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THE DIOPHANTINE EQUATION $y^2 = Dx^4 + 1$, III

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Ljunggren [6] has shown that the equation of the title has at most two solutions in positive integers, for any positive integer D, not a square. For special values of D, elementary methods have been employed [1], [3] to specify the solutions more closely. Mordell [8] and subsequently others [2], [4], [7] have found conditions of a simple type under which there are no positive solutions. We shall prove a result which includes that of Mordell as well as the subsequent ones and show how it may be used to deal completely with all values of $D \leqq 400$.

LEMMA 1. *Let* $\alpha = a + bD^{\frac{1}{2}}$ *denote the fundamental solution of* $\xi^2 - D\eta^2 = 1$. *We distinguish three cases*: —

CASE I. *If a is even, then* $2\alpha = \Omega_1^2$ *with* $\Omega_1 = ur^{\frac{1}{2}} + vs^{\frac{1}{2}}$, $D = rs$, r, s, u *and* v *all odd and* $\Omega_1 \Omega_2 = 2$ *where* $\Omega_2 = ur^{\frac{1}{2}} - vs^{\frac{1}{2}}$.

CASE II. *If a is odd and b is even, then* $\alpha = \Omega_1^2$ *where* Ω_1 *and* Ω_2 *are of the above form*, $D = rs$, $r \neq 1$, *and* $\Omega_1 \Omega_2 = 1$.

CASE III. *If a and b are both odd, then* $\alpha = \Omega_1^{2^t}$ *where again* Ω_1 *and* Ω_2 *are of the above form*, $t \geqq 1$, $D = 4^t rs$, $r \neq 1$, u *and* v *are both odd and* $\Omega_1 \Omega_2 = 1$.

PROOF. Case I. If a is even, $Db^2 = a^2 - 1 \equiv 1 \pmod 2$ and so D and b are both odd. Thus $(a+1)(a-1) = Db^2$ gives, since the two factors of the left have no common factor,

$$a+1 = ru^2, \quad a-1 = sv^2 ,$$

with $D = rs$, $b = uv$ and hence r, s, u, v are all odd. Thus

$$2 = ru^2 - sv^2 = \Omega_1 \Omega_2 ,$$

and

$$\Omega_1^2 = ru^2 + 2uv(rs)^{\frac{1}{2}} + sv^2 = 2a + 2bD^{\frac{1}{2}} = 2\alpha .$$

Case II. If a is odd, b even, then $(a+1, a-1) = 2$ and so

$$a+1 = 2ru^2, \quad a-1 = 2sv^2 ,$$

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where $D=rs$, $b=2uv$. Thus $1=ru^2-sv^2$ and here $r\neq 1$, otherwise we should have $s=D$, and since $v<b$, this would contradict the assumption that α was the fundamental solution. Thus $\Omega_1\Omega_2=1$ and

$$\Omega_1^2 = ru^2+2uv(rs)^{\frac{1}{2}}+sv^2 = a+bD^{\frac{1}{2}} = \alpha .$$

Case III. If a and b are both odd, then D is even and since again $(a+1, a-1)=2$ we obtain

$$a+1 = 2D_1b_1^2, \quad a-1 = 2D_2b_2^2 ,$$

where $b=b_1b_2$, $D=4D_1D_2$ and $1=D_1b_1^2-D_2b_2^2$.

Suppose first that $D_1\neq 1$. Then let $D_1=r$, $D_2=s$, $b_1=u$, $b_2=v$ and $t=1$. Then $r\neq 1$, u and v are both odd (since b is odd) and $\Omega_1^2=\alpha$, $\Omega_1\Omega_2=1$.

Secondly, if $D_1=1$, then $D_2=\frac{1}{4}D$ and since b_1 and b_2 are odd we obtain

$$(b_1+1)(b_1-1) = D_2b_2^2$$

where the factors on the left have common factor precisely 2. Thus

$$b_1+1 = 2D_3b_3^2, \quad b_1-1 = 2D_4b_4^2 ,$$

with $b_2=b_3b_4$, $D_2=4D_3D_4$ and $1=D_3b_3^2-D_4b_4^2$. If now $D_3\neq 1$, then let $r=D_3$, $s=D_4$, $u=b_3$, $v=b_4$ and $t=2$. Then $r\neq 1$, u and v are both odd, $\Omega_1\Omega_2=1$ and

$$\Omega_1^2 = D_3b_3^2+2b_3b_4(D_3D_4)^{\frac{1}{2}}+D_4b_4^2 = b_1+b_2D_2^{\frac{1}{2}}$$

whence $\Omega_1^4=\alpha$.

