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A FIXED POINT FORMULA FOR VARIETIES OVER FINITE FIELDS

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In this note we prove a localization theorem for coherent sheaves on algebraic schemes over a finite field $\boldsymbol{F}_q$. This is an analogue for the Frobenius of localization theorems of Nielsen [6] and Quart [8], which are valid for automorphisms of finite order prime to the characteristic. There results a simple proof of the congruence formula for the number of $\boldsymbol{F}_q$-valued points in a proper $\boldsymbol{F}_q$-scheme, a result which has been proved by Deligne [3] and by Katz [5] by other methods.

Let X be an algebraic $\boldsymbol{F}_q$-scheme, $\mathscr{M}$ a coherent sheaf on X, and $\varphi\colon \mathscr{M} \to \mathscr{M}$ a q-linear endomorphism, i.e. φ is additive and $\varphi(am)=a^q\varphi(m)$ for $a \in \Gamma(U,\mathcal{O}_X)$, $m \in \Gamma(U,\mathscr{M})$, U open in X. If x is an $\boldsymbol{F}_q$-valued point of X, i.e. the residue field $\varkappa(x)$ is $\boldsymbol{F}_q$, then the fibre $\mathscr{M}(x)=\mathscr{M}\otimes_{O_X}\varkappa(x)$ is a finite dimensional vector space over $\boldsymbol{F}_q$, and φ induces an $\boldsymbol{F}_q$-linear endomorphism $\varphi(x)$ of $\mathscr{M}(x)$. If X is proper over $\boldsymbol{F}_q$, then the cohomology groups $H^i(X,\mathscr{M})$ are finite dimensional over $\boldsymbol{F}_q$, and φ induces $\boldsymbol{F}_q$-linear endomorphisms $H^i(\varphi)$ of $H^i(X,\mathscr{M})$.

FIXED POINT FORMULA. *If X is proper over $\boldsymbol{F}_q$, then*

$$\sum_{x\in|X|} \operatorname{trace}(\varphi(x)) = \sum_{i\geqq 0} (-1)^i \operatorname{trace}(H^i(\varphi))$$

where $|X|$ denotes the set of $\boldsymbol{F}_q$-valued points of X.

When $\mathscr{M}=\mathcal{O}_X$, φ the qth power map, let $F_i=H^i(\varphi)$ be the Frobenius action on $H^i(X,\mathcal{O}_X)$. If we let $N_q(X)=\#|X|$ be the number of $\boldsymbol{F}_q$-valued points of X, the formula specializes to a result of [3] and [5]:

$$N_q(X) \equiv \sum (-1)^i \operatorname{trace}(F_i) \bmod p$$

where q is a power of the prime p. In this case Deligne and Katz prove a stronger congruence formula for the zeta function of X.

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If X is the subscheme of projective n-space defined by homogeneous polynomials $F_1,\ldots,F_r$ in $\boldsymbol{F}_q[X_0,\ldots,X_n]$, with $\dim X = n-r$ and $\sum \deg F_i \leqq n$, then $H^i(X,O_X)=0$ for $i>0$ and $H^0(X,O_X)=\boldsymbol{F}_q$ [7], and we have the Chevalley–Warning formula [10]:

$$N_q(X) \equiv 1 \bmod p .$$

If V is a complete scheme over an algebraically closed field k, f: $V \to V$ an endomorphism, $\mathscr{F}$ a coherent sheaf on V, and $f^*\mathscr{F} \to \mathscr{F}$ is a homomorphism of $\mathcal{O}_V$-sheaves, a fixed point formula should give a formula for the alternating sum of the traces of the induced maps on $H^i(V,\mathscr{F})$ in terms of contributions localized at the fixed components (See [2], [8], [9] for some recent results under various conditions on V, f and $\mathscr{F}$.) If we take k to be the algebraic closure of $\boldsymbol{F}_q$, $V = X \otimes_{\boldsymbol{F}_q} k$, f: $V \to V$ the (geometric) Frobenius, $\mathscr{F} = \mathscr{M} \otimes_{\boldsymbol{F}_q} k$, then a q-linear endomorphism of $\mathscr{M}$ induces a homomorphism $f^*\mathscr{F} \to \mathscr{F}$, and our result becomes a fixed point formula of this type. There is not yet one theorem or approach which yields these various fixed point formulae for coherent sheaves.

Note that for endomorphisms other than the Frobenius, the contribution at an isolated fixed point which is singular is usually quite different from the contribution at a non-singular point (cf. [2]).

