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SOME C*-DYNAMICAL SYSTEMS WITH A SINGLE KMS STATE

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Introduction.

We show that a C*-dynamical system is not approximately inner, if the C*-algebra contains an isometry which is an eigenoperator for the dynamics, corresponding to a non-zero character. The simple C*untz-algebras $\mathcal{O}_n$, $2 \leqq n \leqq \infty$, offer examples of such dynamical systems. We show that each C*-algebra $\mathcal{O}_n$ has exactly one KMS state φ_n. For $n<\infty$ the corresponding β-value (inverse temperature) is $\log n$. For $n=\infty$ the value for β is ∞ (i.e. φ_∞ is a ground state). These examples solve problems 11 and 13 in [11]. An example of a system with a unique invariant trace (corresponding to $\beta=0$), but no ground states (thus not approximately inner) was given in [6].

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Approximately inner dynamics.

A C*-dynamical system $(\mathscr{A}, \mathsf{R}, \varrho)$ consists of a C*-algebra $\mathscr{A}$ and a representation ϱ of R as *-automorphisms of $\mathscr{A}$, such that each function $t \to \varrho_t(A)$, $A \in \mathscr{A}$, is (norm) continuous. We denote by $\mathscr{A}^{\mathrm{a}}$ the dense *-subalgebra of $\mathscr{A}$ consisting of analytic elements. Thus $A \in \mathscr{A}^{\mathrm{a}}$ if the function $t \to \varrho_t(A)$ has an extension (necessarily unique) to an entire analytic (operator-valued) function. If $0 \leqq \beta < \infty$ we say that an invariant state φ of $\mathscr{A}$ is a β-KMS state if

$$\text{(i)} \qquad \varphi(\varrho_t(A)B) = \varphi(B\varrho_{t+i\beta}(A))$$

for all t, all B in $\mathscr{A}$ and A in $\mathscr{A}^{\mathrm{a}}$. Note that the limit case $\beta=0$ (chaotic state) means that φ is an invariant trace. In the other limit, $\beta=\infty$ (ground state), we modify the definition (i) to the demand that each analytic function

$$\text{(ii)} \qquad \zeta \to \varphi(B\varrho_\zeta(A)), \quad A \in \mathscr{A}^{\mathrm{a}},\ B \in \mathscr{A}\ ,$$

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is bounded (by $\|A\|\,\|B\|$) in the upper half plane. This version of the KMS conditions differs slightly from the traditional approach. It is, however, equivalent to it (cf. [8, 8.12.13 Lemma]) and better suited to the present situation.

We say that ϱ is approximately inner if there is a net $\{H_j\}$ in $\mathscr{A}_{sa}$ such that

$$\text{(iii)} \qquad \|\exp(itH_j)A\exp(-itH_j)-\varrho_t(A)\| \to 0\,,$$

uniformly in t on compact subsets of R for each A in $\mathscr{A}$. It was proved in [9, Theorem 2.3] that if $\mathscr{A}$ has a unit and ϱ is approximately inner there is a ground state for the system $(\mathscr{A}, \mathsf{R}, \varrho)$. Moreover, by [9, Theorem 3.2], if there is an invariant finite trace on $\mathscr{A}$ (corresponding to $\beta=0$) there is also a β-KMS state for any $\beta>0$. Under mild extra assumptions on ϱ (e.g. (iii) being valid for complex scalars if $A \in \mathscr{A}^a$ [8, 8.12.6], or even less, see [5] for the strongest known result) the existence of some β_0-KMS state, $\beta_0>0$, will imply the existence of β-KMS states for all $\beta \geqq 0$.

The condition of being approximately inner is satisfied in C*-dynamical systems of interest in C*-physics. Indeed, it is expected that every continuous one-parameter group of automorphisms of the Fermion algebra is approximately inner. Our first result put certain restrictions on the systems that are approximately inner.

