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ALGEBRAIC TUBULAR NEIGHBORHOODS I

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For varieties over the complex numbers **C**, the notion of a neighborhood (in the usual complex topology) of a subvariety is quite useful. For example, if everything is smooth and compact, this leads to the notion of a tubular neighborhood of a subvariety. In this paper, we want to see if there is a corresponding notion of "nbd" (our abbreviation for neighborhood) in abstract algebraic geometry.

Let Y be a closed subscheme of a scheme X. One well known candidate for a "nbd" of Y in X is the formal completion of X along Y, written $X\hat{}$ (the correct notation is $X\hat{}_Y$, but we drop the Y unless some confusion might result). The reader is referred to [7, I § 10] and [9, V] for definitions and some interesting results.

But we would like a more geometric notion of "nbd" than $X\hat{}$. Since the Zariski topology is too coarse, we turn to the étale topology. The natural definition to make is:

DEFINITION 1. An étale nbd of Y in X is an étale map $f\colon W \to X$ such that $W \times_X Y \to Y$ is an isomorphism.

We will regard Y as naturally sitting in any étale nbd W.

However, sometimes there are not many étale nbds of Y in X. For example, if Y is a closed connected subvariety of $X = \boldsymbol{P}^n(k)$, with $\dim Y \geqq 1$, then any separated, connected étale nbd of Y in X is a Zariski open of Y in X (see [9, V § 3]). Thus, we might want something more local. There are lots of these étale nbds locally on X. Since they do not patch to give a variety over X (as the above example shows), we have to do the patching in a weaker sense. This leads to the notion of the henselization of X along Y, written X^h (see [10, Ch. 7], [6] and section 1 below).

Thus, we have three ways to define a "nbd" of Y in X: étale nbds of Y in X, X^h and $X\hat{}$. We will study these three notions of "nbd" in the following two situations. First, given a morphism $g_0\colon Y \to Z$, where Y is closed in X, when does g_0 extend to a "nbd" of Y in X? Second, given two closed embeddings

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$Y \subseteqq X_1$ and $Y \subseteqq X_2$, when are they isomorphic in some "nbd" of Y? We find, quite surprisingly, that étale nbds of Y in X in X^h behave in exactly the same way under very mild restrictions (Theorems 3 and 4). Then we give an example, due to Arthur Ogus, to show how X^h and $X^\wedge$ can differ in these situations. Over the complex numbers, we do *not* know the exact relation between the usual complex nbds and our three notions.

Part II of this paper will introduce a geometric model of X^h which, from the point of view of étale homotopy theory, behaves exactly like a tubular nbd in differential topology.

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1. Hensel schemes.

We will briefly review the definitions of Hensel couple and Hensel scheme (see [10] and [6]).

A couple is a pair (A, I) consisting of a ring A and an ideal $I \subseteqq A$. A morphism of couples φ: $(A, I) \to (B, J)$ is a ring homomorphism $\varphi: A \to B$ such that $\varphi(I) \subseteqq J$. An ideal $J \subseteqq A$ is an ideal of definition of the couple (A, I) if $\sqrt{I} = \sqrt{J}$.

A couple (A, I) is Hensel if any étale map $X \to \operatorname{Spec}(A)$ which is an isomorphism over $V(I)$ has a section. Note that I is contained in the Jacobson radical of A. See [5], [10] and [15] for some equivalent definitions and properties of Hensel couples.

For any couple (A, I), one can find a ring A^h, called the henselization of A with respect to I such that (A^h, IA^h) is a Hensel couple (with a certain universal property). See [5], [10] and [15] for the properties of henselization. For our purposes, the important fact is that $\operatorname{Spec}(A^h) = \text{lim-proj}\, X$, where X ranges over the inverse system of affine étale maps $X \to \operatorname{Spec}(A)$ which induce an isomorphism over $V(I)$.

For any Hensel couple (A, I), the henselization of $A[x_1, \ldots, x_n]$ with respect to $IA[x_1, \ldots, x_n]$ is denoted $A\{x_1, \ldots, x_n\}$. This is the ring of restricted algebraic power series with coefficients in A.

A Hensel scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine Hensel scheme $\operatorname{Sph}(A, I)$, which is defined as follows. Let (A, I) be a Hensel couple. For $f \in A$, let $\mathcal{O}(D(f) \cap V(I))$ be the henselization of A_f with respect to IA_f. One sees easily that this gives a presheaf $\mathcal{O}$ on $V(I) \subseteqq \operatorname{Spec}(A)$. One can prove that $\mathcal{O}$ is actually a sheaf (see [6]), and the ringed space $(V(I), \mathcal{O})$ is $\operatorname{Sph}(A, I)$.

