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SMALL VALUES OF ZETA-FUNCTIONS OF QUADRATIC FIELDS

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1.

Let χ denote a real primitive Dirichlet character of modulus q, $L(s,\chi)$ the corresponding L-function, and K the quadratic field with Dedekind zeta-function $\zeta_K(s)=\zeta(s)L(s,\chi)$.

In view of the class number formulae of Dirichlet [3, pp. 51–52], one sees that if the regulator of K is not too large (e.g. when K is complex) a lower bound for $L(1,\chi)$ will lead to a lower bound for the class number of K. Classically there have been two approaches for attacking this problem by use of information about zeros of L-functions. The first, begun by Hecke [8], gives a lower bound for $L(1,\chi)$ which depends on the *absence* of real zeros near $s=1$. The latter, initiated by Deuring [4], gives a lower bound which depends on the *presence* of certain zeros off the critical line. Although neither of these approaches has independently yielded substantial results for the original problem, they have been productively combined [7, 11] to yield unconditional bounds. The resulting estimates are ineffective, due to their dependence on Deuring's idea, since no suitable zero off the critical line has been found.

More recently Stark [13] has used the first approach to give effective bounds in the case of totally imaginary quadratic extensions of totally real number fields other than the rationals.

With regard to the second approach, it has been noticed by several authors [2, 5, 6, 9, 14] that the presence of certain zeros on the critical line may be used to give lower bounds for $L(1,\chi)$. Moreover, values other than zero, may sometimes be useful.

It is the purpose of this paper to discuss some aspects of this latter phenomenon. Although the results are of a conditional nature they do seem to offer some hope since, for instance, one may feel more comfortable about searching for zeros on the critical line rather than off it. Most of the bounds derived herein are not the strongest possible; this for the sake of simplifying the exposition. Presumably, at this stage, one should be more interested in the hypotheses themselves. It should also be mentioned that similar results may be

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derived by other means, for example, from the formula of Deuring [4], and its descendants.

2.

Throughout, $s=\sigma+it$ will denote a complex variable and $\varrho=\beta+i\gamma$ will be fixed; δ and ε will be fixed positive reals. All positive constants $c, c_1, c_2 \ldots$, as well as those implied, will be computable. Let $X=q/\log q$ and $a(m)=\sum_{d|m}\chi(d)$. Further,

$$S = S(X,\varrho) = \sum_{m\leqq X^{\frac{1}{2}}} m^{-2\beta} \quad \text{and} \quad S^* = S^*(X,\varrho) = \sum_{m\leqq X} a(m)m^{-\beta}(X/m)^{i\gamma}e^{-m/X} .$$

Let $D(T_1, T_2)$ denote the rectangular region

$$\{s \mid \tfrac{1}{2}\leqq\sigma\leqq 1-1/\log X,\ T_1\leqq t\leqq T_2\} .$$

THEOREM 1. *Assume that, for some ϱ in $D(0,1)$, we have*

$$|S^*-X^{i\gamma}\zeta_K(\varrho)| > c_1 S .$$

Then,

$$L(1,\chi) > c_2 q^{-\frac{1}{2}}(\log q)^{\frac{3}{2}}$$

where c_2 is computable in terms of c_1.

PROOF. S^* may be expressed as a contour integral in the classical fashion

$$S^*(X,\varrho) = \frac{X^{i\gamma}}{2\pi i}\int_{(2)} \zeta_K(s+\varrho)\Gamma(s)X^s\,ds .$$

Shifting the contour to $\sigma=-\frac{7}{8}$ gives

$$S^* = X^{1-\beta}\Gamma(1-\varrho)L(1,\chi)+X^{i\gamma}\zeta_K(\varrho)+O(R) ,$$

where

$$R \ll q^{\frac{1}{2}-\beta}(\log q)^{\frac{7}{8}} \quad \text{if } \beta<\tfrac{3}{4} ,$$

and

$$R \ll q^{-\frac{1}{4}}(\log q)^{\frac{7}{8}} \quad \text{if } \beta\geqq\tfrac{3}{4} .$$

S may be bounded below as follows.