THEOREM 1. *Let $(\mathscr{A}, \mathsf{R}, \varrho)$ be a C*-dynamical system. If there is an isometry V in $\mathscr{A}$ (i.e. $V^*V=1$) which is an eigenoperator for ϱ, but not a fixed point (i.e. $\varrho_t(V)=\exp(ist)V$, $s \neq 0$), then ϱ is not approximately inner.*

PROOF. Assume, to obtain a contradiction, that there is a net $\{H_j\}$ in $\mathscr{A}_{sa}$ satisfying (iii). Let V be an isometry of $\mathscr{A}$ such that $\varrho_t(V)=\exp(ist)V$, $s \neq 0$, and without loss of generality assume that $s<0$. Adding a suitable scalar multiple of 1 to each H_j we may further assume that $H_j \geqq 0$ and $0 \in \mathrm{Sp}(H_j)$ for every j. From (iii) we obtain

$$\text{(iv)} \qquad \|\exp(itH_j)V - V\exp(it(H_j+s1))\| \to 0$$

uniformly on compact subsets of R. Since the expression in (iv) is bounded as a function of t this implies that

$$\left\|\int (\exp(itH_j)V - V\exp(it(H_j+s1)))f(t)\,dt\right\| = \|\hat{f}(H_j)V - V\hat{f}(H_j+s1)\| \to 0$$

for every f in $L^1(\mathsf{R})$. Since $V^*V=1$ this gives

$$\text{(v)} \qquad \|V^*\hat{f}(H_j)V - \hat{f}(H_j+s1)\| \to 0\,.$$

Choose f such that $\hat{f}(s)=1$ and $\operatorname{supp}\hat{f}\subset]-\infty,0[$. Then $\hat{f}(H_j)=0$ since $H_j\geqq 0$, but $\|\hat{f}(H_j+s1\|\geqq 1$ because $s\in \operatorname{Sp}(H_j+s1)$. This contradicts (v).

The theorem above is valid in more general situations. Indeed, let G be any locally compact abelian group, and consider a C*-dynamical system $(\mathscr{A},G,\varrho)$. We say that ϱ is approximately uniformly continuous if there is a net $\{\varrho^{(j)}\}$ of uniformly continuous representations of G in $\operatorname{Aut}(\mathscr{A})$ such that

$$\|\varrho^{(j)}{}_t(A)-\varrho_t(A)\| \to 0.$$

uniformly on compact subsets of G for each A in $\mathscr{A}$. In the one-parameter case this condition, although formally weaker, is equivalent to ϱ being approximately inner (cf. [8, 8.12.7 Proposition]).

THEOREM 1′. *Let $(\mathscr{A},G,\varrho)$ be a C*-dynamical system where G is connected. If there is an isometry V in $\mathscr{A}$ such that $\varrho_t(V)=(t,\gamma)V$ for some non-zero γ in $\hat{G}$, then ϱ is not approximately uniformly continuous.*

PROOF. By van Kampen's theorem $G=\mathbf{R}^n\times G_0$, where G_0 is compact. Write $\gamma=(\gamma_1,\gamma_0)$ with γ_1 in $\hat{\mathbf{R}}^n$ and γ_0 in $\hat{G}_0$. If $\gamma_1\neq 0$ the result follows from Theorem 1 by restricting ϱ first to $\mathbf{R}^n$ and then further to $\mathbf{R}$, for a suitably chosen one-parameter subgroup of $\mathbf{R}^n$.

If $\gamma_1=0$ we may assume that G is compact. Since G is connected, $\hat{G}$ has no elements of finite order. Thus $\hat{G}$ can be totally ordered by a "positive" semi-group $\hat{G}_+$ such that

$$-\hat{G}_+\cap\hat{G}_+ = \{0\}, \qquad -\hat{G}_+\cup\hat{G}_+ = \hat{G}$$

(see e.g. [8, 8.4.2 Lemma]). Without loss of generality we may assume that $\gamma\notin\hat{G}_+$. It follows from Arveson's theory of spectral subspaces that each uniformly continuous representation $\varrho^{(j)}$ of G in $\operatorname{Aut}(\mathscr{A})$ is implemented by a uniformly continuous unitary group $\{U^{(j)}{}_t \mid t\in G\}$ in $\mathscr{A}''$ with $\operatorname{Sp}(U^{(j)})\subset\hat{G}_+$ (see [7, Proposition 5.1] or [8, 8.5.2 Theorem]). Since $U^{(j)}$ is the minimal positive representation of G implementing $\varrho^{(j)}$, and $\operatorname{Sp}(U^{(j)})$ is finite it follows that $0\in\operatorname{Sp}(U^{(j)})$. Now assume that $\varrho^{(j)}\to\varrho$ pointwise in norm on $\mathscr{A}$, and proceed to obtain a contradiction exactly as in the proof of Theorem 1.