If $D_3=1$ we continue in like fashion, the process terminating after a finite number of steps.

THEOREM 1. *Let D denote a positive integer, not a square. Using the notation of Lemma 1, let p denote any odd divisor of $a-1$ and q any divisor of $a+1$ with $q\equiv 1 \pmod 4$. Then the equation $y^2=Dx^4+1$ has no solution in positive integers unless*

(a) *in case* I, (i) $(u\,|\,p)=(v\,|\,q)=1$
and (ii) *the equation $rX^4-sY^4=2$ has solution in positive integers; in particular*

$$r-s \equiv 2 \pmod{16}$$

and $r\equiv 2 \pmod 5$ or $s\equiv 3 \pmod 5$ *or* $r-s\equiv 2 \pmod 5$.

(b) *in case* II *at least one of the following holds, viz,*
(b$_1$) (i) $(u\,|\,p)=(v\,|\,q)=1$
and (ii) *the equations $rX^4-sY^4=1$, $Z^2=2sY^4+1$ hold simultaneously in positive integers; in particular*

$$r \equiv 1 \pmod{16} \text{ or } r-s \equiv 1 \pmod{16},\ 4\,|\,s$$

and $r\equiv 1 \pmod 5$ *or* $s\equiv -1 \pmod 5$ *or* $r\equiv 3 \pmod 5$, $s \equiv 2 \pmod 5$.

or (b_2) (i) $(2u\,|\,p)=(v\,|\,q)=1$

and (ii) *the equation* $4rX^4-sY^4=1$ *has a solution in positive integers; in particular*

$$4r-s \equiv 1 \pmod{16} \text{ or } s \equiv -1 \pmod{16}$$

and $r\equiv -1 \pmod 5$ *or* $s\equiv -1 \pmod 5$ *or* $r+s\equiv -1 \pmod 5$

and (iii) $2^{2k+1}\|u$ *for some integer* $k\geqq 0$.

or (b_3) (i) $(u\,|\,p)=(2v\,|\,q)=1$

and (ii) *the equation* $rX^4-4sY^4=1$ *has a solution in positive integers; in particular*

$$r-4s \equiv 1 \pmod{16} \text{ or } r \equiv 1 \pmod{16}$$

and $r\equiv 1 \pmod 5$ *or* $s\equiv 1 \pmod 5$ *or* $r+s\equiv 1 \pmod 5$

and (iii) $2^{2k+1}\|v$ *for some integer* $k\geqq 0$.

(c) *in case* III, *one of the following holds, viz.,*

(c_1) (i) $t=1$

and (ii) $(u\,|\,p)=(v\,|\,q)=1$

and (iii) *the equation* $rX^4-sY^4=1$ *has a solution in odd integers* X *and* Y; *in particular*

$$r-s \equiv 1 \pmod{16}$$

and $r\equiv 1 \pmod 5$ *or* $s\equiv -1 \pmod 5$ *or* $r-s\equiv 1 \pmod 5$

or (c_2) (i) $t=2$

and (ii) $(u\,|\,p)=(v\,|\,q)=1$,

and (iii) *the equations* $Z^2=2rX^4-1=2sY^4+1$ *hold simultaneously for odd integers* X, Y *and* Z; *in particular*

$$r-s \equiv 1 \pmod{16} \text{ and } 4\,|\,s$$

and $r\equiv 1 \pmod 5$ *or* $s\equiv -1 \pmod 5$ *or* $r\equiv 3$, $s\equiv 2 \pmod 5$

or (c_3) $D=456, 960$.

PROOF. Suppose that $y^2=Dx^4+1$, $x>0$ and that x is minimal with this property. Then for some integer $n>0$, $y+x^2D^{\frac{1}{2}}=\alpha^n$ whence

$$x^2 = \frac{\alpha^n-\alpha'^n}{2D^{\frac{1}{2}}}$$

where α' denotes $a-bD^{\frac{1}{2}}$, and n is minimal with this property.

