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A NON-SEPARABLE MEASURABLE CHOICE PRINCIPLE RELATED TO INDUCED REPRESENTATIONS

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1. Introduction.

Let G be a locally compact group, H a closed subgroup, K a topological group, and u a continuous homomorphism of H into K. We shall call a map P of G into K a ϱ-extension of u, if it is an extension of u satisfying

$$\forall g \in G\ \forall h \in H: \quad P(gh) = P(g)u(h) .$$

If T is a transversal containing the identity for the quotient map $\pi: G \to G/H$, then any identity preserving map of T into K is the restriction of a unique ϱ-extension, and conversely, if P is a ϱ-extension of an isomorphism, then the counter image of the identity is a transversal.

In some cases ϱ-extensions with useful continuity or measurability properties are known to exist, e.g. if G has countable basis for the topology, see [19, p. 104] and [20, p. 289], or if G is abelian and $K = \mathsf{T}$ or $K =]0,\infty[$, cf. also [9]. The main result of the paper is an existence theorem of this type in the case where K is the unitary group of a von Neumann algebra with the ultraweak topology, and in the case where K is a closed subgroup of the unitary group of a von Neumann algebra with separable predual.

In some cases we can deduce the existence of measurable cross sections; and ϱ-extensions can be utilized in the theory of induced representations in much the same way as Borel cross sections are used in the case of second countable groups, cf. [19], [20]. We give some examples, among these a generalization of Theorem 8.2 of [20], cf. [2] Theorem 2.3, dropping the condition of countable basis on the groups but keeping it on the Hilbert space. This combined with results of Blattner [3] makes the little group method, [20] Theorem 8.4, work for σ-compact groups.

In a following paper we shall give another proof of the Mackey–Blattner–Nielsen theorem [24], cf. [25].

Most of our methods are borrowed from I. Segal's proof of the imprimitivity theorem [26, pp. 441–447], used here on an induced representation, and some

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of our results are implicitly given there. We also use the Aumann–von Neumann measurable choice principle [1], used in similar contexts in [14], [28] and [12]. We use freely [6], [7], [8], [10], and [11]. For the definition of induced representations in the non-separable case, see e.g. [24] section 2.

Professor Tatsuuma has kindly informed me that Takesaki and he has obtained (unpublished) Theorem 1 (a) of this paper some years ago, utilizing approximate cross sections, and not using the Aumann–von Neumann theorem.

2. Preliminaries.

If T is a locally compact space, μ a positive Radon measure on T, and S a topological space, we call a map f of T into S Lusin measurable if for any $\varepsilon > 0$ any compact subset of T is the union of a compact set on which f is continuous and a set with measure less than ε, cf. [6]. If S is metrizable with countable basis, we usually just write measurable. If h is a Hilbert space and $S = \mathscr{L}(h)$, we call f a measurable field when $t \mapsto f(t)\xi$ is Lusin measurable for each $\xi \in h$. If A is a measurable subset of T, the essential measure of A is the supremum of the measures of the compact subsets of A, cf. [7]; we call A essential, if any non-empty relatively open subset has positive essential measure.

An essential value of a Lusin measurable map f of T into S is a point $s \in S$ such that the counter image of any open neighbourhood of s has positive essential measure. The set of essential values of f is the closure of the set of values of f on essential compact (or measurable) sets on which f is continuous. Locally equivalent maps have the same essential values, and any Lusin measurable map is locally equivalent to a map taking essential values only. As in [6] $\mathscr{L}^\infty(\mu)$ denotes the set of measurable complex functions on T with a bounded set of essential values, and $L^\infty(\mu)$ denotes the C*-algebra of classes with respect to equality locally almost everywhere (l.a.e.) of functions in $\mathscr{L}^\infty(\mu)$.

Now assume given a locally compact group G with left Haar measure dg and module Δ_G, and a closed subgroup H with left Haar measure dh and module Δ_H. Let π denote the quotient map of G onto G/H.

Choose a continuous ϱ-extension $\varrho : G \to \,]0,\infty[$ of $h \mapsto \Delta_H(h)\Delta_G(h)^{-1}$, $h \in H$; define a continuous function $\varkappa$: $G \times G/H \to \,]0,\infty[$ by

$$\varkappa(g,\pi(k)) = \varrho(gk)\varrho(k)^{-1}, \quad g,k \in G,$$

and define a Radon measure λ on G/H by

$$\int_{G/H}\int_H f(gh)\,dh\,d\lambda(\pi(g)) = \int_G f(g)\varrho(g)\,dg, \quad f \in \mathscr{K}(G).$$

The measure λ is quasi-invariant and

$$\int_{G/H} \varphi(g^{-1}x)\,d\lambda(x) = \int_{G/H} \varphi(x)\varkappa(g,x)\,d\lambda(x), \qquad \varphi \in \mathscr{K}(G/H),\ g \in G,$$

cf. [9], [17], or [8].

Let u be a strongly continuous unitary representation of H on the non-zero Hilbert space $h(u)$.

Let $\mathscr{F}(u)$ denote the space of Lusin measurable functions $f\colon G \to h(u)$ satisfying

$$\forall g \in G\ \forall h \in H\colon f(gh) = u(h)^{-1}f(g)$$

and

$$\int_{G/H} \|f(g)\|^2\,d\lambda(\pi(g)) < \infty .$$

Let $h(\operatorname{ind} u)$ denote the Hilbert space of classes modulo equality l.a.e. of functions in $\mathscr{F}(u)$. We use the same notation for operators on $\mathscr{F}(u)$ respecting the equivalence classes and the corresponding operators on $h(\operatorname{ind} u)$.