The most important Hensel schemes arise as henselizations along a subscheme. To define this, let Y be a closed subscheme of a scheme X. For an affine open U of X, let $\mathcal{O}_{X^h}(U \cap Y)$ be the henselization of $\mathcal{O}_X(U)$ with respect to $\mathcal{I}(V)$, where $\mathcal{I}$ is the ideal sheaf of Y. $\mathcal{O}_{X^h}$ is a presheaf on Y and it is actually a sheaf (see [10, 7.1, 2.1]). Then $X^h = (Y, \mathcal{O}_{X^h})$ is a Hensel scheme, and is called the henselization of X along Y. A more correct notation is X^h_Y, but we will drop the subscript Y unless there is a possibility of confusion. Note that there is a map of ringed spaces $j\colon X^h \to X$.

A sheaf $\mathcal{F}$ on a Hensel scheme $(X, \mathcal{O}_X)$ is called quasi-coherent if every point of X has an affine nbd $U = \mathrm{Sph}\,(A, I)$ so that $\mathcal{F}|_U = M^{\sim}$, where M is an A module and $M^{\sim}$ is the presheaf on $V(I)$ defined by

$$M^{\sim}(D(f) \cap V(I)) = M \otimes_A (A_f)^h \quad \text{for } f \in A$$

($M^{\sim}$ is actually a sheaf — see [10, 7.1.3.1]).

An ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$ is called an ideal of definition of a Hensel scheme $(X, \mathcal{O}_X)$ is $\mathcal{I}$ is quasi coherent and for every affine open $U = \mathrm{Sph}\,(A, I)$, $\mathcal{I}(U)$ is an ideal of definition of (A, I). Note that $(X, \mathcal{O}_X/\mathcal{I})$ is then just an ordinary scheme. One can prove that every Hensel scheme has an ideal of definition (see [11]).

For example, let Y be a closed subscheme of a scheme X defined by the ideal sheaf $\mathcal{I}$, and let $\mathcal{F}$ be a quasi coherent sheaf on X. Using the above map j: $X^h \to X$, we get the sheaf

$$\mathcal{F}^h = j^*\mathcal{F} = \mathcal{F}|_Y \otimes_{\mathcal{O}_X|_Y} \mathcal{O}_{X^h}\,.$$

One can show that for $U \subseteqq X$ an affine open,

$$\mathcal{F}^h(U \cap Y) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U)^h$$

(see [10, 7.5.3]). Thus, $\mathcal{F}^h$ is a quasi coherent sheaf on X^h. Note that $\mathcal{I}^h$ is an ideal of definition of X^h, and that $\mathcal{O}_{X^h}/\mathcal{I}^h \cong \mathcal{O}_Y$.

A map of Hensel schemes is just a map of local ringes spaces f: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. One proves (see [10, 7.1.2.4]) that f can be locally described as the map $\tilde{\varphi}$: Sph $(B, J) \to$ Sph (A, I) induced by a map of Hensel couples $\varphi\colon (A, I) \to (B, J)$. We say that f is adic if $f^*(\mathcal{I})$ is an ideal of definition for $(X, \mathcal{O}_X)$ whenever $\mathcal{I}$ is an ideal of definition of $(Y, \mathcal{O}_Y)$. An adic map induces a map of schemes

$$(X, \mathcal{O}_X/f^*(\mathcal{I})\mathcal{O}_X) \to (Y, \mathcal{O}_Y/\mathcal{I})\,.$$

A map f: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is locally of finite presentation if f can be locally described as

$$\tilde{\varphi}\colon \mathrm{Sph}\,(B, J) \to \mathrm{Sph}\,(A, I)$$

(coming from φ: $(A,I) \to (B,J)$), where $B \cong A\{X_1,\ldots,X_n\}/(g_1,\ldots,g_m)$ and $\sqrt{J} = \sqrt{IB}$. Note that such an f is adic.

For an example of all this, take a commutative diagram of schemes:

$$\text{(1)} \qquad \begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

where the vertical maps are closed immersions. Then we obviously get a map of Hensel schemes f^h: $X'^h \to X^h$. f^h is adic precisely when $f^{-1}(Y) = Y'$ (as sets). If, in addition, f is locally of finite presentation in a nbd of Y', then f^h is locally of finite presentation.

Another important map of Hensel schemes comes from the fact that any closed immersion of schemes $Y \hookrightarrow X$ factors $Y \hookrightarrow X^h \to X$. In fact, it factors $Y \hookrightarrow X\hat{} \to X^h \to X$, where $X\hat{}$ denotes the formal completion of X along Y.

