MATH. SCAND. 42 (1978), 150–160

THE DISCRETE SKELETON METHOD AND A TOTAL VARIATION LIMIT THEOREM FOR CONTINUOUS-TIME MARKOV PROCESSES

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Abstract.

In this paper we consider the discrete skeleton Markov chains of continuous-time Markov processes and give sufficient conditions for the recurrence of the skeleton chains. As an application of these results we consider a total variation convergence theorem for the transition probability function of a continuous-time Markov process.

0. Introduction.

Kingman [6] studied for a continuous-time Markov proces $\{X_t, t \geqq 0\}$ on a countable state space the discrete skeleton Markov chain $\{X_{n\delta}, n=0,1,\ldots\}$, $\delta>0$, and deduced from known convergence results of the latter process convergence results for the continuous-time process. In a recent work Winkler [13] uses discrete skeletons and known decomposition theorems for Markov chains when proving decomposition theorems for continuous-time Markov processes.

The main purpose of the present paper is to prove a total variation convergence result for a continuous-time, recurrent Markov process $\{X_t\}$ on a general locally compact separable state space S. More specifically, we shall give sufficient conditions for the convergence of the integral $\int_0^\infty \|\lambda P_t\| \, dt$, where λ is a finite signed measure on the Borel σ-field $\mathscr{B}$ of S with total mass $\lambda(S)$ zero, and P_t denotes the transition probability function of the process (Theorem 2). We also construct a recurrent potential kernel G satisfying

$$\lim_{t\to\infty} \left\| \lambda \int_0^t P_u \, du - \lambda G \right\| = 0 .$$

On one hand our results complement the earlier results of Duflo and Revuz [3], who proved that $\lim_{t\to\infty} \|\lambda P^t\| = 0$ under certain recurrence and regularity conditions, and on the other hand the corresponding results (convergence of

Received April 25, 1977

sums of transition probabilities) for discrete-time Markov chains (see Cogburn [2], Griffeath [4], Nummelin [8]).

We also give (see Theorem 1) sufficient conditions for the recurrence of the skeleton chains $\{X_{n\delta};\, n=0,1,\dots\}$ and for the regularity (in the sense of Nummelin [8]; see also Proposition 1.2 below) of probability measures with respect to the skeleton chains. For other studies on the recurrence of the skeleton chains see Winkler [13] and Arjas, Nummelin and Tweedie [1].

1. Notation and preliminaries.

Let S be a locally compact separable topological space and let $\mathscr{B}$ denote the Borel σ-field of S. Denote $\mathsf{R}_+=[0,\infty)$, $\mathsf{N}_+=\{1,2,\dots\}$, $\mathsf{N}=\{0,1,2,\dots\}$, $\mathscr{R}_+$ $=$the Borel σ-field of R_+, $l=$the Lebesgue measure. Let $\{X_t,\, t\in\mathsf{R}_+\}$ be a strong Markov process on $(S,\mathscr{B})$ with transition probability function $P_t(x,E)$, $(t\in\mathsf{R}_+,\, x\in S,\, E\in\mathscr{B})$. P_t satisfies

(1.1) $P_0(x,E) = 1_E(x) = 1(0) \quad \text{if } x\in E \ (x\notin E)$,

(1.2) for fixed t,x, $P_t(x,\cdot)$ is a probability measure on $\mathscr{B}$,

(1.3) for fixed t,E, $P_t(\cdot,E)$ is a measurable function on S,

(1.4) for all t,s,x,e, $P_{t+s}(x,E) = \int_S P_t(x,dy)P_s(y,E)$.

We denote by $(\Omega,\sum)$ the canonical sample space $(\otimes_{\mathsf{R}_+} S, \otimes_{\mathsf{R}_+}\mathscr{B})$ of $\{X_t\}$, and by $\theta_s\colon \Omega\to\Omega$ the translation operator

$$\theta_s X_t(\omega) = X_{t+s}(\omega), \qquad t,s\in\mathsf{R}_+,\ \omega\in\Omega .$$

We denote by $\boldsymbol{P}_\mu$ the canonical probability measure on $\sum$ corresponding to the initial distribution μ and transition probability P_t; for $\mu=\varepsilon_x$, the probability measure assigning unit mass to $x\in S$, we write $\boldsymbol{P}_{\varepsilon_x}=\boldsymbol{P}_x$. In the following the abbreviation "a.s." means "$\boldsymbol{P}_x$ − a.s. for all $x\in S$". We shall assume that $\{X_t\}$ has right-continuous sample paths a.s.