Suppose first that $n=2m$ is even. Then

$$x^2 = 2\cdot\frac{\alpha^m+\alpha'^m}{2}\cdot\frac{\alpha^m-\alpha'^m}{2D^{\frac{1}{2}}} = 2hk, \quad \text{say}.$$

Clearly both h and k are rational integers and since $h^2-Dk^2=1$, $(h,k)=1$. We cannot have $k=x_1^2$ since this would contradict the assumption that n was minimal with the property. Hence we must have $k=2x_1^2$. We cannot now have that $m=2M$ is even, for it would imply

$$x_1^2 = \frac{\alpha^M+\alpha'^M}{2}\cdot\frac{\alpha^M-\alpha'^M}{2D^{\frac{1}{2}}} = HK, \quad \text{say},$$

yielding $K=x_2^2$ which would again contradict the minimal property. Thus

$$2x_1^2 = \frac{\alpha^m-\alpha'^m}{2D^{\frac{1}{2}}} = \sigma_m, \quad \text{say, with } m\equiv 1 \pmod 2.$$

But $\sigma_{m+2}-(\alpha+\alpha')\sigma_{m+1}+\alpha\alpha'\sigma_m=0$, that is,

$$\sigma_{m+2} = 2a\sigma_{m+1}-\sigma_m \equiv \sigma_m \pmod 2$$

and so $2x_1^2=\sigma_m$ with $m\equiv 1 \pmod 2$ implies $2\mid\sigma_1=b$, and so this can occur only in Case II. We have in this case $\alpha=\Omega_1^2$ and $\alpha'=\Omega_2^2$ and so

$$x_1^2 = \frac{\Omega_1^{2m}-\Omega_2^{2m}}{4(rs)^{\frac{1}{2}}} = \frac{\Omega_1^m+\Omega_2^m}{2r^{\frac{1}{2}}}\cdot\frac{\Omega_1^m-\Omega_2^m}{2s^{\frac{1}{2}}} = \lambda\mu, \quad \text{say},$$

it being readily verified that λ and μ are rational integers, since m is odd. Moreover $r\lambda^2-s\mu^2=1$ and so $(\lambda,\mu)=1$. Thus $\lambda=X^2$ and $\mu=Y^2$ yielding $rX^4-sY^4=1$, and

$$rX^4+sY^4 = \tfrac{1}{2}\{\Omega_1^{2m}+\Omega_2^{2m}\} = h = Z^2,$$

whence $Z^2=2sY^4+1$.

In this case if Y is even, then X is odd whence $r\equiv 1 \pmod{16}$ and if Y is odd then $4\mid s$ and so again X is odd with $r-s\equiv 1 \pmod{16}$. Similarly

$$\text{if } 5\mid rX \text{ then } 5\nmid Y \text{ and so } s \equiv -1 \pmod 5$$
$$\text{if } 5\mid sY \text{ then } 5\nmid X \text{ and so } r \equiv 1 \pmod 5$$

and if $5\nmid rsXY$ then $r-s\equiv 1 \pmod 5$ and $2s+1\equiv 0$ or 1 or $4 \pmod 5$. But $2s+1\equiv 1 \pmod 5$ is impossible since $5\nmid s$; $2s+1\equiv 4 \pmod 5$ is impossible since it would imply $r\equiv s+1\equiv 0 \pmod 5$ and so we obtain only $r\equiv 3 \pmod 5$, $s\equiv 2 \pmod 5$. This completes the proof of condition (b_1) (ii). To prove (b_1) (i) in this case we observe that we have here

$$X^2 r^{\frac{1}{2}} + Y^2 s^{\frac{1}{2}} = \lambda r^{\frac{1}{2}} + \mu s^{\frac{1}{2}} = \Omega_1^m$$
$$= (ur^{\frac{1}{2}} + vs^{\frac{1}{2}})^m .$$

Thus

$$X^2 = u^m r^{\frac{1}{2}(m-1)} + \binom{m}{2} u^{m-2} r^{\frac{1}{2}(m-3)} v^2 s + \ldots + \binom{m}{m-1} u (v^2 s)^{\frac{1}{2}(m-1)}$$
$$\equiv u(u^2 r)^{\frac{1}{2}(m-1)} \pmod{sv^2}$$
$$\equiv u \qquad \pmod{sv^2}, \quad \text{since } \Omega_1 \Omega_2 = ru^2 - sv^2 = 1 .$$

Now $a-1=2sv^2$ and so $X^2 \equiv u \pmod p$ yields $(u\,|\,p)=1$. Similarly $Y^2 \equiv \pm v \pmod{ru^2}$ and so $(v\,|\,q)=1$ since $q \equiv 1 \pmod 4$.