The induced representation $\operatorname{ind}_{H\to G} u = U$ is defined by

$$(U(g)f)(k) = \varkappa(g^{-1},\pi(k))^{\frac{1}{2}} f(g^{-1}k), \qquad f \in \mathscr{F}(u),\ g,k \in G .$$

Define

$$(\varphi\xi)^u(g) = \int_H \varphi(gh)u(h)\xi\,dh, \qquad \varphi \in \mathscr{K}(G),\ \xi \in h(u) .$$

Then $(\varphi\xi)^u$ is continuous and belongs to $\mathscr{F}(u)$, the corresponding elements in $h(\operatorname{ind} u)$ span $h(\operatorname{ind} u)$, and for each $g \in G$ the values $(\varphi\xi)^u(g)$ span $h(u)$ [19], [5].

LEMMA 1. *Let A be a bounded measurable field*: $G \to \mathscr{L}(h(u))$ *with the property*

$$\forall g \in G,\ \forall h \in H\colon \quad A(gh) = u(h)^{-1}A(g)u(h) .$$

If $A(g)f(g)=0$ l.a.e. *for every* $f \in \mathscr{F}(u)$, $A(g)\xi=0$ l.a.e. *for every* $\xi \in h(u)$, *and* $A(g)=0$ l.a.e. *if also* $h(u)$ *is separable.*

PROOF. For $\varphi \in \mathscr{K}(G)$, $\xi \in h(u)$ and F continuous in $\mathscr{F}(u)$ we have

$$\begin{aligned} 0 &= \int_{G/H} (A(g)(\varphi\xi)^u(g) \mid F(g))\,d\lambda(\pi(g)) \\ &= \int_{G/H}\int_H \varphi(gh)(u(h)A(gh)\xi \mid F(g))\,dh\,d\lambda(\pi(g)) , \end{aligned}$$

so

$$0 = \int_G \varphi(g)(A(g)\xi \mid F(g))\varrho(g)\,dg .$$

Since $g \mapsto (A(g)\xi \mid F(g))\varrho(g)$ is locally integrable, the last formula holds for any $\varphi \in \mathscr{L}^\infty(G)$ with compact support. So $A(g)\xi = 0$ on any essential compact set K, on which $g \mapsto A(g)\xi$ is continuous.

Define

$$p_u(\varphi)f = (\varphi \circ \pi)f, \qquad \varphi \in \mathscr{L}^\infty(\lambda),\ f \in h(\operatorname{ind} u)\ .$$

Then $p_u(\varphi) = 0$ if and, by Lemma 1, only if $\varphi = 0$ λ l.a.e.

Thus p_u defines an isomorphism and isometry (also denoted p_u) of $L^\infty(\lambda)$ onto a subalgebra $\mathscr{A}$ of $\mathscr{L}(h(\operatorname{ind} u))$. Since p_u is continuous from $\sigma(L^\infty(\lambda), L^1(\lambda))$ to weak operator topology, the unit sphere in $\mathscr{A}$ is strongly closed and $\mathscr{A}$ is a von Neumann algebra.

Since $U(g)\mathscr{A}\ U(g)^{-1} = \mathscr{A}$, $g \in G$, and since the only $U(g)$ invariant projections in $\mathscr{A}$ are 0 and 1, because λ is ergodic, $\mathscr{A}'$ is homogeneous, say of type I_n. It follows from Segal's proof of the imprimitivity theorem [26] combined with the Mackey–Blattner theorem [20], [4] that the multiplicity n of $\mathscr{A}$ is equal to the dimension of $h(u)$. We obtain it here as a corollary of Theorem 1.

Let h_n denote some Hilbert space with dimension n; there exists a unitary map D of $L^2(\lambda, h_n)$ onto $h(\operatorname{ind} u)$ intertwining the representation of $L^\infty(\lambda)$ as multiplication operators on $L^2(\lambda, h_n)$ and p_u, that is

$$D(\varphi f) = (\varphi \circ \pi)Df, \qquad \varphi \in L^\infty(\lambda),\ f \in L^2(\lambda, h_n)\ .$$

To any operator A in the commutant $u(H)'$ of $u(H)$ we define an operator $\hat{A} \in \mathscr{L}(h(\operatorname{ind} u))$ by $(\hat{A}f)(g) = A(f(g))$, $f \in \mathscr{F}(u)$. Then $\hat{A} \in U(G)' \cap \mathscr{A}'$, and $A \mapsto \hat{A}$ is an injective *-homomorphism, onto $U(G)' \cap \mathscr{A}'$ by the Mackey–Blattner theorem [20], [4]. We sketch a proof. It is enough to show that to any closed $U(G) \cup \mathscr{A}$-invariant subspace L of $h(\operatorname{ind} u)$ there exists a projection $E \in u(H)'$, such that $\hat{E}$ is the projection on L.

If K is a $u(H)$-invariant closed subspace of $h(u)$, let $K^\wedge$ denote the set of classes of functions in $\mathscr{F}(u)$ with all essential values in K. If $f \in \mathscr{F}(u)$ and L is a compact set on which f is continuous, then f is continuous on LH; if L is essential then so is LH, since if O is open any compact subset of $OH \cap L$ is covered by finitely many translates of $O \cap LH$, so $OH \cap L$ has essential measure zero if $O \cap LH$ has. Thus any function in $\mathscr{F}(u)$ is locally equivalent to a function in $\mathscr{F}(u)$ taking essential values only. This implies that if $F \in \mathscr{L}(h(u))$ is the projection on K, then $\hat{F}$ is the projection on $K^\wedge$. If L is a $U(G) \cup \mathscr{A}$-invariant closed subspace of $h(\operatorname{ind} u)$ let $L^\vee$ denote the closed subspace of $h(u)$ spanned by the essential values of functions in $\mathscr{F}(u)$ with equivalence classes in L. Then $L^\vee$ is $u(H)$-invariant and $L \subseteqq L^{\vee\wedge}$. To show equality it is enough to show that any essential value of any function $h \in \mathscr{F}(u)$ with class orthogonal to L is orthogonal to $L^\vee$. So assume that there exists $f \in \mathscr{F}(u)$ with class in L, and essential values x of f and y of h, with $(x \mid y) \neq 0$; then there exist compact sets K and M with positive measure, such that $(f(k) \mid h(m)) \neq 0$ when $k^{-1} \in K$ and $m \in M$; since $\|1_M * 1_K\|_1 \neq 0$ there exists $a \in G$ such that $N = aK^{-1} \cap M$ has positive measure, but this contradicts