Let τ be an arbitrary stopping time (relative to $\{X_t\}$). We define the iterates of τ by

(1.5) $$\tau_0 = 0, \qquad \tau_{n+1} = \tau_n+\tau\circ\theta_{\tau_n} .$$

We denote for all $A\in\mathscr{B}$, $\delta>0$,

(1.6) $$T_A = T_A^{(0)} = \inf\{t\in\mathsf{R}_+ ;\ X_t\in A\} ,$$

(1.7) $$T_A^{(\delta)} = \inf\{t\in\mathsf{R}_+ ;\ X_{\delta+t}\in A\} ,$$

(1.8) $$\tau_A^{(\delta)} = \inf\{n\in\mathsf{N}_+ ;\ X_{n\delta}\in A\} .$$

DEFINITION 1.1. A set A is called *recurrent*, provided that the random set $\{t \in \mathbf{R}_+ ; X_t \in A\}$ is unbounded a.s., or equivalently, provided that $T_A^{(\delta)}$ is finite a.s. for some (hence for all) $\delta > 0$ (note that in our definition we do not require the more usual condition $\int_{\mathbf{R}_+} 1_A(X_t)\, dt = \infty$).

In Nummelin [8] (cf. also Griffeath [4, Chapter 3.3]) the following result was proved. It is the result we need from Markov chain theory when proving our continuous-time limit theorem.

PROPOSITION 1.2. (Theorem 6.6 of Nummelin [8]). *Let* $\{X_n, n \in \mathbf{N}\}$ *be an aperiodic, φ-recurrent* (cf. Orey [11]) *Markov chain on a general measurable state space* $(S, \mathscr{B})$ *($\mathscr{B}$ is assumed to be countably generated) with transition probability* $P(x, A)$ $(x \in S,\ A \in \mathscr{B})$ *and invariant measure* π. *Call a finite measure* μ *on* $\mathscr{B}$ *regular, provided that* $E_\mu[\inf\{n \in \mathbf{N}_+ ; X_n \in A\}]$ *is finite for all* A *with* $\pi(A) > 0$. *If* $\{X_n\}$ *is positive recurrent (that is,* $\pi(S) < \infty$*), and if* λ *is a finite signed measure on* $\mathscr{B}$ *with total mass* $\lambda(S)$ *zero such that* $|\lambda|$ *is regular, then*

$$\sum_{n=0}^{\infty} \|\lambda P^n\| < \infty ,$$

and there exists a recurrent potential kernel $G \colon S \times \mathscr{B} \to [0, \infty]$ *such that for* μ *a finite regular measure,* μG *is a finite measure, and*

$$\lim_{N \to \infty} \left\| \sum_{n=0}^{N} \lambda P^n - \lambda G \right\| = 0 .$$

REMARK. Note that the above concept of a regular measure should not be confused with the usual topological regularity of a measure on a topological space.

2. The main results.

At first we formulate a minorization condition, called (MC). In Theorem 1 we prove that it implies the recurrence (in the sense of Harris [5]) of the skeleton chains. It is also needed when studying the regularity of a given measure with respect to the skeleton chains. In Theorem 2 we combine Proposition 1.2 and Theorem 1 in order to get the total variation convergence results mentioned in Section 0. Recall from Revuz [12, p. 90], the definition of spread-outness: a finite measure F on $\mathscr{R}_+$ is called *spread-out*, provided that some convolution multiple F^{*k} of F has a non-trivial absolutely continuous component with respect to the Lebesgue measure.

(MC): There exist a stopping time τ relative to the process $\{X_t, t \in \mathbf{R}_+\}$, a constant $\alpha > 0$, a recurrent, (topologically) closed set $C \in \mathscr{B}$ and a probability measure ν on $\mathscr{B} \times \mathscr{R}_+$ such that the measure $\nu(C \times \cdot)$ on $\mathscr{R}_+$ is spread-out and

$$\text{(2.1)} \qquad \boldsymbol{P}_x\{X_\tau \in A,\ \tau \in \Gamma\} \geqq \alpha 1_C(x)\nu(A \times \Gamma), \qquad x \in S,\ A \in \mathscr{B},\ \Gamma \in \mathscr{R}_+ .$$

Sometimes a suitable minorization condition, satisfied by the transition probability function P_t, implies (MC):