This concludes the consideration of this case. In what follows n must be odd. To reduce the amount of working we shall in each case only obtain the relevant quartic Diophantine equation(s), supressing the remaining details which may be verified just as above.

Case I. Here $\alpha = \frac{1}{2}\Omega_1^2$, $\alpha' = \frac{1}{2}\Omega_2^2$ and so

$$x^2 = \frac{\Omega_1^{2n} - \Omega_2^{2n}}{2^{n+1}(rs)^{\frac{1}{2}}} = \frac{\Omega_1^n + \Omega_2^n}{2^{\frac{1}{2}(n+1)} r^{\frac{1}{2}}} \cdot \frac{\Omega_1^n - \Omega_2^n}{2^{\frac{1}{2}(n+1)} s^{\frac{1}{2}}} = \lambda\mu, \quad \text{say} ,$$

where it is readily verified that for n odd both λ and μ are rational integers. Also $r\lambda^2 - s\mu^2 = 2$ and so $(\lambda, \mu) = 1$. Thus $\lambda = X^2$, $\mu = Y^2$ and so $rX^4 - sY^4 = 2$, as required.

Case II. Here $\alpha = \Omega_1^2$, $\alpha' = \Omega_2^2$ and so

$$x^2 = \frac{\Omega_1^{2n} - \Omega_2^{2n}}{2(rs)^{\frac{1}{2}}} = 2 \cdot \frac{\Omega_1^n + \Omega_2^n}{2r^{\frac{1}{2}}} \cdot \frac{\Omega_1^n - \Omega_2^n}{2s^{\frac{1}{2}}} = 2\lambda\mu, \quad \text{say} ,$$

Here $r\lambda^2 - s\mu^2 = 1$ whence $(\lambda, \mu) = 1$ and so we obtain

either $\lambda = X^2$, $\mu = 2Y^2$ whence $rX^4 - 4sY^4 = 1$,

or $\lambda = 2X^2$, $\mu = Y^2$ whence $4rX^4 - sY^4 = 1$.

Case III. Here $\alpha = \Omega_1^{2^t}$, $\alpha' = \Omega_2^{2^t}$ and so

$$x^2 = \frac{\alpha^n - \alpha'^n}{2^{t+1}(rs)^{\frac{1}{2}}} = z_1 z_2 \ldots z_{t-1} \lambda\mu ,$$

where

$$z_i = \tfrac{1}{2}(\Omega_1^{n \cdot 2^i} + \Omega_2^{n \cdot 2^i}), \quad 1 \leqq i \leqq t-1 ,$$

$$\lambda = \frac{\Omega_1^n + \Omega_2^n}{2r^{\frac{1}{2}}}, \quad \text{and} \quad \mu = \frac{\Omega_1^n - \Omega_2^n}{2s^{\frac{1}{2}}} .$$

Since n is odd, $\lambda, \mu, z_1, z_2, \ldots, z_{t-1}$ are all rational integers with

$$r\lambda^2 - s\mu^2 = 1$$

$$z_1^2 - 4rs(\lambda\mu)^2 = 1$$

$$z_2^2 - 4^2 rs(\lambda\mu z_1)^2 = 1$$

$$\ldots$$

$$z_{t-1}^2 - 4^{t-1} rs(\lambda\mu z_1 z_2 \ldots z_{t-2})^2 = 1 .$$

$$z_1 = 2r\lambda^2 - 1 = 2s\mu^2 + 1 .$$

Thus $\lambda, \mu, z_1, z_2, \ldots, z_{t-1}$ are coprime in pairs, and so since their product is a perfect square, each one must be a perfect square. Let $\lambda = X^2$, $\mu = Y^2$ and $z_i = Z_i^2$ for $i = 1, 2, \ldots, (t-1)$. Now to prove that Y is odd, let

$$\varrho_n = \frac{\Omega_1^n - \Omega_2^n}{2s^{\frac{1}{2}}} .$$

Then

$$\varrho_{n+4} = (\Omega_1^2 + \Omega_2^2)\varrho_{n+2} - \Omega_1^2\Omega_2^2\varrho_n$$
$$= 2(ru^2 + sv^2)\varrho_{n+2} - \varrho_n \equiv \varrho_n \pmod 2$$

and so ϱ_n is odd for every odd n, since both $\varrho_1 = v$ and $\varrho_3 = v(4ru^2 - 1)$ are. Thus μ and therefore Y is odd, and similarly X is odd.

