MATH. SCAND. 42 (1978), 64–70.

THE IVERSEN THEOREM IN A POLYDISC

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1. Introduction.

One of the fundamental results in the study of the boundary behavior of meromorphic functions of a single complex variable is the Iversen Theorem [5]. This result, which may be viewed as an extended form of a local maximum principle, states that the boundary of the cluster set is contained in the radial boundary cluster set modulo a set of measure zero. An excellent account of the extensions of the Iversen Theorem and its significance in the theory of functions may be found in Collingwood and Lohwater [2, Chapter 5] (cf. also Lohwater [6, Chapter 5] and the references contained therein).

It was Max Weiss [7] who first pointed out the relationship between the Iversen Theorem and its analogue in the algebra of bounded analytic functions of a single complex variable. Recently, T. W. Gamelin [3] studied the connection between the Iversen Theorem and its abstract analogues in a function algebraic setting. In particular, in [3] Gamelin proved the Iversen Theorem for bounded analytic functions defined on polydomains. Subsequently, J. B. Garnett [4] used the Vitushkin localization operator in order to simplify Gamelin's proof.

The purpose of this paper is to provide a direct and geometric proof of the Iversen Theorem for meromorphic functions defined on a polydisc. Our proof is based on a result which is of independent interest in itself. Theorem 2 is an extension of a result of Carathéodory [1] to the polydisc. This theorem asserts that the cluster set of a bounded analytic function f at a point P on the distinguished boundary of the polydisc is contained in the closure of the convex hull of the set of radial limit values of f in a vicinity of P.

For the sake of simplicity, we shall present our theorems in the unit polydisc in $\mathbf{C}^2$. We remark, however, that our results remain valid for a polydisc in $\mathbf{C}^n$, where $n \geqq 2$.

2.

Throughout this paper, U^2 will denote the unit polydisc in $\mathbf{C}^2$, T^2 will denote the distinguished boundary of U^2, and m_2 will denote the normalized Lebesgue measure on T^2, i.e., $m_2(T^2)=1$.

Received May 20, 1977.

We begin with a lemma.

LEMMA 1. *Let $f(z,w)$ be analytic and bounded, $|f|<M$, in U^2, and let $\varepsilon>0$. Let G be a subset of T^2, with $m_2(G)<\varepsilon$, such that the radial limit*

$$f^\star(e^{i\theta},e^{i\varphi}) = \lim_{r\to 1} f(re^{i\theta},re^{i\varphi})$$

exists for $(e^{i\theta},e^{i\varphi}) \in G^c = T^2 - G$. Then $f(0,0)=\xi_1+\xi_2$, where ξ_1 lies in $\bar{\hat{S}}$, the closure of the convex hull of

$$S = \{f^\star(e^{i\theta},e^{i\varphi}) \mid (e^{i\theta},e^{i\varphi}) \in G^c\} ,$$

and where $|\xi_2|<2\varepsilon M$.

PROOF. We shall prove the lemma with the aid of an auxiliary function g on T^2 defined by

$$g = \begin{cases} f^\star & \text{on } G^c \\ f^\star(1,1) & \text{on } G \end{cases},$$

where we have tacitly assumed that the point $(1,1)$ lies in G^c. Since $f(z,w)$ is analytic and bounded in U^2, it follows that

$$\begin{aligned} f(0,0) &= \int_{T^2} f^\star\, dm_2 \\ &= \int_{T^2} g\, dm_2 + \int_{T^2} (f^\star - g)\, dm_2 \\ &= \int_{T^2} g\, dm_2 + \int_G (f^\star - g)\, dm_2 . \end{aligned}$$

Next we set $f(0,0)=\xi_1+\xi_2$, where

$$\xi_1 = \int_{T^2} g\, dm_2 \quad \text{and} \quad \xi_2 = \int_G (f^\star - g)\, dm_2 .$$

Then the assumptions $|f|<M$ and $m_2(G)<\varepsilon$ imply that $|\xi_2|<2\varepsilon M$. Thus, in order to complete the proof of the lemma, it suffices to show that ξ_1 lies in $\bar{\hat{S}}$, the closure of the convex hull of S.

Consider a closed half-plane H which contains $\bar{\hat{S}}$. It suffices to show that ξ_1 lies in H. We may choose real numbers θ_0 and α such that $e^{i\theta_0}H+i\alpha$ is the closed upper half-plane. Since the range of g lies in $S \subseteqq H$, $h=e^{i\theta_0}g+i\alpha$ satisfies $\operatorname{Im} h \geqq 0$. Since m_2 is a probability measure,

$$
\begin{aligned}
0 &\leqq \int_{T^2} (\operatorname{Im} h)\, dm_2 \\
&= \operatorname{Im} \int_{T^2} h\, dm_2 \\
&= \operatorname{Im}\left(e^{i\theta_0} \int_{T^2} g\, dm_2 + i\alpha\right).
\end{aligned}
$$

It follows that $\xi_1 = \int_{T^2} g\, dm_2$ lies in H, and so the lemma is proved.