EXAMPLE 2.1. Suppose that there exist $\alpha' > 0$, a transition kernel η from $(\mathsf{R}_+, \mathscr{R}_+)$ into $(S, \mathscr{B})$, and a recurrent, closed set $C \in \mathscr{B}$ such that

$$\int_{\mathsf{R}_+} \eta(t, C)\, dt > 0$$

and

$$P_t(x, A) \geqq \alpha' 1_C(x)\eta(t, A) \qquad \text{for all } t \in \mathsf{R}_+,\ x \in S,\ A \in \mathscr{B} .$$

Then we can choose τ to be any non-negative random variable, independent of $\{X_t\}$ and with a strictly positive density function $f(t)$, and (MC) holds with

$$\alpha = \alpha' \int_{\mathsf{R}_+} f(u)\eta(u, S)\, du,$$

$$\nu(dy \times dt) = \left[\int_{\mathsf{R}_+} f(u)\eta(u, S)\, du\right]^{-1} f(t)\eta(t, dy)\, dt .$$

In the following example we give the formulation of (MC) in the most important special case: that is the case, when C can be chosen to be a one-point set.

EXAMPLE 2.2. Assume that there exist a *recurrent point* $a \in S$ (that is, $\{a\}$ is a recurrent set), and a stopping time τ relative to $\{X_t\}$ such that $X_\tau = a$ a.s. and the $\boldsymbol{P}_a$-distribution of τ,

$$F(\Gamma) = \boldsymbol{P}_a\{\tau \in \Gamma\}, \qquad \Gamma \in \mathscr{R}_+ ,$$

is spread out. Then (MC) holds with $C = \{a\}$, $\nu = \varepsilon_a \times F$.

The following proposition and its corollary give sufficient conditions for (MC).

PROPOSITION 2.3. *Suppose that there exists a non-trivial finite measure φ on $\mathscr{B}$ such that every set $A \in \mathscr{B}$ satisfying $\varphi(A) > 0$ is recurrent. Assume in addition that there exist a stopping time τ' relative to $\{X_t\}$, $E \in \mathscr{B}$ with $\varphi(E) > 0$ and an* *interval* $\Gamma_0 \in \mathscr{R}_+$ *such that* $l(\Gamma_0)>0$ *and for all* $x \in E$, $\Gamma \in \Gamma_0 \cap \mathscr{R}_+$ *with* $l(\Gamma)>0$ *and all* $\varphi \times l$*-negligible* $N \subset E \times \Gamma_0$

$$\text{(2.2a)} \qquad \boldsymbol{P}_x\{(X_{\tau'_n}, \tau'_n) \in E \times \Gamma \backslash N \textit{ for some } n \in \mathbf{N}_+\} > 0 ,$$

and for all $y \in S$, $A \subset E$ *with* $\varphi(A)>0$,

$$\text{(2.2b)} \qquad \boldsymbol{P}_y\{X_{\tau'_n} \in A \textit{ for some } n \in \mathbf{N}_+\} > 0 .$$

Then there exist τ, α, C *and* ν *such that* (MC) *holds.*

PROOF. The process $\{(X_{\tau'_n}, \tau'_n),\ n \in \mathbf{N}\}$ is a Markov renewal process corresponding to the semi-Markov kernel

$$Q(x, A \times \Gamma) = \boldsymbol{P}_x\{X_{\tau'} \in A,\ \tau' \in \Gamma\}, \qquad x \in S,\ A \in \mathscr{B},\ \Gamma \in \mathscr{R}_+ .$$

According to Proposition 3.1 of Nummelin [9], and since φ as a finite measure on a locally compact separable space is known to be topologically regular, there exist $k \in \mathbf{N}_+$, $\alpha>0$, a closed set $C \subset E$ with $\varphi(C)>0$ (hence C is recurrent) and a probability measure ν on $\mathscr{B} \times \mathscr{R}_+$ such that $\nu(C \times \cdot) \ll l$ and

$$Q^{*k}(x, A \times \Gamma) = \boldsymbol{P}_x\{X_{\tau'_k} \in A,\ \tau'_k \in \Gamma\} \geqq \alpha 1_C(x) \nu(A \times \Gamma) .$$

By defining $\tau = \tau'_k$ we get (2.1).