Finally we observe that $z_{i+1} = 2z_i^2 - 1$, and so if $t \geqq 4$, then

$$Z_3^2 = z_3 = 2z_2^2 - 1 = 2(2z_1^2 - 1)^2 - 1 = 8z_1^4 - 8z_1^2 + 1 ,$$

and it is shown in [5] that this equation has no solution unless $z_1 = 0, 1$ neither of which gives a solution in positive integers for the given equation. Indeed, even for $t = 3$ we get

$$Z_2^2 = z_2 = 2z_1^2 - 1 = 2Z_1^4 - 1 ,$$

and by a result of Ljunggren the only solutions of this are $Z_1 = 1$ and $Z_1 = 13$.

The former gives no value, whereas $Z_1 = 13$ yields $4rs(XY)^4 = 28560 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 17$, $rX^4 - sY^4 = 1$ satisfied only by $r = 85$, $s = 84$, $X = Y = 1$ yielding the single value $D = 4^3 rs = 456,960$ with solution $x = 3107$, $y = 6,525,617,281$.

This concludes the proof of the theorem.

Given a numerical value for D the above theorem can be used in practice in two ways. As is well known a and b can be found by the usual continued fraction algorithm and then the lemma will determine which of the three cases has occurred and find u, v, r, s and if applicable, t. In many cases the theorem will show that the given equation has no solution for example if $D=7$ we find $a=8$, $b=3$ and so we have Case I with $r=1$, $u=3$, $s=7$, $v=1$, and the theorem shows that no solution can occur, since the condition $r-s\equiv 2 \pmod{16}$ fails to hold. In some cases on the other hand, the theorem may lead to a solution, where one exists, by indicating which equations need to be solved; the example $D=5$ yields $a=9$, $b=4$, case II, $r=5$, $u=1$, $s=1$, $v=2$. Here (b_1) and (b_2) are both impossible modulo 16, but (b_3) leads to $5X^4-4Y^4=1$ with the obvious solution $X=Y=1$ leading to $x=2$ for the given equation. Of course this last example is very simple, but less trivial examples do occur.

However, to construct a table of values for which no solutions occur, it is worthwhile to reduce the amount of calculation by proving the following

LEMMA 2. *The equation of the title has no solutions in positive integers in any of the following cases:* —

(1) $D=D_1^2$,
(2) $D=D_1^4D_2$ *and* $y^2=D_2x^4+1$ *has no such solutions,*
(3) $D=p$, *a prime, except if* $p=5$ *or possibly if* $p\equiv 3$, 63, 67 *or* 79 (mod 80),
(4) $D=4p$, *except if* $p=2$ *or* 5 *or possibly if* $p\equiv 1$ *or* 17 (mod 80),
(5) $D=2p$, *except perhaps if* $p\equiv 1$ *or* 7 *or* 9 *or* 39 (mod 40),
(6) $D=8p$, *except perhaps if* $p\equiv 1$, 3, 31, 33, 41, 49, 51, 73 *or* 79 (mod 80).

PROOF. (1) and (2) are clearly trivial.

(3) If $p\equiv 1 \pmod 4$ this follows by a result of Ljunggren [7]. If $p\equiv 3 \pmod 8$, then $u^2-pv^2=-2$ possesses solutions and so the result follows by the corollary in [4]; if $p\equiv 7 \pmod 8$ then $u^2-pv^2=2$ possesses solutions and the result follows similarly. If $p=2$ it is well known that $y^2=2x^4+1$ possesses no solutions.

(4) Solutions exist for $D=8, 20$ given by $x=1, 6$ respectively. Now suppose $(p, 10)=1$. Consider first $p\equiv 1 \pmod 4$. Then $\xi^2-p\eta^2=-1$ possesses solutions. Let $A+Bp^{\frac{1}{2}}$ be the fundamental solution. Then

$$(A+B\cdot p^{\frac{1}{2}})^2 = A^2+pB^2+ABD^{\frac{1}{2}}$$

where $D=4p$ gives the fundamental solution of $x^2-Dy^2=1$. Thus $a=A^2+pB^2$, $b=AB$ and since a is odd and b even we have Case II with