$$\int_{G/H} \varphi \circ \pi(g)((U(a)f)(g) \mid h(g))\, d\lambda(\pi(g)) = 0 ,$$

if $\varphi \in \mathscr{L}^\infty(\lambda)$ is chosen such that

$$\varphi \circ \pi(g) = \overline{\operatorname{sign}\left((U(a)f)(g) \mid h(g)\right)} 1_N(\pi(g)) .$$

3. The choice principle.

THEOREM 1. *Let G be a locally compact group, H a closed subgroup, and u a strongly continuous unitary representation of H on a Hilbert space $h(u)$.*

(a) *There exists a ϱ-extension P of u with values in $u(H)''$, and with $g \mapsto P(g)\xi$ and $g \mapsto P(g)^*\xi$ Lusin measurable for each $\xi \in h(u)$.*

(b) *If $h(u)$ is separable, or if H is σ-compact and the center of $u(H)''$ is of countable type, then P can be chosen as a Lusin measurable map into the ultraweak closure $\overline{u(H)}$ of $u(H)$ in $u(H)''$.*

REMARK. We can obtain P with the following measurability property:
To each compact $K \subseteq G$ there exists a family $(h_i)_{i \in I}$ of pairwise orthogonal, separable closed subspaces of $h(u)$, each invariant under $P(K)$ and $P(K)^*$, with sum $h(u)$, and such that each map $k \mapsto P(k) \mid h_i$ is measurable.

Before the proof of Theorem 1 we mention some consequences.

COROLLARY 1. *$(\tilde{P}f)(\pi(g)) = P(g)(f(g))$ defines a map of $\mathscr{F}(u)$ onto $\mathscr{L}^2(\lambda, h(u))$ and a unitary map of $h(\operatorname{ind} u)$ on $L^2(\lambda, h(u))$, transforming p_u into the representation of $L^\infty(\lambda)$ as multiplication operators on $L^2(\lambda, h(u))$, and for each $A \in u(H)'$ transforming the operator $\hat{A}$ given by $(\hat{A}f)(g) = A(f(g))$, into the operator on $L^2(\lambda, h(u))$ corresponding to the constant field $x \mapsto A$ on G/H. The multiplicity n of $p_u(L^\infty(\lambda))$ is equal to the dimension of $h(u)$.*

PROOF. Straight forward verification on basis on Theorem 1(a).

COROLLARY 2. *Assume given a continuous homomorphism ψ of H into a locally compact group K with countable basis. ψ has a measurable ϱ-extension.*

PROOF. Let Φ be a homeomorphism and isomorphism of K with a closed subgroup of the unitary group on a separable Hilbert space, e.g. the left regular representation. Choose a measurable ϱ-extension P of $\Phi \circ \psi$ and use $\Phi^{-1} \circ P$.

COROLLARY 3. *Let N be a closed normal subgroup of H, and assume H/N has countable basis. There exists a measurable transversal and a Lusin measurable cross section for the natural map $G/N \to G/H$.*

PROOF. Let φ denote the natural map $G \to G/N$. Choose a measurable ϱ-extension $P: G \to H/N$ of $\varphi | H$. Then $P^{-1}(eN)$ has the form $\varphi^{-1}(T)$ with T measurable in G/N; $\varphi(\pi^{-1}(x))$ intersects T in a unique point $c(x)$ for each $x \in G/H$. As $g \mapsto gP(g)^{-1}$ is continuous where P is, and constant on H cosets, it defines a Lusin measurable cross section $G/H \to G/N$.

REMARK. Conversely, given a Lusin measurable cross section c with $c(\pi(e)) = \varphi(e)$, $g \mapsto [g^{-1}c(\pi(g))]^{-1}$ defines a Lusin measurable ϱ-extension. (To K compact in G we can choose a compact subset L of $\pi(K)$ such that $\pi^{-1}(L) \cap K$ has almost the same measure as K and $c | L$ is continuous, so $c \circ \pi$ is measurable.).

The case $N = \{e\}$ should be compared with [23] and [13].

COROLLARY 4. *If H has countable basis, any continuous homomorphism ψ of H into a topological group allows a Lusin measurable ϱ-extension.*

PROOF. Choose a Lusin measurable ϱ-extension P of $h \mapsto h$ and use $\psi \circ P$.

PROOF OF THEOREM 1. We split the proof in 7 steps.

STEP 1. We prove, after I. Segal [26, p. 445], that if G is σ-compact and $h(u)$ is separable, then the multiplicity n of $\mathscr{A}$ is $\leqq \aleph_0$.

By the Mackey–Blattner theorem $\mathscr{A}' \cap U(G)'$ is of countable type, and so has a separating sequence $(x_i)_{i \in \mathbf{N}}$ of vectors. Then $\{U(g)x_i \mid g \in G, i \in \mathbf{N}\}$ is σ-compact and metric and so contains a dense sequence $(y_j)_{j \in \mathbf{N}}$.