COROLLARY. *Suppose that there exists a non-trivial σ-finite measure φ on $\mathscr{B}$ such that every set $A \in \mathscr{B}$ satisfying $\varphi(A)>0$ is recurrent. Denote by $p_t(x, \cdot)$ the density of $P_t(x, \cdot)$ with respect to φ. If there exist a set $E \in \mathscr{B}$ and an interval $\Gamma_0 \in \mathscr{R}_+$ satisfying $\varphi(E)>0$ and $l(\Gamma_0)>0$, such that for every $x \in E$,*

$$\text{(2.3a)} \qquad \int_E p_t(x, y) \varphi(dy) > 0 \textit{ for } l\textit{-almost all } t \in \Gamma_0 ,$$

and for every $y \in S$, $A \subset E$ *with* $\varphi(A)>0$,

$$\text{(2.3b)} \qquad \int_{\mathbf{R}_+} P_t(y, A)\, dt > 0 ,$$

then there exist τ, α, C *and* ν *such that* (2.1) *holds.*

PROOF. Let τ' be an exponentially distributed random variable with parameter λ, say, and assume that τ' is independent of the process $\{X_t\}$. Then for any $x \in S$, $E \in \mathscr{B}$, $\Gamma \in \mathscr{R}_+$,

$$\text{(2.4)} \qquad \boldsymbol{E}_x[\#\{n \in \mathbf{N}_+ \ ;\ (X_{\tau'_n}, \tau'_n) \in E \times \Gamma\}] = \int_\Gamma \lambda P_t(x, E)\, dt .$$

Now (2.3) and (2.4) imply (2.2), and Proposition 2.3 can be applied.

Next we come to the main results of this paper:

THEOREM 1. (i) (cf. Winkler [13, Theorem 2.1 and Lemma 2.7]) *Assume that* (MC) *holds. Then the skeleton chains* $\{X_{n\delta}\}$, $\delta>0$, *possess an essentially unique invariant measure, which we denote by* π, *and for any* δ, $\{X_{n\delta}\}$ *is aperiodic and* π-*recurrent.*

(ii) *Assume* (MC), *and that for some (hence for all)* $\delta>0$

$$\sup_{x\in C} E_x T_C^{(\delta)} < \infty \,.. \tag{2.5}$$

Then for all $\delta>0$, *the skeleton chain* $\{X_{n\delta}\}$ *is positive recurrent (that is,* $\pi(S)<\infty$), *and for any probability measure* μ *on* $\mathscr{B}$ *satisfying*

$$E_\mu T_C < \infty \,, \tag{2.6}$$

we have

$$E_\mu \tau_A^{(\delta)} < \infty \text{ for all } A\in\mathscr{B} \text{ with } \pi(A)>0; \tag{2.7}$$

that is, μ *is regular with respect to* $\{X_{n\delta}\}$.

REMARK. It is clear that we have not necessarily $\pi(C)>0$, that is, C may well be a transient set for all the skeleton chains $\{X_{n\delta}\}$, $\delta>0$. As an example consider the Ornstein–Uhlenbeck process on $\mathbf{R}$. Any point $a\in\mathbf{R}$ serves as a recurrent point for the process although the stationary measure π (the normal distribution) satisfies $\pi(\{a\})=0$ for any $a\in\mathbf{R}$.

THEOREM 2. (i) *If for some* $\delta>0$, *the skeleton chain* $\{X_{n\delta}\}$ *is positive recurrent and if* λ *is a finite, signed measure on* $\mathscr{B}$ *with* $\lambda(S)=0$ *and such that* $|\lambda|$ *is regular with respect to* $\{X_{n\delta}\}$, *then*

$$\int_{\mathbf{R}_+} \|\lambda P_t\|\, dt < \infty \,, \tag{2.8}$$

and there exists a non-negative transition kernel G *on* $(S,\mathscr{B})$ *satisfying* $|\lambda|G(S)<\infty$ *and*

$$\lim_{t\to\infty}\left\|\int_0^t \lambda P_u\,du-\lambda G\right\| = 0\,. \tag{2.9}$$

(ii) *In particular, if* (MC), *and for some* $\delta>0$, (2.5) *hold, and if* λ *is a finite, signed measure satisfying* $\lambda(S)=0$ *and* $E_{|\lambda|}T_C<\infty$, *then the conclusions of part* (i) *are valid.*

PROOF OF THEOREM 1. (i) Let $\delta,\varrho>0$ be arbitrary and fixed. At first we shall prove that $\{X_{n\delta}\}$ is φ-recurrent with φ defined by

$$\varphi(A) = \int_\delta^\infty e^{-\varrho t}(\nu * P)_t(A)\,dt, \qquad A \in \mathscr{B}, \tag{2.10}$$

where we have used the notation $(\nu * P)_t(A) = \int_S \int_0^t \nu(dy \times du) P_{t-u}(y, A)$.

