smallest T containing a given class of smooth functions, say exponentials or trigonometric functions. These ideas are essentially due to Lawvere [10].

For the sake of the lemma below we introduce the condition on the theory T that whenever a smooth functions belongs to T, then all its partial derivatives do also.

LEMMA 1. *Let B be a local* R*-algebra with residue field* R *and maximal ideal m. If m is a nil ideal, then B has a unique T-model structure.*

PROOF. We have $B = \mathsf{R} \oplus m$, with every element of m nilpotent. So every n-ple of elements of B can be written as $\boldsymbol{b}+\boldsymbol{\eta}$, where $\boldsymbol{b} = \langle b_1, \ldots, b_n\rangle$ is a sequence of elements of R and $\boldsymbol{\eta} = \langle \eta_1, \ldots, \eta_n\rangle$ is a sequence of elements of m. If f is an n-ary operation of T, we simply define $f(\boldsymbol{b}+\boldsymbol{\eta})$ by means of a Taylor expansion about $\boldsymbol{b}$, which terminates because the η_i are nilpotents.

If X is a T-model, we denote by $\mathrm{Spec}(X)$ the corresponding object of the dual category $(T\text{-mod})^{\mathrm{op}}$.

LEMMA 2. *Let B be a local* R*-algebra as above and let X be a T-model. Then the underlying* R*-algebra of the co-ordinate T-model of* $\mathrm{Spec}(X) \times \mathrm{Spec}(B)$ *is* $X \otimes_{\mathsf{R}} B$.

PROOF. First we can show that $X \otimes_{\mathbf{R}} B$ has a unique T-model structure, by exactly the same techniques as above. It is then straight forward to show that this T-model is the co-product of X and B in T-mod.

LEMMA 3. *Let B a finite dimensional local* $\mathbf{R}$*-algebra with residue field* $\mathbf{R}$. *Then* Spec (B) *is exponentiable in* $(T\text{-mod})^{\mathrm{op}}$.

PROOF. Follow exactly the same lines as that for the case of $\mathbf{R}$-algebras.

We then get

$$\begin{aligned} A &= \operatorname{Spec}(C^\infty(\mathbf{R})) \\ D_0 &= \operatorname{Spec}(\mathbf{R}) \\ D_1 &= \operatorname{Spec}(\mathbf{R}[X]/(X^2)) \\ D_2 &= \operatorname{Spec}(\mathbf{R}[X, Y]/(X^2, Y^2, XY)) \end{aligned}$$

To show that D_1 is given as above, we use the fact that every smooth function f can be written in the form $f(x) = a + bx + x^2 g(x)$.

Then every object of $(T\text{-mod})^{\mathrm{op}}$ satisfies axioms 1) and 2). Indeed, the use of Lemma 2 makes the proof identical to that for Example 1.

EXAMPLE 4. Let $\boldsymbol{E}$ be the category of formal k-schemes, where k is a field. A formal k-scheme is a finite $\varprojlim$ preserving set-valued functor on the category of finite dimensional k-algebras. The category $\boldsymbol{E}$ is equivalent to the category of commutative k-coalgebras and is equivalent to the dual of the category of linearly compact topological k-algebraic and continuous maps (cf. Demazure [3]). It is cartesian closed.

We let A be the forgetful functor from finite dimensional k-algebras to sets. It is a ring object of line type. As in the example 1), we have

$$\begin{cases} D_0 = h^k \\ D_1 = h^{k[x]/(x^2)} \\ D_2 = h^{k[x,y]/(x^2, y^2, xy)} \end{cases}$$

For any formal k-scheme X and any finite dimensional k-algebra B, we have

$$X^{h^B}(-) = X(B \otimes_k -) .$$

The verification of axioms 1) and 2) proceeds precisely as for affine schemes.

REFERENCES

1. A. Boileau, *Types versus Topos*, Ph. D. Thesis, Université de Montréal, 1976.
2. M. Coste, *Logique d'ordre supérieur dans les topos élèmentaires*, Séminaire Bénabou, Centre Scientifique et Polytechnique, St. Denis, 1974.
3. *M. Demazure*, *Lectures on p-divisible groups* (Lecture Notes in Mathematics 302), Springer-Verlag, Berlin - Heidelberg - New York, 1972.
4. M. Demazure et P. Gabriel, *Groupes Algébriques* I, North Holland Publ. co., Amsterdam, 1970.
5. M. Demazure et A. Grothendieck, *Seminaire de Geometrie Algebrique* 1963, *Schemas en Groupes* (S.G.A.D.), fasc. 1, exposé II, Institute des Hautes Études Scientifique, 1963.
6. M. Hakim, *Topos annéles et schemas relatifs* (Ergebnisse Math. 64), Springer-Verlag, Berlin - Heidelberg - New York, 1972.
7. A. Kock, *A simple axiomatics for differentiation*, Math. Scand. (to appear).
8. A. Kock, *Taylor series calculus for ring objects of line type*, Preprint No. 4 1976/77, Aarhus Universitet, 1976.
9. A. Kock and G. C. Wraith, *Elementary toposes*, Lecture Notes No. 30, Aarhus Universitet, 1971.
10. F. W. Lawvere, *Categorical dynamics*, Lecture at Chicago University, 1967.
11. G. Osius, *Logical and set theoretical methods in elementary topoi*, in *Model Theory and Topoi*, ed. F. W. Lawvere, C. Maurer, and G. C. Wraith (Lecture Notes in Mathematics 445), Springer-Verlag, Berlin - Heidelberg - New York, 1975.
12. R. S. Palais, *Equivariant Real Algebraic Differential Topology*, Lectures dedicated to J. Nash. Preprint, Brandeis University, 1972.
13. J. P. Serre, *Lie Algebras and Lie Groups*, W. A. Benjamin, New York, 1975.
14. E. H. Spanier, *Algebraic Topology*, McGraw–Hill, New York - London - Sydney, 1966.

DEPARTMENT OF MATHEMATICS
UNIVERSITÉ DE MONTRÉAL
CANADA

AND

SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES
UNIVERSITY OF SUSSEX
ENGLAND