Let T be a projection in $\mathscr{A}'$, and assume $Ty_j = 0$, $j \in \mathbf{N}$. Define $S \in \mathscr{A}' \cap U(G)'$ by

$$S = \sup_{g \in G} U(g)TU(g)^{-1};$$

since $U(g)TU(g^{-1})x_i = 0$ for each $g \in G$ and $i \in \mathbf{N}$, $Sx_i = 0$ for each i, $S = 0$, and $T = 0$. Thus $\mathscr{A}'$ is of countable type, and $n \leqq \aleph_0$.

STEP 2. If $h(u)$ is separable and $n \leqq \aleph_0$, then $n = \dim h(u)$ and u has a Lusin measurable ϱ-extension with values in $\overline{u(H)}$.

The operator $D: L^2(\lambda, h_n) \to h(\operatorname{ind} u)$ introduced in Section 2 can be disintegrated. We follow the proof in [10, pp. 167–168], making all the choices of functions in the start of the proof in $\mathscr{F}(u)$, making all exceptional local null sets counter images under π of local λ null sets, and using the axiom of choice to get cross sections over these. This way we obtain a bounded measurable map $\tilde{D}: G \to \mathscr{L}(h_n, h(u))$ with the properties: if $f \in \mathscr{L}^2(\lambda, h_n)$, then $g \mapsto \tilde{D}(g)(f \circ \pi(g))$ is a function $G \to h(u)$, whose class is the image under D of the class of f, and

$$\forall g \in G\ \forall h \in H: \quad \tilde{D}(gh) = u(h)^{-1}\tilde{D}(g)\,.$$

Then $g \mapsto \tilde{D}(g)^*$ is a bounded measurable map $\tilde{D}^*: G \to \mathscr{L}(h(u), h_n)$; if $f \in \mathscr{F}(u)$, then $\pi(g) \mapsto \tilde{D}^*(g)(f(g))$ is a well-defined function in $\mathscr{L}^2(\lambda, h_n)$ whose class in the image under D^* of the class of f [10, p. 161]. Now $g \mapsto \tilde{D}^*(g)\tilde{D}(g)$ is a disintegration of $1 \in \mathscr{L}(h_n)$, so $\tilde{D}(g)$ is an isometry l.a.e. [10, p. 160] and $g \mapsto \tilde{D}(g)\tilde{D}^*(g)$ is a disintegration of $1 \in \mathscr{L}(h(u))$, satisfying

$$\forall g \in G\ \forall h \in H: \quad \tilde{D}(gh)\tilde{D}^*(gh) = u(h)^{-1}\tilde{D}(g)\tilde{D}^*(g)u(h)\,.$$

Subtracting the constant field $g \mapsto 1$, and using Lemma 1 we see that $\tilde{D}(g)$ is unitary outside the counter image of a local λ null set, which we may assume empty.

This proves $n = \dim h(u)$, when both are $\leq \aleph_0$, and so by Step 1 when $h(u)$ is separable and G is σ-compact.

Define $C(g) = \tilde{D}(e)\tilde{D}(g)^{-1}$; then C is a measurable unitary ϱ-extension of u.

LEMMA 2. *Let S be a locally compact space and ν a Radon measure on S. Let T be a topological space homeomorphic to a Borel set in a Polish space. Let M be a metrizable space, F a Borel set in M, and f a map: $S \times T \to M$. Assume that*

$$\forall s \in S\ \exists t \in T: f(s,t) \in F, \text{ and}$$

$$\forall s \in S: t \mapsto f(s,t) \text{ is continuous, and}$$

$$\forall t \in T: s \mapsto f(s,t) \text{ is measurable}\,.$$

Then there exists a measurable map $\varphi: S \to T$, with the property

$$\forall s \in S: f(s,\varphi(s)) \in F\,.$$

PROOF. By a result of Mackey [19, Lemma 9.2] $f^{-1}(F)$ is Borel in the product Borel structure on $S \times T$, where we use the Borel structure of measurable sets on S. Choose a local null set N and a locally countable family $(K_i)_{i \in I}$ of pairwise disjoint compact sets in S, with $S = \bigcup_{i \in I} K_i \cup N$, and choose φ on each K_i by the Aumann–von Neumann measurable choice principle [1], and on N by the axiom of choice.

Now let T denote the unitary group on $h(u)$, and let $f\colon G/H \times T \to T/\overline{u(H)}$ be defined by $f(\pi(g), Q) = QC(g)\overline{u(H)}$, $g \in G$, $Q \in T$. The lemma ensures the existence of a map $A\colon G/H \to T$, such that $A(\pi(g))C(g)\overline{u(H)} = \overline{u(H)}$ for all $g \in G$, and $A(\pi(e)) = 1$.

The map $g \mapsto P(g) = A(\pi(g))C(g)$ is a Lusin measurable ϱ-extension of u with values in $\overline{u(H)}$.

STEP 3. If G is σ-compact, there exists a ϱ-extension P of u with values in $u(H)''$, and with $g \mapsto P(g)\xi$ and $g \mapsto P(g)^*\xi$ Lusin measurable for each $\xi \in h(u)$.

First note that H is σ-compact.

If $u(H)''$ has separable predual, u is quasi-equivalent to a representation on a separable Hilbert space k, i.e. there exists an isomorphism Φ of $u(H)''$ with a sub von Neumann algebra of $\mathscr{L}(k)$. By Step 1 and Step 2 there exists a measurable ϱ-extension R of $\Phi \circ u$ with values in $\overline{\Phi \circ u(H)}$; then $\Phi^{-1} \circ R$ is a measurable ϱ-extension of u with values in $\overline{u(H)}$.

If the center of $u(H)'$ is of countable type then $u(H)'$ contains a projection E of countable type with central carrier 1 [10, Lemme 7, p. 236]; $Eh(u)$ is separable since cyclic subspaces under $u(H)$ are, so $A \mapsto A \mid Eh(u)$ defines a quasi-equivalence as wanted.