(I) For
$$\frac{1}{2} \leqq \beta < \frac{1}{2}+\frac{\log\log\log q}{\log q} ,$$

$$S \gg q^{\frac{1}{2}-\beta} \sum_{m \leqq X^{\frac{1}{2}}} m^{-1} \gg q^{\frac{1}{2}-\beta} \log q .$$

(II) For
$$\frac{1}{2}+\frac{\log\log\log q}{\log q} \leqq \beta < \frac{3}{4},$$
$$S \gg \int_1^{X^{\frac{1}{2}}} t^{-2\beta}\, dt \gg (\beta-\tfrac{1}{2})^{-1} .$$

(III) For
$$\frac{3}{4} \leqq \beta < 1-\frac{1}{\log X}, \quad S \gg 1 .$$

From these estimates it follows that, uniformly for ϱ in $D(0,1)$, we have

$$RS^{-1} \ll (\log q)^{-\frac{1}{8}} .$$

From the hypothesis, $|S^* - X^{i\gamma}\zeta_K(\varrho)| > c_1 S$, it follows that

$$L(1,\chi) \gg X^{\beta-1} S |\Gamma(1-\varrho)|^{-1} .$$

It is easily checked that in each of the three cases, this yields the bound stated in the theorem.

One may remark that, since $a(m) \geqq 0$ and $a(m^2) \geqq 1$, it follows that, for ϱ in $D(0, (\frac{1}{2}\pi-\delta)/\log X)$,

$$\operatorname{Re}(S^*(X,\varrho)) \gg S ,$$

and, for ϱ in $D(\delta/\log X, (\frac{1}{2}\pi-\delta)/\log X)$, we also have

$$\operatorname{Im}(S^*(X,\varrho)) \gg S .$$

These lead to the following conditions, any one of which gives satisfaction to the hypothesis of Theorem 1.

(A) If, for some ϱ in $D(0, (\frac{1}{2}\pi-\delta)/\log X)$, $\zeta_K(\varrho)=0$, the hypothesis of Theorem 1 is amply satisfied.

(B_1) There exists $c>0$ such that if, for some γ with $0 \leqq \gamma \leqq (\frac{1}{2}\pi-\delta)/\log X$, we have $|\zeta_K(\frac{1}{2}+i\gamma)| < c \log q$, then the hypothesis is satisfied.

(B_2) There exists $c>0$ such that if, for some ϱ in $D(0, (\frac{1}{2}\pi-\delta)/\log X)$, we have $|\zeta_K(\varrho)| < c$, then the hypothesis is satisfied.

(C) If, for some ϱ in $D(\delta/\log X, \frac{1}{2}\pi-\delta/\log X)$, either

$$\operatorname{Re}(X^{i\gamma}\zeta_K(\varrho)) \leqq 0, \quad \text{or} \quad \operatorname{Im}(X^{i\gamma}\zeta_K(\varrho)) \leqq 0 ,$$

then the hypothesis is satisfied.

In connection with (A) one might mention the result of Siegel [12] that, if $L(s,\chi)$ has no zeros ϱ with $T_1 \leqq \gamma \leqq T_2$ where $0 \leqq T_1 \leqq T_2 \leqq 1$, then

$$T_2 - T_1 \ll \frac{1}{\log\log\log q}.$$

In connection with (B_2) one can come a little closer although still far short of the mark by adapting Siegel's method to prove the following result.

THEOREM 2. *If* $0 \leqq T_1 \leqq T_2 \leqq 1$ *and if* $|\zeta_K(\varrho)| > c$ *for all* ϱ *in* $D(T_1, T_2)$, *then*

$$T_2 - T_1 \ll \frac{1}{\log\log q}.$$

Theorem 2 is proved by application of the following lemma.

LEMMA (Siegel). *Let* $\lambda > 0$, $0 < \xi < 1$, $0 < M_0 < M$ *and let* $f(z)$ *be a function of* $z = x + iy$, *analytic in the rectangle,* $0 \leqq x \leqq 1$, $-\frac{1}{2}\lambda \leqq y \leqq \frac{1}{2}\lambda$. *Assume that* $\operatorname{Re}(f(z)) \leqq M$ *in the whole rectangle and that* $|f(z)| \leqq M_0$ *on the right side* $x = 1$.
It then follows that

$$\log\left|\frac{2M}{f(\xi)} - 1\right| \Big/ \log\left(\frac{2M}{M_0} - 1\right) \geqq \frac{\sinh(\pi\lambda\xi)}{\sinh(\pi\lambda)}.$$

PROOF. See [12, p. 420].