REMARK. To each C*-dynamical system $(\mathscr{A},G,\varrho)$ corresponds a dual system $(\mathscr{A}\times_\varrho G,\hat{G},\hat{\varrho})$ by [12, Proposition 3.1]. Here $\mathscr{A}\times_\varrho G$ is the crossed product of $\mathscr{A}$ with G and $\hat{\varrho}$ is the dual action of G on $\mathscr{A}\times_\varrho G$ given by

$$\hat{\varrho}_\gamma(B)(t) = (t,\gamma)B(t)$$

for each continuous function $B\colon G\to\mathscr{A}$ with compact support (and these elements lie dense in $\mathscr{A} \times_\varrho G$). Each element t in G is embedded as a unitary multiplier U_t of $\mathscr{A} \times_\varrho G$, where U_t may be regarded as a δ-function on G. It follows that $\hat{\varrho}_\gamma(U_t) = (t,\gamma)U_t$. From Theorem 1′ we see that when $\hat{G}$ is connected (i.e. G is torsion free), then the dual action $\hat{\varrho}$ is never approximately uniformly continuous on $\mathscr{A} \times_\varrho G$.

Cuntz's C*-Algebras.

We recall some of the results from [2]. Let $2 \leqq n \leqq \infty$. Given n isometries $\{S_k\}$ on a Hilbert space $\mathscr{H}$ such that $\sum S_k S_k^* = 1$ if $n < \infty$ and $\sum S_k S_k^* \leqq 1$ if $n = \infty$, we denote by $\mathcal{O}_n$ the C*-subalgebra of $\mathscr{B}(\mathscr{H})$ generated by $\{S_k\}$. Each $\mathcal{O}_n$ is a simple, separable C*-algebra with unit, admitting no non-zero finite (or semi-finite) traces ($\mathcal{O}_n$ is a C*-algebra of type III in the language of [3]). Moreover, $\mathcal{O}_n$ is independent of $\mathscr{H}$.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a multiindex of length $l = l(\alpha)$, where $1 \leqq \alpha_k < n+1$ for each k, we define

$$S_\alpha = S_{\alpha_1} S_{\alpha_2} \ldots S_{\alpha_l} .$$

The C*-subalgebra $\mathscr{F}_n$ of $\mathcal{O}_n$ generated by 1 together with elements of the form $S_\alpha S_\beta^*$, where $l(\alpha) = l(\beta)$, is isomorphic to the Glimm algebra $\otimes^\infty M_n$ if $n < \infty$. If $n = \infty$, the algebra $\mathscr{F}_\infty$ is the example $\mathscr{A}_{\mathbf{N}}$ in [1] (see also [4, 4.7.17]). Thus $\mathscr{F}_\infty$ is primitive, but not simple, and its non-trivial closed ideals form a decreasing sequence $(\mathscr{I}_n)$ such that

$$\mathscr{F}_\infty / \mathscr{I}_1 = \mathbf{C}, \quad \mathscr{I}_n / \mathscr{I}_{n+1} = \mathscr{C}(\mathscr{H}), \quad n \in \mathbf{N} .$$

In any case there is a unique tracial state τ_n on $\mathscr{F}_n$. If $n < \infty$ we have $\tau_n = \otimes^\infty (\frac{1}{n}\mathrm{Tr})$. If $n = \infty$, τ_∞ is the complex homomorphism $\mathscr{F}_\infty \to \mathscr{F}_\infty / \mathscr{I}_1$.