Let $\zeta = f(z, w)$ be an extended complex-valued function defined on the polydisc U^2, and let P be a point on ∂U^2, the boundary of U^2. The *cluster set* $C(f, P)$ *of* f *at the point* P is the set of all values ζ in the extended complex plane such that there exists a sequence $\{(z_n, w_n)\}$ in U^2 with the properties that $(z_n, w_n) \to P$ and $f(z_n, w_n) \to \zeta$.

Preliminaries aside, we shall now apply our lemma to prove the following extension of a theorem of Carathéodory [1] (cf. also Collingwood and Lohwater [2, p. 96]).

THEOREM 2. *Let $f(z, w)$ be analytic and bounded, $|f| < M$, in U^2. Let A denote the set*

$$\{(e^{i\theta}, e^{i\varphi}) \mid \theta_1 < \theta < \theta_2, \varphi_1 < \varphi < \varphi_2\}.$$

If for every point $(e^{i\theta}, e^{i\varphi}) \in A - E$, where $m_2(E) = 0$, the radial limit

$$f^\star(e^{i\theta}, e^{i\varphi}) = \lim_{r \to 1} f(re^{i\theta}, re^{i\varphi})$$

exists and lies in a set V, then, for every point $P \in A$, the cluster set $C(f, P)$ is contained in $\bar{\bar{V}}$, the closure of the convex hull of V.

PROOF. Since f is analytic and bounded in U^2, it follows from a theorem of Marcinkiewicz and Zygmund [8, p. 316] that there is a set $E_1 \subseteqq T^2$ of measure zero such that $f(z, w) \to f^\star(e^{i\theta}, e^{i\varphi})$ uniformly, whenever $(z, w) \to (e^{i\theta}, e^{i\varphi})$ from inside any fixed Stolz domain whose vertex is at the point $(e^{i\theta}, e^{i\varphi}) \in E_1^c$. We shall assume, without any circumlocution, that the negligible set E, $m_2(E) = 0$, mentioned in the theorem includes the set E_1.

For a suitable choice of the point (z_0, w_0) in U^2, the map

$$(\zeta, \eta) = \Phi(z, w) = \left(\frac{z_0 - z}{1 - \bar{z}_0 z}, \frac{w_0 - w}{1 - \bar{w}_0 w}\right)$$

transforms the set A into a product set $B \subseteqq T^2$, whose area we may write as

$1-\varepsilon/2$. Moreover, Φ takes the set E, $m_2(E)=0$, into a set $F \subseteqq T^2$ whose measure is also zero.

We shall next cover the set $F \cup (T^2 - B)$ with an open set G such that $m_2(G) < \varepsilon$, and consider the function

$$g(\zeta,\eta) = f \circ \Phi^{-1}(\zeta,\eta) = f\left(\frac{z_0-\zeta}{1-\bar{z}_0\zeta}, \frac{w_0-\eta}{1-\bar{w}_0\eta}\right).$$

Lemma 1 allows us to write

$$f(z_0, w_0) = g(0,0) = \xi_1 + \xi_2 ,$$

where $|\xi_2| < 2\varepsilon M$, and where ξ_1 lies in the closure of the convex hull of the set

$$S = \{g^\star(e^{i\alpha}, e^{i\beta}) \mid (e^{i\alpha}, e^{i\beta}) \in G^c\} .$$

We will show that $S \subseteqq V$. Then $\xi_1 \in \bar{V}$. The proof will then be complete, because ε tends to zero with $|P-(z_0, w_0)|$.

If $(e^{i\alpha}, e^{i\beta}) \in G^c$, then the point $(e^{i\theta}, e^{i\varphi}) = \Phi^{-1}(e^{i\alpha}, e^{i\beta})$ is an element of $A-E$. Moreover, $g(re^{i\alpha}, re^{i\beta}) = f(r_1 e^{i\theta}, r_2 e^{i\varphi})$, where r_1 and r_2 tend to 1 as r tends to 1. Now it follows from the result of Marcinkiewicz and Zygmund, cited at the beginning of this proof, that

$$g^\star(e^{i\alpha}, e^{i\beta}) = \lim_{r\to 1} g(re^{i\alpha}, re^{i\beta}) = f^\star(e^{i\theta}, e^{i\varphi}) .$$

Since $f^\star(e^{i\theta}, e^{i\varphi})$ lies in V by hypothesis, $S \subseteqq V$, and the proof of the theorem is complete.