Denote $\nu_A = \nu(A \times \cdot)$, $A \in \mathscr{B}$. Since ν_C is spread out by (MC), we can choose $\beta > 0$, $k, m_0 \in \mathbf{N}_+$ such that

$$\nu_C^{*k}(du) \geqq \beta 1_{[m_0\delta - \delta,\, m_0\delta + \delta]}(u)\,du \tag{2.11}$$

(see Revuz [12, p. 90]). From (MC) we get for all $x \in C$, $A \in \mathscr{B}$, $\Gamma \in \mathscr{R}_+$,

$$\boldsymbol{P}_x\{X_{\tau_k} \in A,\ \tau_k \in \Gamma\} \geqq \alpha^k \nu_C^{*(k-1)} * \nu_A(\Gamma). \tag{2.12}$$

Since by assumption $\{X_t\}$ is strong Markov, we have from (2.1) for all $t \in \mathbf{R}_+$, $x \in S$, $A \in \mathscr{B}$,

$$\begin{aligned} P_t(x, A) &\geqq \int_S \int_0^t \boldsymbol{P}_x\{X_\tau \in dy,\ \tau \in du,\ X_t \in A\} \\ &\geqq 1_C(x)(\nu * P)_t(A). \end{aligned} \tag{2.13}$$

Let now $A \in \mathscr{B}$ with $\varphi(A) > 0$ be arbitrary and fixed, and let $n_0 \in \mathbf{N}_+$ be such that

$$\gamma_A = \int_{n_0\delta}^{n_0\delta + \delta} (\nu * P)_t(A)\,dt > 0. \tag{2.14}$$

Denote $q_0 = m_0 + n_0 + 1$. Then we have for all $x \in C$, $u \in [0, \delta)$,

$$\begin{aligned} P_{q_0\delta - u}(x, A) &\geqq \int_S \int_0^{q_0\delta - u} \boldsymbol{P}_x\{X_{\tau_k} \in dy,\ \tau_k \in dv,\ X_{q_0\delta - u - v} \in A\} \\ &\geqq \alpha^k \int_S \int_0^{q_0\delta - u} \nu_C^{*(k-1)} * \nu_{dy}(dv) P_{q_0\delta - u - v}(y, A) \\ &\geqq \alpha^k \int_0^{q_0\delta - u} \nu_C^{*k}(dv)(\nu * P)_{q_0\delta - u - v}(A) \qquad \text{by (2.13)}, \\ &\geqq \alpha^k \beta \int_{m_0\delta - \delta}^{m_0\delta + \delta} (\nu * P)_{q_0\delta - u - v}(A)\,dv \qquad \text{by (2.11)}, \\ &\geqq \alpha^k \beta \gamma_A > 0. \end{aligned} \tag{2.15}$$

It is easily seen that the bivariate process $\{(X_{n\delta}, T_C^{(n\delta)}),\ n \in \mathbf{N}\}$ is a Markov chain with state space $(S \times \mathbf{R}_+,\ \mathscr{B} \times \mathscr{R}_+)$, and that it visits the set $S \times [0, \delta)$ infinitely often a.s., because C is a recurrent set for $\{X_t\}$. From (2.15) we get for any $m \in \mathbf{N}$, $y \in S$, $u \in [0, \delta)$

$$
\begin{aligned}
(2.16) \qquad & P_x\{X_{(q_0+m)\delta} \in A \mid X_{m\delta}=y,\ T_C^{(m\delta)}=u\} \\
& \geqq E_x[P_{(q_0+m)\delta-(m\delta+T_C^{(m\delta)})}(X_{m\delta+T_C^{(m\delta)}}, A) \mid T_C^{(m\delta)}=u] \\
& \geqq \alpha^k \beta \gamma_A > 0 ,
\end{aligned}
$$

since by the right-continuity of the sample paths $X_{m\delta+T_C^{(m\delta)}} \in C$ P_x – a.s.. From this we conclude that $\{X_{n\delta}, n \in \mathbf{N}\}$ visits the set A infinitely often a.s., and so we have proved the φ-recurrence of $\{X_{n\delta}\}$.