The *-algebra generated algebraically by $\{S_k\}$ is denoted by $\mathscr{P}_n$. Each element A in $\mathscr{P}_n$ has a unique representation.

$$\text{(vi)} \qquad A = \sum_k S_1^{k*} A_{-k} + A_0 + \sum_k A_k S_1^k ,$$

where $A_k \in \mathscr{P}_n \cap \mathscr{F}_n$ for all k in $\mathbf{N}$. Moreover, $\|A_k\| \leqq \|A\|$ for every k and the map $\pi_0 \colon \mathcal{O}_n \to \mathscr{F}_n$ determined by $\pi_0(A) = A_0$ if $A \in \mathscr{P}_n$, is a projection of norm one. Note that if A and B are elements in $\mathscr{P}_n$ represented as in (vi) then

$$\text{(vii)} \qquad \pi_0(AB) = \sum_k S_1^{k*} A_{-k} B_k S_1^k + A_0 B_0 + \sum_k A_k S_1^k S_1^{k*} B_{-k} .$$

LEMMA 1. (cf. [2, 1.10. Proof of Proposition]). *For each point e^{it} on the circle $\mathbf{T}$ define $\varrho_t(S_k) = e^{it} S_k$ for all k. Then ϱ_t has a unique extension to a *-automorphism of $\mathcal{O}_n$, and we obtain a C*-dynamical system $(\mathcal{O}_n, \mathbf{T}, \varrho)$. Moreover, $\mathscr{F}_n$ is the fixed-point algebra of ϱ in $\mathcal{O}_n$ and $\mathscr{P}_n \subset (\mathcal{O}_n)^a$.*

PROOF. The only choice of $\varrho_t(A)$, if $A \in \mathscr{P}_n$ represented as in (vi) is

$$\varrho_t(A) = \sum_k e^{-ikt} S_1^{k*} A_{-k} + A_0 + \sum_k e^{ikt} A_k S_1^k \tag{viii}$$

With this definition ϱ_t becomes a linear *-preserving continuous map of $\mathscr{P}_n$ onto itself with inverse ϱ_{-t}. Take A, B in $\mathscr{P}_n$ represented as in (vi), and show by straightforward computations that $\varrho_t(AB) = \varrho_t(A)\varrho_t(B)$. Thus each ϱ_t is a *-automorphism of $\mathscr{P}_n$ and extends by continuity to a *-automorphism ϱ_t of $\mathcal{O}_n$.

Clearly the function $t \to \varrho_t$ is a representation of T (identified with $\mathbf{R}/2\pi\mathbf{Z}$) in Aut $(\mathcal{O}_n)$. Moreover, since $\|A_k\| \leq \|A\|$ for all k and each A in $\mathscr{P}_n$, it follows from the expression (viii) that $t \to \varrho_t(A)$ is continuous for each A in $\mathscr{P}_n$, and therefore also for each A in $\mathcal{O}_n$. Thus $(\mathcal{O}_n, \mathrm{T}, \varrho)$ is a C*-dynamical system.

If $B \in \mathscr{P}_n \cap \mathscr{F}_n$ then B is a fixed-point for ϱ. By continuity $\mathscr{F}_n \subset (\mathcal{O}_n)^\varrho$. Conversely, if $B \in (\mathcal{O}_n)^\varrho$, take $\varepsilon > 0$ and choose A in $\mathscr{P}_n$ such that $\|B-A\| < \varepsilon$. Using the expression (viii) we get

$$\|B - A_0\| = \left\| \int_{\mathrm{T}} \varrho_t(B-A)\, dt \right\| \leq \|B-A\| < \varepsilon .$$

Since ε is arbitrary it follows that $B \in \mathscr{F}_n$. Finally, from the expression (viii) it is immediate that every A in $\mathscr{P}_n$ is analytic for ϱ, i.e. $\mathscr{P}_n \subset (\mathcal{O}_n)^{\mathrm{a}}$.

LEMMA 2. *Let $\pi_0 \colon \mathcal{O}_n \to \mathscr{F}_n$ be the projection of norm one mentioned above. If τ_n is the unique tracial state on $\mathscr{F}_n$, define $\varphi_n = \tau_n \circ \pi_0$ on $\mathcal{O}_n$. The state φ_n satisfies the equations*:

$$\varphi_n(S_1 A S_1^*) = n^{-1}\varphi_n(A), \qquad A \in \mathcal{O}_n \qquad (\textit{with } \infty^{-1} = 0); \tag{ix}$$

$$\varphi_n(S_1^{k*} A S_1^k) = n^k \varphi_n(E_k A E_k), \qquad A \in \mathcal{O}_n \tag{x}$$

where $E_k = S_1^k S_1^{k*}$ and $n < \infty$.