In any case there exists a family $(E_i)_{i \in I}$ of pairwise orthogonal projections of countable type in the center of $u(H)'$ with sum 1. The direct sums $P(g)$ of the corresponding unitary operators $P_i(g)$ on $E_i h(u)$ define a ϱ-extension P of u with values in $u(H)''$, such that $g \mapsto P(g)\xi$ and $g \mapsto P(g)^*\xi$ are Lusin measurable for each $\xi \in h(u)$.

STEP 4. Completion of the proof of Theorem 1 (a).

Let K be a compact set in G/H. Choose an open σ-compact subgroup G_0 of G with $\pi(G_0) \supseteq K$. Let $H_0 = G_0 \cap H$ and $u_0 = u \mid H_0$. Choose a ϱ-extension P_0 of u_0 with values in $u_0(H_0)'' \subseteq u(H)''$, such that $g \mapsto P_0(g)\xi$ and $g \mapsto P_0(g)^*\xi$ are Lusin measurable on G_0 for each $\xi \in h(u)$.

Then $gh \mapsto P_0(g)u(h)$ is a well-defined map $P\colon G_0H \to u(H)''$, and $P(ghk) = P(gh)u(k)$ when $k \in H$. Let $\xi \in h(u)$ and a compact set $L \subseteq G_0H$ be given; choose L_0 compact in G_0 with $L_0H = LH$, and put $M = (L_0^{-1}L) \cap H$; choose a dense sequence $(x_i)_{i \in \mathbb{N}}$ in $u(M)\xi$ and a compact set $Q_0 \subseteq L_0$, such that $Q = (Q_0H) \cap L$ has almost the same measure as L and $g \mapsto P_0(g)x_i$ is continuous on Q_0 for each i; then $g \mapsto P_0(g)u(h)\xi$ is continuous on Q_0 for each $h \in M$, $(g, h) \mapsto P(gh)\xi$ is continuous on $Q_0 \times M$, and $gh \mapsto P(gh)\xi$ is continuous on Q.

So $g \mapsto P(g)\xi$ and (similarly) $g \mapsto P(g)^*\xi$ are Lusin measurable on $G_0H \supseteq \pi^{-1}(K)$ for each $\xi \in h(u)$.

Now write G/H as $\bigcup_{i\in I} K_i \cup N$, where $(K_i)_{i\in I}$ is a locally countable family of pairwise disjoint compact sets and N is a local λ null set. Define a map Q as P above on $\pi^{-1}(K_i)$ for each $i \in I$, and by choice of an arbitrary cross section on $\pi^{-1}(N)$. Finally define $P = Q(e)^{-1}Q$.

STEP 5. $n = \dim h(u)$.

This is contained in Corollary 1, which is an immediate consequence of Theorem 1 (a).

STEP 6. If $h(u)$ is separable, then u has a Lusin measurable ϱ-extension with values in $\overline{u(H)}$.

In fact $n \leqq \aleph_0$ by Step 5, so Step 2 applies.

STEP 7. Completion of the proof of Theorem 1 (b).

Assume H is σ-compact and the center of $u(H)''$ is of countable type. As in Step 3 there exists an isomorphism Φ of $u(H)''$ with a von Neumann algebra on some separable Hilbert space. By Step 6 there exists a measurable ϱ-extension R of $\Phi \circ u$ with values in $\overline{\Phi \circ u(H)}$; then $\Phi^{-1} \circ R$ is a measurable ϱ-extension of u with values in $\overline{u(H)}$.

4. The groupoid viewpoint. Some commutants.

Let the locally compact group G, the closed subgroup H, and the representation u of H be given as above. Let a ϱ-extension P of u be given with values in $u(H)''$ and such that P and P^* are measurable fields.

It is wellknown that products of measurable fields are measurable fields (cf. the proof in the beginning of Step 4 of the proof of Theorem 1). Also if Q is any Lusin measurable map on G, then $(g,k) \mapsto Q(gk)$ is Lusin measurable on $G \times G$.

So $(g_1, g_2) \mapsto P(g_1g_2)P(g_2)^{-1}$ is a measurable field on $G \times G$, constant on $\{e\} \times H$ cosets, and defines a measurable field K on $G \times G/H$ with values in $u(H)''$, also measurable in the variables separately. K satisfies:

$$\forall g_1, g_2 \in G\ \forall x \in G/H: \quad K(g_1g_2, x) = K(g_1, g_2x)K(g_2x)\,,$$

i.e. K is a representation of the associated groupoid, cf. [18].

Define the operator $K(g,\lambda)$ on $L^2(\lambda, h(u))$ by

$$(K(g,\lambda)f)(x) = K(g,x)f(x), \quad f \in \mathscr{L}^2(\lambda, h(u)),\ g \in G,\ x \in G/H\,,$$

and define $V(g)$ on $L^2(\lambda, h(u))$ by

$$(V(g)f)(x) = \varkappa(g^{-1}, x)^{\frac{1}{2}} f(g^{-1}x) .$$

Then just as in the second countable case

$$V(g)K(g,\lambda) = \tilde{P} \operatorname{ind} u(g) \tilde{P}^{-1}.$$

PROPOSITION 1. *The von Neumann algebra generated by* $\operatorname{ind} u(G)$ *and* $p_u(L^\infty(\lambda))$ *is spatially isomorphic to the von Neumann tensor product* $\mathscr{L}(L^2(\lambda)) \otimes u(H)''$.

PROOF. By Corollary 1 of Theorem 1, and the Mackey–Blattner Theorem (cf. Section 2) $\tilde{P}$ takes $\operatorname{ind} u(G)' \cap p_u(L^\infty(\lambda))'$ onto $1 \otimes u(H)'$.