Let now π_δ be the essentially unique invariant measure of $\{X_{n\delta}\}$, which is known to exist by Harris [5]. As in Winkler [13] we get from the uniqueness of π_δ that

$$
(2.17) \qquad \pi_\delta P_{m\delta/n} = \pi_\delta \quad \text{for all } m,n \in \mathbf{N}_+ .
$$

Let now $t>0$ be arbitrary, and let the sequence

$$
\{t_i\} \subset \left\{\frac{m\delta}{n} ;\ m,n \in \mathbf{N}_+\right\}
$$

converge to t. As in Winkler [13, the proof of Lemma 2.7], we get from Fatou's lemma for any $A \in \mathscr{B}$

$$
(2.18) \qquad \pi_\delta P_t(A) \leqq \liminf_{n\to\infty} \int_S \pi_\delta(dy) P_{t_n}(y, A) = \pi_\delta(A) .
$$

Hence π_δ is a subinvariant measure for P_t. From Nummelin and Arjas [10, Lemma 1], (cf. also Neveu [7, p. 198]) we know that then $\pi = \pi_\delta$ satisfies (2.18) with equality. From Orey [11, Theorem 7.2 (ii)], we get that, for any $\delta>0$, $\{X_{n\delta}\}$ is π-recurrent.

It remains to prove the aperiodicity of every skeleton chain $\{X_{n\delta}\}$, $\delta>0$. Assume the contrary: for some $\delta>0$, $d \in \{2,3,\ldots\}$, $\{X_{n\delta}\}$ is periodic with period d. Let $\{C_1, C_2, \ldots, C_d\}$ be a cycle of $\{X_{n\delta}\}$ satisfying the conditions of Theorem 3.1 of Orey [11, p. 13]. Then the skeleton chain $\{X_{nd\delta}, n \in \mathbf{N}\}$ would no more be π-recurrent, since the disjoint sets $C_1, \ldots, C_d$ are closed for this chain and a π-recurrent chain is necessarily indecomposable (Orey [11, p. 34]). From this contradiction we conclude the aperiodicity of every skeleton chain $\{X_{n\delta}\}$, $\delta>0$.

(ii) From the definition of $T_C^{(m\delta)}$, and from (2.5) and (2.6) we get

$$
\begin{aligned}
(2.19) \qquad & \sup_{0\leqq u<\delta,\ x\in S} E_x[\inf\{n \in \mathbf{N}_+ ;\ T_C^{(n\delta)} \in [0,\delta)\} \mid T_C^{(0)}=u] \\
& \leqq \delta^{-1} \sup_{z\in C} E_z T_C^{(\delta)} + 1 < \infty ,
\end{aligned}
$$

and

$$
(2.20) \qquad E_\mu[\inf\{n \in \mathbf{N} ;\ T_C^{(n\delta)} \in [0,\delta)\}] \leqq \delta^{-1} E_\mu T_C < \infty .
$$

Let $A \in \mathscr{B}$ with $\varphi(A)>0$ be arbitrary. Applying Lemma 5.7 of Nummelin [8] with

$$\tau = \inf\{n>q_0\ ;\ T_C^{(n\delta)} \in [0,\delta)\}\ , \tag{2.21}$$

$$Z_n = 1_{\{X_{(\tau_{n-1}+q_0)\delta} \in A\}}\ , \tag{2.22}$$

we get from (2.16), (2.19) and (2.20),

$$E_\mu \tau_A^{(\delta)} < \infty \qquad \text{for all } A \in \mathscr{B} \text{ with } \varphi(A)>0\ , \tag{2.23}$$

From (2.13) we get for any $A \in \mathscr{B}$,

$$\pi(A) \geqq \pi(C)(\nu * P)_t(A)\ .$$

Choosing A such that $\pi(A)<\infty$ and $\varphi(A)>0$ we conclude that $\pi(C)<\infty$. Similarly as we derived (2.21), we get, denoting by π_C the restriction of π to C,

$$E_{\pi_C} \tau_C^{(\delta)} < \infty\ , \tag{2.24}$$

which proves that $\{X_{n\delta}\}$ is positive recurrent (cf. Cogburn [2]). Let now $A \in \mathscr{B}$ with $\pi(A)>0$ be arbitrary. We should prove that (2.7) holds. By Cogburn [2, Proposition 3.1], there exists a set $E \in \mathscr{B}$ with $\varphi(E)>0$ such that

$$\sup_{x\in E} E_x \tau_A^{(\delta)} < \infty\ .$$

The inequality (2.7) now follows from (2.23) and from Cogburn [2, Lemma 3.1]:

$$E_\mu \tau_A^{(\delta)} \leqq E_\mu \tau_E^{(\delta)} + \sup_{x\in E} E_x \tau_A^{(\delta)}\ . \tag{2.25}$$