PROOF. Since τ_n is a trace on $\mathscr{F}_n$ we have for all A, B in $\mathscr{F}_n$

$$\tau_n(S_1 A B S_1^*) = \tau_n((S_1 A S_1^*)(S_1 B S_1^*)) = \tau_n(S_1 B A S_1^*) .$$

Thus $\tau_n(S_1 \cdot S_1^*)$ is also a trace on $\mathscr{F}_n$, and the unicity of τ_n implies that $\tau_n(S_1 \cdot S_1^*) = \tau_n(S_1 S_1^*)\tau_n(\cdot)$. However, each element $V_k = S_1 S_k^*$ is a partial isometry in $\mathscr{F}_n$ with

$$V_k V_k^* = S_1 S_1^*, \qquad V_k^* V_k = S_k S_k^* .$$

If $n < \infty$ this implies that

$$1 = \sum \tau_n(S_k S_k^*) = n\tau_n(S_1 S_1^*) ,$$

whence $\tau_n(S_1S_1^*)=n^{-1}$. If $n=\infty$ the same reasoning gives $\tau_n(S_1S_1^*)=0$. In both cases we have (ix).

Assume now that $n<\infty$. Applying (ix) successively we obtain (x) since

$$\begin{aligned}\varphi_n(S_1^{k*}AS_1^k) &= n\varphi_n(S_1S_1^{k*}AS_1^kS_1^*)\\ = \ldots &= n^k\varphi_n(S_1^kS_1^{k*}AS_1^kS_1^{k*}) = n^k\varphi_n(E_kAE_k)\,.\end{aligned}$$

PROPOSITION. *Consider the* C*-*dynamical system* $(\mathcal{O}_n, \mathsf{T}, \varrho)$ *defined in Lemma* 1 *and let* φ_n *be the invariant state of* $\mathcal{O}_n$ *defined in Lemma* 2. *Then* φ_n *satisfies the* KMS *condition at* $\beta=\log n$ *if* $n<\infty$, *and* φ_∞ *is a ground state* (*i.e.* $\beta=\infty$) *for* $\mathcal{O}_\infty$.

PROOF. Assume first that $n<\infty$. Take A,B in $\mathcal{P}_n$, represented as in (vi). Since this representation is unique we have $E_kA_{-k}=A_{-k}$ and $A_kE_k=A_k$ for all $k>0$, where $E_k=S_1^kS_1^{k*}$. Thus by (vii) and (x) we get

$$\begin{aligned}\varphi_n(AB) &= \tau_n\Big(\sum_k S_1^{k*}A_{-k}B_kS_1^k+A_0B_0+\sum_k A_kE_kB_{-k}\Big)\\ &= \sum_k n^k\tau_n(A_{-k}B_k)+\tau_n(A_0B_0)+\sum_k \tau_n(A_kB_{-k})\,.\end{aligned}$$

Put $\zeta=i\log n$. Since $\varrho_t(S_1)=e^{it}S_1$ it follows that $\varrho_\zeta(S_1)=n^{-1}S_1$ and $\varrho_\zeta(S_1^*)=nS_1^*$. Thus by (vii) and (x) we get

$$\begin{aligned}\varphi_n(B\varrho_\zeta(A)) &= \tau_n\Big(\sum_k S_1^{k*}B_{-k}A_kn^{-k}S_1^k+B_0A_0+\sum_k B_kS_1^kn^kS_1^{k*}A_{-k}\Big)\\ &= \sum_k \tau_n(B_{-k}A_k)+\tau_n(B_0A_0)+\sum_k n^k\tau_n(B_kA_{-k})\,.\end{aligned}$$

Since τ_n is a trace on $\mathcal{F}_n$ we see from these results that $\varphi_n(AB)=\varphi_n(B\varrho_\zeta(A))$. Replacing A with $\varrho_t(A)$ we get

$$\varphi_n(\varrho_t(A)B) = \varphi_n(B\varrho_{t+i\log n}(A))\,.$$

Since $\mathcal{P}_n$ is dense in $(\mathcal{O}_n)^{\mathrm{a}}$ it follows that the equation above is valid for any A in $(\mathcal{O}_n)^{\mathrm{a}}$ and B in $\mathcal{O}_n$, so that φ_n is a $\log n$–KMS state.