A special case of this is [27, Lemma 10.1].

PROPOSITION 2. (cf. [27], [24]). *To any operator* $A \in \mathscr{L}(h(\operatorname{ind} u))$ *commuting with* $p_u(L^\infty(\lambda))$ *there exists a bounded measurable field* $a\colon G \to \mathscr{L}(h(u))$, *such that for each* $f \in \mathscr{F}(u)$ *the function* $g \mapsto a(g)(f(g))$ *is a function in* $\mathscr{F}(u)$ *the class of which is the image under* A *of the class of* f, *and such that*

$$\forall g \in G \;\; \forall h \in H\colon \; a(gh) = u(h)^{-1} a(g) u(h) .$$

PROOF. By [21] or [29] any operator on $L^2(\lambda, h(u))$ commuting with all multiplication operators can be disintegrated. Transformation with $\tilde{P}$ gives the proposition.

5. Mackey's Theorem 8.2.

In this section H is a locally compact group; $U = U(h)$ will denote the unitary group on a Hilbert space h, with ultraweak topology; $\mathsf{T} = U(\mathsf{C})$ will be identified with the center of $U(h)$.

By a multiplier on $H \times H$ we understand a map $\omega\colon H \times H \to \mathsf{T}$ measurable with respect to Haar measure on $H \times H$, and satisfying

$$\forall h_1, h_2, h_3 \in H\colon \omega(h_1, h_2)\omega(h_1 h_2, h_3) = \omega(h_1, h_2 h_3)\omega(h_2, h_3)$$

and

$$\omega(e,e) = 1 .$$

We review some known facts, cf. [20], [16].

From $\omega(h,e)\omega(h,k) = \omega(h,k)\omega(e,k)$ we see that $\omega(h,e) = \omega(e,h) = 1$ for all $h \in H$.

The map $k \mapsto \omega(h, k)$ is measurable on H for each $h \in H$. Without lack of generality we assume, to show this, that H is σ-compact. Let H_0 denote the set

$$H_0 = \{h \in H \mid k \mapsto \omega(h,k) \text{ is measurable}\} .$$

Since $\omega(h_1 h_2, k) = \omega(h_1, h_2)^{-1}\omega(h_1, h_2 k)\omega(h_2, k)$, and $\omega(h^{-1}, k) = \omega(h, h^{-1}k)^{-1}\omega(h, h^{-1})$, H_0 is a subgroup of H. By the Fubini theorem the complement of H_0 in H is a local null set, so $H_0 = H$.

An ω-representation on a Hilbert space $h = h(u)$ is a map $u: H \to U(h)$ satisfying $\forall h, k \in H$: $u(h)u(k) = \omega(h,k)u(hk)$, and $\forall \xi \in h(u)$: $h \mapsto u(h)\xi$ is Lusin measurable.

When ω is a multiplier on $H \times H$, γ_ω defined by

$$(\gamma_\omega(h)f)(k) = \omega(h, h^{-1}k)f(h^{-1}k), \quad h,k \in H,\ f \in \mathscr{L}^2(H) ,$$

is an ω-representation of H. The Lusin measurability of $h \mapsto \gamma_\omega(h)f$ follows from

LEMMA 3. *Let S and T be locally compact spaces with Radon measures α and β respectively. Let φ be a bounded measurable complex function on $S \times T$, and assume that $t \mapsto \varphi(s,t)$ is measurable for each $s \in S$. Define the operator $\varphi(s,\beta)$ on $L^2(\beta)$ by*

$$(\varphi(s,\beta)f)(t) = \varphi(s,t)f(t), \quad f \in \mathscr{L}^2(\beta) .$$

Then $s \mapsto \varphi(s,\beta)f$ is Lusin measurable for each $f \in L^2(\beta)$.

PROOF. It is enough to observe that if K is a compact subset of S and L is a compact subset of T, then there exists a bounded sequence $(\varphi_n)_{n \in \mathsf{N}}$ of functions in $\mathscr{K}(S \times T)$ tending to φ almost everywhere on $K \times L$. (We owe this simple proof to C. Berg).

A projective representation of H on h is a continuous homomorphism: $H \to U(h)/\mathsf{T}$ (with quotient topology).

When v is an ω-representation, $h \mapsto v(h)\mathsf{T}$ is a projective representation, see [16] Theorem 3. For completeness we sketch a proof.

It is enough to show that to any $\xi \in h(v)$ there exists a set $A \subseteq G$ with positive essential measure, such that v maps $A^{-1}A$ into $\{\eta \in h(v) \mid \|\eta - \mathsf{T}\xi\| < 1\}$; choose a compact set $K \subseteq G$ with positive measure, such that $g \mapsto v(g)\xi$ is continuous on K, and a sequence $(g_i)_{i \in \mathsf{N}}$ of elements in K with $\{g_i\xi \mid i \in \mathsf{N}\}$ dense in $K\xi$; define

$$A = \{y \in G \mid \exists t \in \mathsf{T}: \|v(g)\xi - t\xi\| < \tfrac{1}{2}\};$$

then A is measurable, and has positive essential measure because $K \subseteq \bigcup_{i \in \mathsf{N}} g_i A$; and if $a, b \in A$, $s, t \in \mathsf{T}$ then

$$\|\dot{v}(a^{-1}b)\xi - t^{-1}s\omega(a^{-1},b)^{-1}\omega(a,a^{-1})\xi\| \leqq \|v(b)\xi - s\xi\| + \|t\xi - v(a)\xi\| .$$

When w is a projective representation, $\tilde{H} = \{(h,u) \in H \times U \mid w(h) = u\mathsf{T}\}$ is a closed subgroup of $H \times U$; $(h,u) \mapsto h$ is a continuous homomorphism of $\tilde{H}$ onto H with kernel $\{(e,t) \mid t \in \mathsf{T}\}$, open since the image of $(V \times W) \cap \tilde{H}$ is $V \cap w^{-1}(W)$, so $\tilde{H}$ is an extension of T by H, cf. [15]. The homomorphism $\tilde{w}$: $(h,u) \mapsto u$ restricts to $t \mapsto t1$ on (the copy in $\tilde{H}$ of) T, and $\tilde{w}(\tilde{h})\mathsf{T} = w(\tilde{h}\mathsf{T})$, $\tilde{h} \in \tilde{H}$. $\tilde{H}$ is locally compact by [22, p. 52].