PROOF OF THEOREM 2. Let

$$T = \frac{1}{2}(T_1 + T_2), \qquad \lambda = \frac{1}{T_2 - T_1}\left(1 - \frac{2}{\log q}\right),$$

$$z = \left(1 - \frac{2}{\log q}\right)^{-1}\left(s - \frac{1}{\log q} - Ti\right) \quad \text{and} \quad \xi = \left(1 - \frac{2}{\log q}\right)^{-1}\left(\frac{1}{4} - \frac{1}{\log q}\right).$$

$f(z) = -\log \zeta_K(s)$ is regular in the required rectangle. On the right hand side,

$$|\zeta(s)| \ll \log q, \qquad |L(s,\chi)| \ll \log q,$$

and, by hypothesis, $|\zeta_K(s)|^{-1} \ll 1$. Thus we can take $M_0 = c_1 \log\log q$, for some c_1. Inside the rectangle, we have for $\sigma \geqq \frac{1}{2}$, by hypothesis,

$$\operatorname{Re}(f(z)) = -\log|\zeta_K(s)| < c_2.$$

Recalling the functional equation,

$$\zeta_K(s) = \left(\frac{q}{\pi^2}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+a}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)\Gamma\left(\frac{s}{2}\right)} \zeta_K(1-s)$$

where $a=\frac{1}{2}\{\chi(1)-\chi(-1)\}$, we have, for $\sigma \leqq \frac{1}{2}$,

$$\operatorname{Re}(f(z)) = (\sigma-\tfrac{1}{2})\log\left(\frac{q}{\pi^2}\right) - \log\left|\frac{\Gamma\left(\frac{1-s+a}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)\Gamma\left(\frac{s}{2}\right)}\right| - \log|\zeta_K(1-\bar{s})| .$$

The second term offers no problem except near the left side of the rectangle where it is overwhelmed by the first term. Thus $\operatorname{Re}(f(z)) \leqq c_3$ throughout the rectangle and we choose $M=2M_0$.

Moreover, $f(\xi) = -\log(\zeta_K(\frac{1}{4}+Ti))$, so

$$\operatorname{Re}(f(\xi)) < -\tfrac{1}{4}\log q + O(1) .$$

Thus,

$$\log\left|\frac{2M}{f(\xi)}-1\right| \ll \frac{\log\log q}{\log q} \quad \text{and} \quad \log\left|\frac{2M}{M_0}-1\right| \gg 1 .$$

Hence,

$$\frac{\log\log q}{\log q} \gg \frac{\sinh(\pi\lambda/4)}{\sinh(\pi\lambda)} \geqq e^{-\pi\lambda}\sinh\left(\frac{\pi\lambda}{4}\right) \geqq \frac{\pi\lambda}{2}e^{-\pi\lambda} ,$$

from which it follows that $\lambda \gg \log\log q$, and this gives the result.

In connection with (C) one can again adapt Siegel's method and now come within a constant multiple of the required bound. Unfortunately the best constant I can obtain in this manner seems to be even worse than one that follows trivially from the functional equation.

Indeed, let $0 \leqq \gamma < c/\log q$ and assume that $\zeta_K(\frac{1}{2}+i\gamma) \neq 0$. From the functional equation we have, for some integer n,

$$\begin{aligned}\arg\zeta_K(\tfrac{1}{2}+i\gamma) &= -\gamma\log q + o(1) + \arg\zeta_K(\tfrac{1}{2}-i\gamma) \\ &= -\gamma\log q + o(1) + 2n\pi - \arg\zeta_K(\tfrac{1}{2}+i\gamma)\end{aligned}$$

and so, in $D(\delta/\log X, (\pi+2\delta)/\log X)$, $X^{i\gamma}\zeta_K(\varrho)$ escapes the first quadrant.

3.

The above results stress values of $\zeta_K(s)$ that are not too large. Looking further from the real axis one finds that it would be of use to know values that are not too small.

THEOREM 3. (I) *Let* $T \geqq 2$ *and* $\delta > 0$ *and assume that* K *is a complex quadratic field with class number one. There exists* $c(\delta)$ *such that*

$$\sup_{1 \leqq \gamma \leqq T} |\zeta_K(\tfrac{1}{2}+i\gamma)| < c(\delta) T^{\frac{1}{4}+\delta} .$$

REMARK. This result seems most likely to be useful if $T = o(q)$.