We define a modified Grothendieck group $K.X$ of coherent sheaves with q-linear endomorphisms on X, and construct a natural transformation $t.$: $K.X \to K.|X|$ which we prove is an isomorphism. We have followed the treatment of [8] rather closely, and this note may be regarded as an elementary variation on, or perhaps introduction to, Quart's work.

We also include a reinterpretation of the result as a theorem of Riemann–Roch type. The homomorphisms $t.$: $K.X \to K.|X|$ is a natural transformation of functors, covariant for proper morphisms. If we construct $K^{\cdot}X$ using locally free sheaves, restriction from X to $|X|$ gives a natural transformation $t^{\cdot}$: $K^{\cdot}X \to K^{\cdot}|X|$ of contravariant functors. Motivated by [1], where a covariant map was constructed dual to the Chern character, we found our results by looking for a covariant $t.$ to accompany $t^{\cdot}$.

1. The localization theorem.

Let X be an $\boldsymbol{F}_q$-scheme, by which we mean a (separated) scheme of finite type over $\boldsymbol{F}_q$. Consider the category whose objects are $(\mathscr{M},\varphi)$, where $\mathscr{M}$ is a coherent sheaf on X and φ: $\mathscr{M} \to \mathscr{M}$ is a q-linear endomorphism. A morphism from $(\mathscr{M},\varphi)$ to $(\mathscr{N},\psi)$ is a homomorphism α: $\mathscr{M} \to \mathscr{N}$ of coherent sheaves such that $\psi\alpha = \alpha\varphi$. We define $K.X$ to be the free abelian group on the isomorphism classes $[\mathscr{M},\varphi]$ of such $(\mathscr{M},\varphi)$, modulo relations:

(i) $$[\mathscr{M},\varphi] = [\mathscr{M}',\varphi'] + [\mathscr{M}'',\varphi'']$$

if there is an exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ compatible with the endomorphisms.

$$\text{(ii)} \qquad [\mathcal{M}, \varphi_1 + \varphi_2] = [\mathcal{M}, \varphi_1] + [\mathcal{M}, \varphi_2]$$

if φ_1 and φ_2 are q-linear endomorphisms of $\mathcal{M}$.

By abuse of notation we write $[\mathcal{M}, \varphi]$, or even $[\mathcal{M}]$ if φ is understood, for the element of $K.X$ determined by $(\mathcal{M}, \varphi)$.

If $X = \operatorname{Spec}(\boldsymbol{F}_q)$ is a point, $\mathcal{M}$ is just a finite-dimensional $\boldsymbol{F}_q$-space, and φ is a linear map; it is easy to see that taking $[\mathcal{M}, \varphi]$ to the trace of φ gives an isomorphism of $K.X$ with $\boldsymbol{F}_q$ (cf. Lemma 1).

When $f: X \to Y$ is a proper morphism, and $\mathcal{M}$ is a coherent sheaf on X with q-linear endomorphism φ, the higher direct image sheaves $R^i f_* \mathcal{M}$ have induced q-linear endomorphisms $R^i f_* \varphi$, and the formula

$$f_*[\mathcal{M}, \varphi] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{M}, R^i f_* \varphi]$$

determines a homomorphism $f_*: K.X \to K.Y$, so that $K.$ becomes a covariant functor for proper morphisms.

We regard the set $|X|$ of $\boldsymbol{F}_q$-valued points as a (reduced) $\boldsymbol{F}_q$-scheme, and let ι_X or ι denote the closed imbedding of $|X|$ in X. If $f: X \to Y$ is a morphism, $|f|: |X| \to |Y|$ denotes the induced morphism. Since $|X|$ is a disjoint union of N_q points, $K.|X|$ is a vector space over $\boldsymbol{F}_q$ of dimension N_q; an element of $K.|X|$ has a unique expression $\sum \lambda_x \langle x \rangle$, where $\lambda_x \in \boldsymbol{F}_q$, and the sum is over $x \in |X|$.

LOCALIZATION THEOREM. *For any $\boldsymbol{F}_q$-scheme X, the inclusion $\iota: |X| \to X$ induces an isomorphism $\iota_*: K.|X| \to K.X$. The inverse isomorphism $t.: K.X \to K.|X|$ is defined by the formula*

$$t.[\mathcal{M}, \varphi] = \sum_{x \in |X|} \operatorname{trace}(\varphi(x)) \langle x \rangle$$

where $\varphi(x)$ is the induced map on the fibre of $\mathcal{M}$ at x.