THEOREM 3. *Let $f(z,w)$ be analytic and bounded in U^2, let $(e^{i\theta_0}, e^{i\varphi_0})$ be an arbitrary point on T^2, and let E be an any subset of T^2 with $m_2(E)=0$. Then*

$$\overline{\lim_{(z,w)\to(e^{i\theta_0}, e^{i\varphi_0})}} |f(z,w)| \leqq \overline{\lim_{(\theta,\varphi)\to(\theta_0,\varphi_0)}} \Big(\overline{\lim_{\substack{(z,w)\to(e^{i\theta}, e^{i\varphi}) \\ (e^{i\theta}, e^{i\varphi}) \notin E}}} |f(z,w)|\Big) .$$

PROOF. Let E_1 denote the set of measure zero on T^2 for which

$$f^\star(e^{i\theta}, e^{i\varphi}) = \lim_{r\to 1} f(re^{i\theta}, re^{i\varphi})$$

fails to exist. If $E_2 = E \cup E_1$, then

$$\begin{aligned} &\overline{\lim_{(\theta,\varphi)\to(\theta_0,\varphi_0)}} \Big(\overline{\lim_{\substack{(z,w)\to(e^{i\theta}, e^{i\varphi}) \\ (e^{i\theta}, e^{i\varphi}) \notin E}}} |f(z,w)|\Big) \\ \geqq\ &\overline{\lim_{(\theta,\varphi)\to(\theta_0,\varphi_0)}} \Big(\overline{\lim_{\substack{(z,w)\to(e^{i\theta}, e^{i\varphi}) \\ (e^{i\theta}, e^{i\varphi}) \notin E_2}}} |f(z,w)|\Big) \\ \geqq\ &\overline{\lim_{\substack{(\theta,\varphi)\to(\theta_0,\varphi_0) \\ (e^{i\theta}, e^{i\varphi}) \notin E_2}}} |f^\star(e^{i\theta}, e^{i\varphi})| . \end{aligned}$$

Let $\delta > 0$, and let A_δ denote the set

$$\{(e^{i\theta}, e^{i\varphi}) \mid \theta_0 - \delta < \theta < \theta_0 + \delta,\ \varphi_0 - \delta < \varphi < \varphi_0 + \delta\}\ .$$

By Theorem 2, the cluster set $C(f, (e^{i\theta_0}, e^{i\varphi_0}))$ is contained in the closure of the convex hull of the set

$$V_\delta = \{f^\star(e^{i\theta}, e^{i\varphi}) \mid (e^{i\theta}, e^{i\varphi}) \in A_\delta - E_2\}\ .$$

Since $\delta > 0$ is arbitrary, it follows that

$$\overline{\lim_{\substack{(\theta,\varphi)\to(\theta_0,\varphi_0)\\ (e^{i\theta}, e^{i\varphi}) \notin E_2}}} |f^\star(e^{i\theta}, e^{i\varphi})| \geqq \overline{\lim_{(z,w)\to(e^{i\theta_0}, e^{i\varphi_0})}} |f(z,w)|\ ,$$

and thus the proof of Theorem 3 is complete.

Alternatively, Theorem 3 may be stated as

THEOREM 4. *Let $f(z,w)$ be analytic and bounded in U^2, let $(e^{i\theta_0}, e^{i\varphi_0})$ be any point on T^2, and let E be an arbitrary set of measure zero on T^2. Then*

$$\overline{\lim_{(z,w)\to(e^{i\theta_0}, e^{i\varphi_0})}} |f(z,w)| \leqq \overline{\lim_{\substack{(\theta,\varphi)\to(\theta_0,\varphi_0)\\ (e^{i\theta}, e^{i\varphi}) \notin E}}} |f^\star(e^{i\theta}, e^{i\varphi})|\ .$$

In order to prove our main result, the Iversen Theorem for meromorphic functions in the polydisc U^2, we introduce some additional terminology and definitions.