(II) *Let* q *be prime and* $g(q)$ *denote the least prime quadratic residue modulo* q. *Let* $c_1 > 0$ *and* $\varepsilon > 0$ *and assume* $g(q) > q^{\varepsilon}$. *There exists* $c_2(c_1, \varepsilon)$ *such that*

$$\sup_{1 \leqq \gamma \leqq q^{c_1}} |\zeta_K(1+i\gamma)| < c_2 .$$

PROOF. Let

$$b(m) = \begin{cases} 1 & \text{if } m \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

and consider the sum

$$(*) \quad \sum_{m \leqq Y} (a(m)-b(m)) m^{-\varrho} e^{-m/Y} = \frac{1}{2\pi i} \int_{(2)} \{\zeta_K(s+\varrho) - \zeta(2(s+\varrho))\} \Gamma(s) Y^s \, ds$$
$$= \Gamma(1-\varrho) L(1,\chi) Y^{1-\varrho} + \zeta_K(\varrho) - \zeta(2\varrho) -$$
$$- \tfrac{1}{2}\Gamma(\tfrac{1}{2}-\varrho) Y^{\frac{1}{2}-\varrho} + \frac{1}{2\pi i} \int_{(\sigma_0)} ,$$

where $\sigma_0 = -\frac{1}{2} - \frac{1}{2}\delta$ and $0 < \delta \leqq \frac{1}{2}$.

Choosing $Y = \frac{1}{4}q$ and assuming K has class number one, we may assume q is prime and then, as is well known, $a(m) = b(m)$ for $m \leqq Y$ so the left side vanishes. Choosing $\beta = \frac{1}{2}$, $1 \leqq \gamma \leqq T$,

$$|\Gamma(1-\varrho) L(1,\chi) Y^{1-\varrho}| \ll 1 ,$$
$$|\zeta(2\varrho)| \ll \log T ,$$

and

$$|\tfrac{1}{2}\Gamma(\tfrac{1}{2}-\varrho) Y^{\frac{1}{2}-\varrho}| \ll 1 .$$

Moreover,

$$\left|\int_{(\sigma_0)}\right| \ll q^{\sigma_0}(q^{-\sigma_0}\gamma^{-\sigma_0}+\gamma^{-\frac{1}{2}-2\sigma_0}) \ll T^{\frac{1}{2}+\delta},$$

by well known estimates, and the first result follows.

Returning to (*), choose $\delta=\frac{1}{2}$, $\beta=1$, $1\leqq\gamma\leqq q^{c_1}$ and $Y=q^{1+c_1}$.

$$\left|\int_{(-\frac{3}{4})}\right| \ll q^{\frac{1}{2}}Y^{-\frac{3}{4}}\int_0^\infty (q^{c_1}+t)^{\frac{1}{2}}e^{-t}\,dt \ll q^{\frac{1}{2}(1+c_1)}Y^{-\frac{3}{4}} \ll 1 .$$

For the left side of (*) we use the estimate

$$\left|\sum_{m\leqq Y}(a(m)-b(m))m^{-\varrho}e^{-m/Y}\right| \ll \sum_{m\leqq Y}\frac{a(m)}{m} .$$

Noting that $a(m)\geqq 0$ and that $a(m)=0$ unless m can be written in the form $m=rs^2$ where r is divisible by no prime $<q^{\varepsilon}$, it follows that $a(m)\leqq c_3(c_1,\varepsilon)$ and thus

$$\sum_{m\leqq Y}\frac{a(m)}{m} \ll \sum_{rs^2\leqq Y}\frac{1}{rs^2} \ll \sum_{r\leqq Y}\frac{1}{r} .$$

The number of integers $r\leqq Y$ of the required form has the upper bound $c_4(c_1,\varepsilon)Y/\log Y$ is well-known [1], and partial summation yields

$$\sum_{m\leqq Y}\frac{a(m)}{m} \leqq c_5(c_1,\varepsilon) .$$

Returning to the right side of (*), one has the obvious bounds

$$|\zeta(2\varrho)| \ll 1 \quad \text{and} \quad |\Gamma(\tfrac{1}{2}-\varrho)Y^{\frac{1}{2}-\varrho}| \ll 1 .$$

For the first term one uses the bound

$$L(1,\chi) \leqq c_6(\varepsilon)/\log q$$

which follows from the hypothesis $g(q)>q^{\varepsilon}$ by a result of Wolke [15]. Combining these estimates, (II) follows.

REMARK. With regard to (I), if T is actually taken to be bounded, the conclusion still follows even if the hypothesis is weakened so that K is merely supposed to be complex with bounded class number. This follows directly from the Selberg–Chowla version [10] of Deuring's formula.

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