Part of the assertion is that the mapping $(\mathcal{M}, \varphi) \mapsto \operatorname{trace}(\varphi(x))$ vanishes on exact sequences as in (i), in spite of the fact that the functor taking $\mathcal{M}$ to its fibre $\mathcal{M} \otimes_{O_X} \varkappa(x)$ is not exact.

If $f: X \to Y$ is a proper morphism, then $f_* \iota_* = \iota_* |f|_*$ since $f \circ \iota = \iota \circ |f|$. Since $t. = \iota_*^{-1}$, it follows that $t.$ is also covariant for proper morphisms. When we take Y to be a point, this assertion is precisely the Fixed Point Theorem stated in the introduction.

LEMMA 1. (a) *If $\varphi: \mathcal{M} \to \mathcal{M}$ is a nilpotent q-linear endomorphism on X, then $[\mathcal{M}, \varphi] = 0$ in $K.X$.*

(b) *If* $\mathcal{M}$ *is a direct sum of coherent sheaves* $\mathcal{M}_1, \ldots, \mathcal{M}_r$ *on* X, φ: $\mathcal{M} \to \mathcal{M}$ *a* q*-linear endomorphism,* φ_{ij} *the induced map from* $\mathcal{M}_j$ *to* $\mathcal{M}_i$, *then* $[\mathcal{M}, \varphi] = \sum_{i=1}^r [\mathcal{M}_i, \varphi_{ii}]$ *in* $K.X$.

PROOF. If $\varphi = 0$, then $[\mathcal{M}, \varphi] = [\mathcal{M}, \varphi] + [\mathcal{M}, \varphi]$ by (ii), so $[\mathcal{M}, \varphi] = 0$. If $\varphi^k = 0$, $k > 1$, let $\mathcal{M}' = \operatorname{Ker}(\varphi^{k-1})$, $\mathcal{M}'' = \mathcal{M}/\mathcal{M}'$, with the induced endomorphisms φ' and φ''. Then $(\varphi')^{k-1} = 0$ and $\varphi'' = 0$, so (a) follows by induction and the relation (i) defining K.

To prove (b), let π_j be the projection of $\mathcal{M}$ on $\mathcal{M}_j$, and ϱ_j the inclusion of $\mathcal{M}_j$ in $\mathcal{M}$. Then $[\mathcal{M}, \varphi] = \sum_{i,j} [\mathcal{M}, \varrho_i \varphi_{ij} \pi_j]$ by (ii). Each term with $i \neq j$ is zero by (a), and $[\mathcal{M}, \varrho_i \varphi_{ii} \pi_i] = [\mathcal{M}_i, \varphi_{ii}]$ by (i).

LEMMA 2. *If* j: $Y \to X$ *is a closed imbedding of* $\mathbf{F}_q$*-schemes, then the formula*

$$j^*[\mathcal{M}, \varphi] = [j^*\mathcal{M}, j^*\varphi]$$

defines a homomorphism j^*: $K.X \to K.Y$, *and* $j^* \circ j_*$ *is the identity on* $K.Y$.

For $Y = |X|$, $j = \iota_X$, this implies that the formula in the statement of the localization theorem defines a well-defined mapping $t.$: $K.X \to K.|X|$; the mapping ι_{X*} is one-to one since $t. \circ \iota_{X*}$ is the identity on $K.|X|$.

If Y is defined by the ideal sheaf $\mathcal{I} \subset O_X$, then $j^*\mathcal{M} = \mathcal{M}/\mathcal{I}\mathcal{M}$, and φ^*j is the induced mapping; note that $\varphi(\mathcal{I}\mathcal{M}) \subset \mathcal{I}^q\mathcal{M}$ by q-linearity.

PROOF OF LEMMA 2. We must show that j^* vanishes on the relations (i) defining $K.X$. From an exact sequences as in (i) we obtain an exact sequence

$$0 \to \mathcal{M}'/\mathcal{I}\mathcal{M} \cap \mathcal{M}' \to \mathcal{M}/\mathcal{I}\mathcal{M} \to \mathcal{M}''/\mathcal{I}\mathcal{M}'' \to 0$$

with induced q-linear endomorphisms. The lemma will follow if we show that

$$[\mathcal{M}'/\mathcal{I}\mathcal{M} \cap \mathcal{M}'] = [\mathcal{M}'/\mathcal{I}\mathcal{M}'] \quad \text{in } K.Y,$$

or equivalently that $[\mathcal{I}\mathcal{M} \cap \mathcal{M}'/\mathcal{I}\mathcal{M}'] = 0$ in $K.Y$. But

$$\varphi^k(\mathcal{I}\mathcal{M} \cap \mathcal{M}') \subset \mathcal{I}^{q^k}\mathcal{M} \cap \mathcal{M}' \subset \mathcal{I}\mathcal{M}'$$

for k large by the Artin–Rees lemma, so the result follows from Lemma 1(a).