When $\tilde{H}$ is an extension of T by H there exists a Lusin measurable cross section c: $H \to \tilde{H}$ with $c(e) = e$, by Corollary 3 of Theorem 1, cf. [16]. We give a direct proof, related to the proof by Takesaki and Tatsuuma of Theorem 1 (a). Choose a function $f \in \mathscr{K}(G)$ with $f(t) = t$, $t \in \mathsf{T}$; define

$$d(g) = \int_{\mathsf{T}} f(gt)t^{-1}\,dt;$$

then $g \mapsto g(\operatorname{sign} d(g))^{-1}$ defines a continuous cross section in a neighbourhood of T; from this it is easy to construct a Lusin measurable cross section.

When c is a Lusin measurable cross section with $c(e) = e$, then $(h_1,h_2) \mapsto c(h_1)c(h_2)c(h_1h_2)^{-1}$ defines a multiplier ω_c.

If v is an ω-representation and $\tilde{H}$ and $\tilde{v}$ are the extension and the representation corresponding to $h \mapsto v(h)\mathsf{T}$, and c is a Lusin measurable cross section $H \to \tilde{H}$ with $c(e) = e$, then $\tilde{v}(c(h))\mathsf{T} = v(h)\mathsf{T}$, and $h \mapsto d(h) = c(h)\tilde{v}(c(h))^{-1}v(h)$ defines a new Lusin measurable cross section with $\omega_d = \omega$ and $\tilde{v} \circ d = v$.

When $\tilde{H}$ is an extension of T by H and c is a Lusin measurable cross section with $c(e) = e$, then $\tilde{h} \mapsto c(\tilde{h}\mathsf{T})^{-1}\tilde{h}$ is measurable $\tilde{H} \to \mathsf{T}$ (see remark after Corollary 3 of Theorem 1), and for any ω_c-representation v of H the map $\tilde{h} \mapsto v(\tilde{h}\mathsf{T})c(\tilde{h}\mathsf{T})^{-1}\tilde{h}$ defines a representation $\tilde{v}$ of $\tilde{H}$ with $\tilde{v}(t) = t \cdot 1$, $t \in \mathsf{T}$, and $\tilde{v} \circ c = v$. Thus $x \mapsto x \circ c$ is a bijection of the set of representations of $\tilde{H}$ restricting to $t \mapsto t \cdot 1$ on T onto the set of ω_c-representations of H. E.g. γ_{ω_c} corresponds to the representation of $\tilde{H}$ induced from the representation $t \mapsto t$ of T on C.

PROPOSITION 3. *Let H be locally compact group, N a closed normal subgroup, and v an irreducible representation of N on a separable Hilbert space $h(v)$. Assume that the class of v is invariant under H, i.e.*

$$\forall h \in H\ \exists u(h) \subset U(h(v))\ \forall n \in N: \quad v(hnh^{-1}) = u(h)v(n)u(h)^{-1} .$$

Then there exists a multiplier representation w of H, which is also a ϱ-extension of v; the corresponding multiplier is constant on $N \times N$ cosets.

Proof. Choose a sequence $(n_i)_{i\in\mathsf{N}}$ of elements in N such that $\{v(n_i) \mid i \in \mathsf{N}\}$ is dense in $v(N)$. Define

$$\varphi_i(h,u) = v(hn_ih^{-1})uv(n_i)^{-1}u^{-1}, \qquad h \in H,\ u \in U\ ,$$

and let φ denote the continuous map $(h,u) \mapsto (\varphi_i(h,u))_{i\in\mathsf{N}}$ of $H \times U$ into the product of countably many copies of U. Since $\{v(n_i) \mid i \in \mathsf{N}\}$ is dense in $v(N)$ and the class of v is invariant,

$$\varphi^{-1}(1) = \{(h,u) \in H \times U \mid \forall n \in N\colon v(hnh^{-1}) = uv(n)u^{-1}\}\ .$$

By Lemma 2 there exists a measurable map $a\colon H \to U$, such that $(h,a(h)) \in \varphi^{-1}(1)$, $h \in H$; we may assume $a \mid N = v$, because N is open or a local null set in H. Since v is irreducible, $h \mapsto a(h)\mathsf{T}$ is a projective representation and $\varphi^{-1}(1)$ is the corresponding extension $\tilde{H}$ of T by H.

Let $\tilde{N}$ denote the counter image of N in $\tilde{H}$; i.e.

$$\tilde{N} = \{(n, v(n)t) \mid n \in N,\ t \in \mathsf{T}\}\ .$$

Then $(n, v(n)t) \mapsto t$ is a continuous homomorphism: $\tilde{N} \to \mathsf{T}$; let $q\colon \tilde{H} \to \mathsf{T}$ be a measurable ϱ-extension (Corollary 2 of Theorem 1). Define $W\colon \tilde{H} \to U$ by $W(h,u) = q(h,u)^{-1}u$. W is measurable and constant on T cosets, and thus defines a measurable map $w\colon H \to U$. Then $w(h) = q(h,a(h))^{-1}a(h)$, $h \in H$, and straight forward computations show that w has the wanted properties.