Proof of Theorem 2. (i). By Proposition 1.2

$$\sum_{n=0}^{\infty} \|\lambda P_{n\delta}\| < \infty\ , \tag{2.26}$$

from which we get by the contractivity of P_t

$$\int_{\mathsf{R}_+} \|\lambda P_t\|\, dt = \sum_{n=0}^{\infty} \int_{n\delta}^{n\delta+\delta} \|\lambda P_t\|\, dt \leqq \delta \sum_{n=0}^{\infty} \|\lambda P_{n\delta}\| < \infty\ . \tag{2.27}$$

By Proposition 1.2 there exists a non-negative kernel G_δ satisfying $|\lambda|G_\delta(S)<\infty$ and

$$\lim_{n\to\infty} \left\| \sum_{m=0}^{n} \lambda P_{m\delta} - \lambda G_\delta \right\| = 0\ . \tag{2.28}$$

Denote

$$G = G_\delta \int_0^\delta P_u du . \tag{2.29}$$

Then

$$|\lambda| G(S) = \delta |\lambda| G_\delta(S) < \infty , \tag{2.30}$$

and for any t, denoting $n(t) = \sup\{n \in \mathbf{N};\ n\delta \leqq t\}$,

$$\begin{aligned}\left\| \lambda \int_0^t P_u du - \lambda G \right\| &\leqq \left\| \lambda \sum_{m=0}^{n(t)-1} P_{m\delta} \int_0^\delta P_u du - \lambda G \right\| + \\ &\quad + \left\| \lambda P_{n(t)\delta} \int_0^{t-n(t)\delta} P_u du \right\| \\ &\leqq \delta \left\| \lambda \sum_{m=0}^{n(t)-1} P_{m\delta} - \lambda G_\delta \right\| + \delta \| \lambda P_{n(t)\delta} \| \to 0 ,\end{aligned}$$

as $t \to \infty$ by (2.27) and (2.28).

Part (ii) follows directly from (i) and Theorem 1.

Theorem 2 has the following corollary.

Corollary 1. (i) *If for some $\delta > 0$, the skeleton chain $\{X_{n\delta}\}$ is positive recurrent with invariant probability measure π, then for π-almost all $x, y \in S$*

$$\int_{\mathbf{R}_+} \| P_t(x, \cdot) - P_t(y, \cdot) \| \, dt < \infty , \tag{2.31}$$

$$\lim_{t \to \infty} \left\| \int_0^t P_u(x, \cdot)\, du - G(x, \cdot) \right\| = 0 . \tag{2.32}$$

(ii) *In particular, if* (MC), *and for some $\delta > 0$,* (2.5) *hold, then we have* (2.31) *and* (2.32) *for π-almost all $x, y \in S$.*

Proof. For π-almost all $x \in S$, the measure ε_x is regular (see Nummelin [8, Corollary 5.16 (iii)]).

Finally, let us formulate the preceding general results in the case when there exists a recurrent point for the Markov process $\{X_t\}$ (cf. Example 2.2).

Corollary 2. *Assume that there exists a recurrent point $a \in S$ satisfying the assumptions made in Example 2.2. Then all skeleton chains $\{X_{n\delta}\}$, $\delta > 0$, possess an essentially unique invariant measure π and they all are π-recurrent. Moreover if $\mathbf{E}_a T_{\{a\}}^{(\delta)}$ is finite for some $\delta > 0$, then $\pi(S) < \infty$ and for any finite, signed measure λ on $\mathscr{B}$ satisfying $\lambda(S) = 0$ and*

(2.33) $$E_{|\lambda|} T_{\{a\}} < \infty ,$$

we have (2.8), *and there exists a non-negative transition kernel G satisfying* $|\lambda| G(S) < \infty$ *and* (2.9).

ACKNOWLEDGEMENT. I would like to thank Professor Elja Arjas for useful comments on the paper.

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