Our next task is to calculate $K.P$ for $P = \mathbf{P}^n$ a projective space. We denote by $\mathcal{O}(k)$ the usual line bundles on $\mathbf{P}^n$, $\mathcal{M}(k) = \mathcal{M} \otimes \mathcal{O}(k)$, and we make use of the correspondence between coherent sheaves $\mathcal{M}$ on P and finitely generated graded $\mathbf{F}_q[T_0, \ldots, T_n]$-modules M; $M = \oplus M_k$, with $M_k = \Gamma(P, \mathcal{M}(k))$ ([7], [4]). A q-linear endomorphism of φ of $\mathcal{M}$ induces a q-linear homomorphism $\mathcal{M}(k)$

$\to \mathcal{M}(qk)$, for all k, by $m \otimes a \to \varphi(m) \otimes a^q$ in local coordinates; this determines a q-linear endomorphism of the $\boldsymbol{F}_q[T_0,\ldots,T_n]$-module M. Conversely, it is easy to see that a q-linear endomorphism of M, with $M_k \to M_{qk}$ for all k, induces a q-linear endomorphism of $\mathcal{M}$. If $L \to M$ is a graded homomorphism of a finitely generated free $\boldsymbol{F}_q[T_0,\ldots,T_n]$-module L onto M, a q-linear endomorphism of M lifts to a q-linear endomorphism of L, since such a mapping is determined by its action on a set of generators. Using the fact that M has a finite free graded resolution, and Lemma 1(b), we see that $K.P$ is generated by $[\mathcal{M},\varphi]$ for $\mathcal{M} = \mathcal{O}(-k)$, $k = 0,1,2,\ldots$, and φ is determined by a homogeneous F in $\boldsymbol{F}_q[T_0,\ldots,T_n]$ of degree $k(q-1)$; write $\varphi = \varphi_F$.

Lemma 3. *The elements* $[\mathcal{O}(-k),\varphi_F]$ *for* $F = T_0^{a_0}\cdot\ldots\cdot T_n^{a_n}$, $0 \leqq a_i \leqq q-1$, *and* $\sum_{i=0}^n a_i = k(q-1)$, *generate* $K.\boldsymbol{P}^n$ *as a vector space over* $\boldsymbol{F}_q$.

Proof. By relation (ii) $K.\boldsymbol{P}^n$ is generated by $[\mathcal{O}(-k),\varphi_F]$, where F is a monomial. It suffices to show that if F is divisible by some T_i^q, then

$$[\mathcal{O}(-k),\varphi_F] = [\mathcal{O}(-k+1),\varphi_G]\,,$$

where $F = T_i^{q-1}G$. In fact, if Q is the hyperplane defined by $T_i = 0$, we have an exact sequence

$$0 \to \mathcal{O}_P(-k) \xrightarrow{\cdot T_i} \mathcal{O}_P(-k+1) \to \mathcal{O}_Q(-k+1) \to 0$$

compatible with the q-linear endomorphisms φ_F, φ_G, and $\varphi_{\bar G}$ respectively, where

$$\bar G = G(T_0,\ldots,T_{i-1},0,T_{i+1},\ldots,T_n);$$

if T_i divides G, then $\bar G = 0$, and the result follows from Lemma 1(a) and relation (i). This proves Lemma 3.

A simple count shows that there are just as many monomials satisfying the conditions of Lemma 3 as there are $\boldsymbol{F}_q$-valued points in $\boldsymbol{P}^n$. Hence the injection $\iota_*\colon K.|\boldsymbol{P}^n| \to K.\boldsymbol{P}^n$ is an isomorphism. The localization theorem (and hence the fixed point theorem) for arbitrary projective $\boldsymbol{F}_q$-schemes follows easily from this result and Lemma 2; if j is a closed imbedding of X in $\boldsymbol{P}^n$, then $j^* \circ \iota_{P^n *} = \iota_{X*} \circ |j|^*$, and the surjectivity of j^* and $\iota_{P^n *}$ implies that of ι_{X*}. In the next section we will give a proof for arbitrary $\boldsymbol{F}_q$-schemes.