Let $\zeta = f(z,w)$ be an extended complex-valued function defined on the polydisc U^2. The *radial cluster set of f at the point $(e^{i\theta}, e^{i\varphi})$ on T^2* is denoted by $C_{\mathrm{rad}}(f, (e^{i\theta}, e^{i\varphi}))$, and is defined as the set of all limiting values of $f(z,w)$ when the defining sequences $\{(z_n, w_n)\}$ satisfy $(z_n/|z_n|, w_n/|w_n|) = (e^{i\theta}, e^{i\varphi})$ for all n. We define the *radial boundary cluster set modulo E, $E \subseteqq T^2$, of f at the point* $(e^{i\theta_0}, e^{i\varphi_0})$ to be the set

$$C_{R-E}(f, (e^{i\theta_0}, e^{i\varphi_0})) = \bigcap_{\eta > 0} \overline{\{\bigcup C_{\mathrm{rad}}(f, (e^{i\theta}, e^{i\varphi}))\}}\ ,$$

where the union is taken over all $(e^{i\theta}, e^{i\varphi})$ such that $0 < |(\theta,\varphi) - (\theta_0, \varphi_0)| < \eta$ and $(e^{i\theta}, e^{i\varphi}) \notin E$, and where the bar denotes the closure operator.

We are now in a position to state and prove the principal result of this paper.

THEOREM 5. *If $f(z,w)$ is a meromorphic function in U^2, and if $E \subseteqq T^2$ is a set of measure zero, then at every point $(e^{i\theta_0}, e^{i\varphi_0}) \in T^2$,*

$$\partial C(f, (e^{i\theta_0}, e^{i\varphi_0})) \subseteqq C_{R-E}(f, (e^{i\theta_0}, e^{i\varphi_0}))\ ,$$

where ∂C denotes the boundary of C.

PROOF. If Theorem 5 were not true, there would be a point $\zeta_0 \in \partial C(f, (e^{i\theta_0}, e^{i\varphi_0}))$ not in $C_{R-E}(f, (e^{i\theta_0}, e^{i\varphi_0}))$. Thus, we would be able to find a $\delta > 0$ such that ζ_0 is at a distance greater than δ from all points of $C_{R-E}(f, (e^{i\theta_0}, e^{i\varphi_0}))$. Since ζ_0 is a boundary point of $C(f, (e^{i\theta_0}, e^{i\varphi_0}))$, we can find a second point ζ_1 not in $C(f, (e^{i\theta_0}, e^{i\varphi_0}))$ such that $|\zeta_1 - \zeta_0| < \frac{1}{2}\delta$; this implies that the distance between ζ_1 and any point of $C_{R-E}(f, (e^{i\theta_0}, e^{i\varphi_0}))$ is greater than $\frac{1}{2}\delta$. Since the function $g = [f - \zeta_1]^{-1}$ is bounded in a neighbourhood of $(e^{i\theta_0}, e^{i\varphi_0})$, we can apply Theorem 4 to g and obtain

$$
\begin{aligned}
\frac{2}{\delta} &< \frac{1}{|\zeta_0 - \zeta_1|} \\
&\leqq \overline{\lim_{(z, w) \to (e^{i\theta_0}, e^{i\varphi_0})}} |g(z, w)| \\
&\leqq \overline{\lim_{\substack{(\theta, \varphi) \to (\theta_0, \varphi_0) \\ (e^{i\theta}, e^{i\varphi}) \notin E}}} |g^\star(e^{i\theta}, e^{i\varphi})| \\
&\leqq \frac{2}{\delta}.
\end{aligned}
$$

This contradiction establishes the theorem.

Since $C_{R-E}(f, (e^{i\theta}, e^{i\varphi}))$ is a subset of $C(f, (e^{i\theta}, e^{i\varphi}))$, the inclusion relation in Theorem 5 is equivalent to the assertion that the difference set $C(f, (e^{i\theta}, e^{i\varphi})) - C_{R-E}(f, (e^{i\theta}, e^{i\varphi}))$ is open.

We remark, in conclusion, that our proof of Theorem 5 yields the following stronger result:

$$
\partial C(f, (e^{i\theta_0}, e^{i\varphi_0})) \subseteqq \bigcap_{\eta > 0} \overline{\Big\{ \bigcup_{\substack{0 < |(\theta, \varphi) - (\theta_0, \varphi_0)| < \eta \\ (e^{i\theta}, e^{i\varphi}) \notin E}} C^*_{\mathrm{rad}}(f, (e^{i\theta}, e^{i\varphi})) \Big\}},
$$

where $C^*_{\mathrm{rad}}(f, (e^{i\theta}, e^{i\varphi}))$ denotes the set of all points ζ such that there is a sequence $\{(z_n, w_n)\}$ in U^2 with the properties that $(z_n, w_n) = (r_n e^{i\theta}, r_n e^{i\varphi})$ for all n, $r_n \to 1$, and $f(z_n, w_n) \to \zeta$.

A more detailed treatment of the relations between the various cluster sets of a function defined on the polydisc will be provided by the authors in a forthcoming paper.

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