If $n=\infty$, we know from (ix) that $\tau_\infty(E_k)=0$ for every k. Consequently, for A,B in $\mathcal{P}_n$ and $\zeta=t+is$ we have

$$\begin{aligned}&\varphi_\infty(B\varrho_\zeta(A))\\ &= \tau_\infty\Big(\sum_k S_1^{k*}B_{-k}A_ke^{ik\zeta}S_1^k+B_0A_0+\sum_k B_kS_1^ke^{-ik\zeta}S_1^{k*}A_{-k}\Big)\end{aligned}$$

$$= \tau_\infty\left(\sum_k e^{ik\zeta} S_1^{k*} B_{-k} A_k S_1^k + B_0 A_0 + \sum_k e^{-ik\zeta} B_k E_k A_{-k}\right)$$

$$= \sum_k e^{ikt} e^{-ks} \tau_\infty (S_1^{k*} B_{-k} A_k S_1^k) + \tau_\infty (B_0 A_0) .$$

Thus the function $\zeta \to \varphi_\infty(B\varrho_\zeta(A))$ is bounded in the upper half plane. By the Phragmen–Lindelöf theorem the supremum of the function in the region is the supremum on the real line, i.e.

$$|\varphi_\infty(B\varrho_\zeta(A))| \leqq \|B\| \, \|A\| .$$

Another application of the Phragmen–Lindelöf theorem shows that each function $\zeta \to \varphi_\infty(B\varrho_\zeta(A))$ is bounded in the upper half plane for every B in $\mathcal{O}_\infty$ and A in $(\mathcal{O}_\infty)^a$; i.e., φ_∞ is a ground state for $\mathcal{O}_\infty$.

THEOREM 2. *Let* $2 \leqq n \leqq \infty$. *The* C*-*dynamical system* $(\mathcal{O}_n, \mathsf{T}, \varrho)$ *defined in Lemma* 1 *has exactly one* KMS *state (viz.* φ_n*). The only admissible* β*-value is* $\log n$ *if* $n < \infty$ *and* ∞ *if* $n = \infty$.

PROOF. Assume first that $n < \infty$. From the Proposition we know that φ_n is a KMS state corresponding to $\beta = \log n$. Assume now that φ is a β-KMS state for some β $(0 \leqq \beta \leqq \infty)$. If $\beta < \infty$ this implies that φ is a trace on $\mathcal{F}_n$ (cf. (i)), whence $\varphi = \varphi_n$ by the unicity of τ_n. If $\beta = \infty$ we take $\zeta = t + is$ and compute

(xi) $$\varphi(S_k \varrho_\zeta(S_k^*)) = e^{-i\zeta} \varphi(S_k S_k^*) = e^{-it} e^s \varphi(S_k S_k^*) .$$

The function $\zeta \to \varphi(S_k \varrho_\zeta(S_k^*))$ is not bounded in the upper half plane unless $\varphi(S_k S_k^*) = 0$. But $\sum S_k S_k^* = 1$ and $\varphi(1) = 1$, a contradiction. Thus $\beta = \log n$ is the only admissible value.

In the case $n = \infty$ we know from the Proposition that φ_∞ is a ground state for $\mathcal{O}_\infty$. Assume now that φ is a β-KMS state for some β $(0 \leqq \beta \leqq \infty)$. If $\beta < \infty$ this gives

$$\varphi(S_k S_k^*) = \varphi(S_k^* \varrho_{i\beta}(S_k)) = e^{-\beta} \varphi(S_k^* S_k) = e^{-\beta} .$$

However, $\sum S_k S_k^* \leqq 1$ so we have a contradiction. If $\beta = \infty$ we see from (xi) that $\varphi(S_k S_k^*) = 0$ for all k. By the Cauchy–Schwarz inequality this implies that $\varphi(S_\alpha S_\beta^*) = 0$ for all multiindices α, β (with $l(\alpha) = l(\beta)$) unless $l(\alpha) = l(\beta) = 0$. But the elements $S_\alpha S_\beta^*$, $1 \leqq l(\alpha) = l(\beta)$, generate the maximal ideal $\mathcal{I}_1$ in $\mathcal{F}_\infty$ and $\mathcal{F}_\infty / \mathcal{I}_1 = \mathbf{C}$. Consequently $\varphi = \varphi_\infty$.

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