2. Locally free sheaves.

For an $\boldsymbol{F}_q$-scheme X, let $K^{\cdot}X$ be the abelian group defined using the same construction as for $K.X$ in section 1, but using only locally free sheaves in place of coherent sheaves. The tensor product makes $K^{\cdot}X$ into a ring, and makes $K.X$ into a module over $K^{\cdot}X$.

If $f\colon X \to Y$ is a morphism, there is a ring homomorphism $f^*\colon K^{\cdot}Y \to K^{\cdot}X$ defined as usual by pulling back locally free sheaves (together with their q-linear endomorphisms).

We now complete the proof of the localization theorem, by showing that $\iota_*\colon K.|X| \to K.X$ is surjective for any $\boldsymbol{F}_q$-scheme X.

If P is a projective space, it follows from the fact that $\iota_*\colon K.|P| \to K.P$ is an isomorphism with inverse ι^* that $[\mathcal{O}_P] = \iota_*[\mathcal{O}_{|P|}]$ in $K.P$. Let

$$0 \to \mathcal{L}_n \to \mathcal{L}_{n-1} \to \dots \to \mathcal{L}_0 \to \iota_*\mathcal{O}_{|P|} \to 0$$

be a resolution of $\iota_*\mathcal{O}_{|P|}$ by a complex of locally free sheaves, with a q-linear endomorphism of the complex $\mathcal{L}.$ extending the identity endomorphism of $\mathcal{O}_{|P|}$; such a resolution exists by the proof of Lemma 3. Then $[\mathcal{O}_P] = \sum(-1)^k[\mathcal{L}_k]$ in $K.P$. The argument preceding Lemma 3 shows that the canonical "Poincaré duality" map from $K^{\cdot}P$ to $K.P$ is an isomorphism. It follows that $[\mathcal{O}_P] = \sum(-1)^k[\mathcal{L}_k]$ in $K^{\cdot}P$, and hence that $[\mathcal{O}_X] = \sum(-1)^k[f^*\mathcal{L}_k]$ in $K^{\cdot}X$ for any morphism $f\colon X \to P$.

We apply this in the case where f imbeds X as a locally closed subscheme in a projective space P over $\boldsymbol{F}_q$ to show that $\iota_*\colon K.|X| \to K.X$ is surjective. If $\mathcal{M}$ is a coherent sheaf on X with q-linear endomorphism, then

$$[\mathcal{M}] = [\mathcal{O}_X \otimes \mathcal{M}] = \sum(-1)^k[f^*\mathcal{L}_k \otimes \mathcal{M}] \quad \text{in } K.X\,,$$

and this last sum equals $\sum(-1)^k[\mathcal{H}_k]$, where $\mathcal{H}_k$ is the kth homology sheaf of the complex $f^*\mathcal{L}.\otimes\mathcal{M}$. But $\mathcal{L}.$ is exact off $|P|$, so these homology sheaves are supported on $f^{-1}(|P|) \cap X = |X|$. If $\mathcal{I}$ is the ideal sheaf of $|X|$ in X, the fact that $\mathcal{I}^r H_k = 0$ for r large shows that

$$[H_k] = \sum_j \iota_*[\mathcal{I}^j H_k/\mathcal{I}^{j+1}H_k]\,,$$

and this shows that ι_* is surjective. (The above argument is due to Quart [8].)

For an arbitrary $\boldsymbol{F}_q$-scheme X, use Chow's lemma [4, § 5.6] to find a proper morphism $f\colon \tilde{X} \to X$ from a quasi-projective $\boldsymbol{F}_q$-scheme $\tilde{X}$ onto X, which is an isomorphism from $f^{-1}(U)$ to U for some open U in X. Let $Y = X - U$, $j\colon Y \to X$ the inclusion. Then if $\mathcal{M}$ is a coherent sheaf on X with q-linear endomorphism, the kernel and cokernel of the canonical map $\mathcal{M} \to f_*f^*\mathcal{M}$, as well as the sheaves $R^if_*f^*\mathcal{M}$, are all supported on Y. It follows that $K.X = f_*K.\tilde{X} + j_*K.Y$, and the fact that $\iota_*\colon K.|X| \to K.X$ is surjective follows from the corresponding result on $\tilde{X}$ (by the quasi-projective case) and Y (by noetherian induction).