$$a+1 = 2pB^2, \quad a-1 = 2A^2 = 2.4(\tfrac{1}{2}A)^2$$

and so we have $r=p$, $s=4$, $u=B$, $v=\frac{1}{2}A$. Then (b_1) gives

$$pX^4-4Y^4 = 1, \quad Z^2 = 8Y^4+1 .$$

The latter holds only for $Y=1$ and so we obtain only $p=5$; (b_2) is impossible modulo 16 and (b_3) implies $p\equiv 1 \pmod{16}$ and $p\equiv 1$ or $2 \pmod 5$, that is, $p\equiv 1, 17 \pmod{80}$. Thus we must show that $p\equiv 3 \pmod 4$ never yields a solution.

Suppose first that $p\equiv 3 \pmod 8$. Then the equation $\xi^2-p\eta^2=-2$ has a solution; let $A+Bp^{\frac{1}{2}}$ be the fundamental solution with both A and B odd. Then $\frac{1}{2}(A^2+pB^2)+AB\cdot p^{\frac{1}{2}}$ is the fundamental solution of $\xi^2-p\eta^2=1$ and so since A and B are both odd we find since $D=4p$ that

$$a+bD^{\frac{1}{2}} = \{\tfrac{1}{2}(A^2+pB^2)+ABp^{\frac{1}{2}}\}^2$$

and so we have Case II with $r=4$, $s=p$, $u=\frac{1}{4}(A^2+pB^2)$ and $v=AB$. Theorem 1 shows that no solution exists.

Similarly if $p\equiv 7 \pmod 8$ and $A+Bp^{\frac{1}{2}}$ is the fundamental solution of $\xi^2-p\eta^2=2$ then A and B are both odd and again we have Case II with $r=4$, $s=p$. In this case, (b_1) and (b_3) are clearly impossible modulo 16 and the only possibility remaining is (b_2) which would require

$$pY^4 = 16X^4-1 = (4X^2+1)(4X^2-1)\,.$$

Since $p\equiv 3 \pmod 4$, $p \nmid 4X^2+1$ and so we should have

$$4X^2-1 = pY_1^4, \quad 4X^2+1 = Y_2^4$$

and the latter is impossible for $X>0$.

(5) If $D=2p$, we can exclude $p=2$ by (1) and then if p is odd it is well known that precisely one of the three equations $\xi^2-2p\eta^2\equiv -1, 2$ and -2 has solutions. We consider these in turn.

If $A+B(2p)^{\frac{1}{2}}$ is the fundamental solution of $\xi^2-2p\eta^2=-1$ then A and B are both odd, and $\alpha=\Omega_1^2$ where $\Omega_1=B(2p)^{\frac{1}{2}}+A$. Here we have Case II with $r=2p$ and $s=1$. Then since $r\equiv 2 \pmod 4$ Theorem 1 shows that no solution exists.

If $A+B(2p)^{\frac{1}{2}}$ is the fundamental solution of $\xi^2-2p\eta^2=+2$ then $2\mid A$ and B is odd and $\alpha=\Omega_1^2$ where $\Omega_1=(\frac{1}{2}A)2^{\frac{1}{2}}+Bp^{\frac{1}{2}}$. Again we have Case II with now $r=2$, $s=p$. In this case by Theorem 1 the only possibility arises in case (b_2) with $p\equiv 7$ or $39 \pmod{40}$. Similarly if $A+B(2p)^{\frac{1}{2}}$ is the fundamental solution of $\xi^2-2p\eta^2=-2$ then we obtain Case II with $r=p$ and $s=2$ leading to $p\equiv 1$ or $9 \pmod{40}$.

(6) The proof here is entirely similar to the previous case, and the details are omitted.

The above results enable the following result to be proved: —

THEOREM 2. *The equation $y^2=Dx^4+1$ possesses a solution in positive integers when $D=$* 3, 5, 8, 14, 15, 18, 20, 24, 33, 35, 39, 48, 60, 63, 65, 68, 79, 80, 83, 95, 99, 105, 120, 138, 143, 150, 156, 168, 183, 189, 195, 203, 224, 248, 254, 255, 258, 264, 288, 315, 320, 323, 325, 328, 333, 360, 390 *and* 399 *and for no other values of* $D \leqq 400$.

PROOF. The details are omitted, but consist of a straightforward application of the previous results save for the values 223, 227, 383 and 387 for which a small refinement is required in each case.

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