Now let M be a closed subgroup of H. We shall say that a multiplier ω on $H \times H$ is adapted to M, if $\omega \mid M \times H$ is measurable with respect to Haar measure on $M \times H$.

Lemma 4. *Let ω be a multiplier on $H \times H$. There exists a measurable function $a\colon H \to \mathsf{T}$, such that the multiplier*

$$(h_1,h_2) \mapsto a(h_1)a(h_2)a(h_1h_2)^{-1}\omega(h_1,h_2)$$

on $H \times H$ is adapted to M. There exists an ω-representation v of H such that $v \mid M$ is a measurable field on M, if and only if ω is adapted to M. If ω is adapted to M, then $\omega \mid M \times M$ is a multiplier on $M \times M$, and the restriction to M of any ω-representation of H is an $\omega \mid M \times M$-representation of M.

Proof. Choose an extension $\tilde{H}$ of T by H and a cross section $d\colon H \to \tilde{H}$ with $\omega_d = \omega$. Let $\tilde{M}$ be the counter image of M in $\tilde{H}$; then $\tilde{m} \mapsto \tilde{m}\mathsf{T}$ is an open mapping of $\tilde{M}$ onto M. Choose a Lusin measurable cross section $c\colon M \to \tilde{M}$ with $c(e) = e$, and define $c(h) = d(h)$, $h \in H \backslash M$, and define $a = d^{-1}c$. Then c is Lusin measurable, and $(m,h) \mapsto c(mh)$ is Lusin measurable on $M \times H$ by a well known argument based on the homeomorphism $(m,h) \mapsto (m,mh)$, so ω_c is adapted to M.

Assume there exists an ω-representation v of H such that $v \mid M$ is a measurable field on M. Then $(m,h) \mapsto v(mh)$ is a measurable field on $M \times H$, and ω is adapted to M.

Now assume ω is adapted to M. Then $\gamma_\omega \mid M$ is a measurable field on M by Lemma 3, and $\omega \mid M \times M$ is measurable. In this case the cross section c can be chosen such that $\omega_{c \mid M} = \omega \mid M \times M$. Any ω-representation w of H has the form $\tilde{w} \circ c$ for some representation $\tilde{w}$ of $\tilde{H}$; hence $w \mid M = \tilde{w} \circ (c \mid M)$ is a measurable field on M.

THEOREM 2. (cf. [20], [2]). *Let H be a locally compact group, N a closed normal subgroup, ω a multiplier on $H \times H$ adapted to N, and v an irreducible $\omega \mid N \times N$-representation of N on a separable Hilbert space $h(v)$. Assume*

$$\forall h \in H \ \exists u(h) \in U(h(v)) \ \forall n \in N:$$

$$\omega(h,n)\omega(hn,h^{-1})\omega(h^{-1},h)^{-1}v(hnh^{-1}) = u(h)v(n)u(h)^{-1} .$$

Then there exist a multiplier ω_1 on $H \times H$ constant on $N \times N$ cosets and an $\omega\overline{\omega_1}$-representation w of H extending v.

PROOF. Let $\tilde{H}$ be an extension of T by H and c a cross section $H \to \tilde{H}$ with $\omega_c = \omega$ and c and $c \mid N$ Lusin measurable. Let $\tilde{N}$ be the counter image of N in $\tilde{H}$, and let $\tilde{v}$: $\tilde{n} \mapsto v(\tilde{n}\mathrm{T})c(\tilde{n}\mathrm{T})^{-1}\tilde{n}$ be the representation of $\tilde{N}$ corresponding to v. The class of $\tilde{v}$ is invariant under $\tilde{H}$, $\tilde{v}(\tilde{h}\tilde{n}\tilde{h}^{-1}) = \tilde{u}(\tilde{h})\tilde{v}(\tilde{n})\tilde{u}(\tilde{h})^{-1}$, where $\tilde{u}(\tilde{h}) = u(\tilde{h}\mathrm{T})c(\tilde{h}\mathrm{T})^{-1}\tilde{h}$. So by Proposition 3 there exists a multiplier $\tilde{\omega}$ on $\tilde{H} \times \tilde{H}$ constant on $\tilde{N} \times \tilde{N}$ cosets and an $\tilde{\omega}$-representation $\tilde{w}$ of $\tilde{H}$ extending $\tilde{v}$.

From $\tilde{w}(\tilde{h}\tilde{n}) = \tilde{w}(\tilde{h})\tilde{v}(\tilde{n})$ we get that $\tilde{h} \mapsto \tilde{w}(c(\tilde{h}\mathrm{T})) = \tilde{w}(\tilde{h})\tilde{h}^{-1}c(\tilde{h}\mathrm{T})$ is measurable and constant on T cosets, so $w = \tilde{w} \circ c$ is measurable on H; also $w(n) = \tilde{v}(c(n)) = v(n)$ when $n \in N$.

Define $\omega_1(h_1,h_2) = \tilde{\omega}(c(h_1),c(h_2))$, $h_1, h_2 \in H$; then ω_1 is measurable and constant on $N \times N$ cosets, and w is an $\omega\overline{\omega_1}$-representation.

As noted in [2] Theorem 8.3 of [20] is easily extended. Concerning Theorem 8.1 of [20], see [3] and [2]. Combination gives a generalization of Theorem 8.4 of [20] valid (in the case of ordinary representations) for a locally compact group G and a closed normal subgroup N of type I with $\hat{N}/G$ almost Hausdorff, provided that for any $\psi \in \hat{N}$ the Hilbert space of ψ is separable and the coset space G/H of G over the isotropy group $H = G_\psi$ of ψ is σ-compact.

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