3. Riemann–Roch theorems.

Let $\mathscr{C}$ be a category. By a *homology-cohomology theory* on $\mathscr{C}$ we mean a covariant functor $K.$ from $\mathscr{C}$ to the category of abelian groups, and a contravariant functor $K^{\cdot}$ from $\mathscr{C}$ to the category of rings, together with a "cap product"

$$K^{\cdot}X \otimes K.X \xrightarrow{\cap} K.X$$

for all X in $\mathscr{C}$ making $K.X$ into a $K^{\cdot}X$-module, and satisfying the "projection formula"

$$f_*(f^*b \cap a) = b \cap f_*a$$

for $f\colon X \to Y$, $b \in K^{\cdot}Y$, $a \in K.X$. We also allow that $K.$ may only be covariant for some of the morphisms in $\mathscr{C}$, e.g. the "proper" morphisms in algebraic geometry or topology.

If $H., H^{\cdot}$ is a homology-cohomology theory on a category $\mathscr{D}$, by a *Riemann–Roch theorem* we mean a functor $|\cdot|\colon \mathscr{C} \to \mathscr{D}$, taking propeı morphisms in $\mathscr{C}$ to proper morphisms in $\mathscr{D}$, together with (1) a natural transformation $t^{\cdot}\colon K^{\cdot} \to H^{\cdot}\circ|\cdot|$ of contravariant functors from $\mathscr{C}$ to rings, and (2) a natural transformation $t.\colon K. \to H.\circ|\cdot|$ of covariant functors from $\mathscr{C}$ to abelian groups. We require the compatibility condition, or "module property":

$$\begin{array}{ccc} K^{\cdot}X \otimes K.X & \xrightarrow{\cap} & K.X \\ \downarrow{\scriptstyle t^{\cdot}\otimes t.} & & \downarrow{\scriptstyle t.} \\ H^{\cdot}|X| \otimes H.|X| & \xrightarrow{\cap} & H.|X| \end{array}$$

should commute for all X in $\mathscr{C}$.

This formalism is abstracted from [1], where $\mathscr{C}$ is the category of complex quasi-projective varieties, $K.X$ and $K^{\cdot}X$ are the Grothendieck groups of coherent sheaves and locally free sheaves on X, $\mathscr{D}$ is the category of topological spaces, and $H.$ and $H^{\cdot}$ are (Borel–Moore) homology and cohomology with rational coefficients. The functor $\mathscr{C} \to \mathscr{D}$ takes a variety X to its underlying topological space $|X|$, $t^{\cdot}\colon K^{\cdot}X \to H^{\cdot}|X|$ is the Chern character; the corresponding map $t.\colon K.X \to H.|X|$ is constructed in [1]. The Hirzebruch–Riemann–Roch formula results by mapping X to a point. Riemann–Roch theorems satisfying this formalism are also common in topology.

To interpret the localization theorem as a Riemann–Roch theorem, let $\mathscr{C}$ be the category of $\boldsymbol{F}_q$-schemes, $K., K^{\cdot}$ the functors defined in sections 1 and 2. Let $\mathscr{D}$ be the category of finite sets. For a finite set S, let $H.S$ be the vector space over $\boldsymbol{F}_q$ with S as a basis, and let $H^{\cdot}S$ be the ring of $\boldsymbol{F}_q$-valued functions on S. Then $H., H^{\cdot}$ form a homology-cohomology theory on $\mathscr{D}$ in an obvious way; in fact if we regard S as a zero-dimensional (reduced) $\boldsymbol{F}_q$-scheme, then $H.S = K.S$

and $H^{\cdot}S = K^{\cdot}S$. The functor $\mathscr{C} \to \mathscr{D}$ takes an F_q-sceme X to its set $|X|$ of F_q-valued points. If $\iota\colon |X| \to X$ is the inclusion as before, then $t^{\cdot}\colon K^{\cdot}X \to H^{\cdot}|X| = K^{\cdot}|X|$ is simply the restriction, i.e. $t^{\cdot} = \iota^*$, and $t_.\colon K^{\cdot}X \to H_.|X| = K_.|X|$ is the homomorphism constructed in section 1. The module property follows easily from the definition. We note that in contrast to the Riemann–Roch theorems of [1] and [2], we have a simple explicit formula for the covariant map $t_.$, and the theorem is proved even for non-projective varieties; it follows easily from Lemma 2 that there are "Gysin maps" $f^*\colon K_.Y \to K_.X$ for *all* morphisms $f\colon X \to Y$, which are compatible with $t_.$, and this too is much more than one can usually expect (cf. [1, § 4.4]).

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