Finally, the category of Hensel schemes has fiber products, which are explicitly constructed in [11].

2. The étale topology of a Hensel scheme.

We first define the notion of étale map:

DEFINITION 2. A map of Hensel schemes f: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is étale if f is locally of finite presentation and locally f can be written as $\tilde{\varphi}$: Sph $(B,J) \to$ Sph (A,I) where B (in the J adic topology) is formally étale over A (in the I adic topology). See [8, 0.19.10.2].

To get an equivalent definition, we will use the fact that any map of Hensel schemes f: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ gives us a sheaf $\Omega^1_{X/Y}$ of $\mathcal{O}_X$ modules in the usual way. From [10, 3.6.2, 3.6.3, 3.6.3.5] we get:

PROPOSITION 1. *Let f: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of Hensel schemes which is locally of finite presentation. Then f is étale iff $\Omega^1_{X/Y} = 0$ and f is flat.*

An application of this is the following:

PROPOSITION 2. *Suppose we have a cartesian diagram of schemes* (1) *where f is locally of finite presentation and the vertical maps are closed immersions. Then f: $X' \to X$ is étale in a* nbd *of Y' iff f^h: $X'^h \to X^h$ is étale.*

PROOF. For x in Y', [10, 7.1.1.2] shows that

$$\mathcal{O}_{X'^h, x} = (\mathcal{O}_{X', x})^h \quad \text{and} \quad \mathcal{O}_{X^h, f(x)} = (\mathcal{O}_{X, f(x)})^h .$$

Since these are local rings, $\mathcal{O}_{X'^h, x}$ and $\mathcal{O}_{X^h, f(x)}$ are faithfully flat over $\mathcal{O}_{X', x}$ and $\mathcal{O}_{X, f(x)}$ respectively. Then the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{X, f(x)} & \xrightarrow{\ f_x\ } & \mathcal{O}_{X', x} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X^h, f(x)} & \xrightarrow{\ f_x^h\ } & \mathcal{O}_{X'^h, x} \end{array}$$

shows that f_x is flat iff f_x^h is.

The argument of [10, 7.4.2.7] shows that $\Omega^1_{X'^h/X^h} \cong (\Omega^1_{X'/X})^h$, so that by section 1, we have

$$\Omega^1_{X'^h/X^h, x} \cong \Omega^1_{X'/X, x} \otimes_{\mathcal{O}_{X', x}} \mathcal{O}_{X'^h, x} \qquad \text{for } x \in Y' .$$

Then faithful flatness shows that $\Omega^1_{X'^h/X^h, x} = 0 \Leftrightarrow \Omega^1_{X'/X, x} = 0$.

Using Proposition 1 and the characterization of étale maps given in [8, IV 17.6.1 and 17.4.1], we are done.

The final topic we consider is the étale topology of a Hensel scheme $(X, \mathcal{O}_X)$. If $\mathcal{I}$ is an ideal of definition of $(X, \mathcal{O}_X)$, then we will use the following abuse of notation: X will stand for the Hensel scheme $(X, \mathcal{O}_X)$ and X_0 will stand for the usual scheme $(X, \mathcal{O}_X/\mathcal{I})$. If f: $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is an adic map of Hensel schemes, we write $Y \times_X X_0$ for the scheme $(Y, \mathcal{O}_Y/f^*(\mathcal{I})\mathcal{O}_Y)$ If f is étale, the induced map $Y \times_X X_0 \to X_0$ is an étale map of schemes. But much more is true:

THEOREM 1. *The functor which sends an étale map* f: $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ *to the étale map* $Y \times_X X_0 \to X_0$ *induces an equivalence between the category of étale maps over* X *and the category of étale maps over* X_0.

PROOF. In [10, 3.6.1], a definition of étale map between local ringed spaces is given, which for schemes is the usual definition and for hensel schemes coincides with our definition. So we need only verify that the hypotheses of Satz 3.6.4 in [10] are verified for the map $(X, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)$. Certainly $(\mathcal{O}_X/\mathcal{I})_x$ is finite over $\mathcal{O}_{X,x}$ for x in X, and for the "universal homeomorphism" requirement, the proof of 3.6.4 shows that we use it only for étale maps $(Y, \mathcal{O}_Y)$ $\to (X, \mathcal{O}_X)$ of Hensel schemes. But étale maps are adic, and for any adic map $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$,

$$Y \times_X X_0 = (Y, \mathcal{O}_Y/f^*(\mathcal{I})\mathcal{O}_Y)$$

so that $Y \times_X X_0 \to Y$ is a homeomorphism since $f^*(\mathscr{I})\mathcal{O}_Y$ is an ideal of definition.

This theorem is the key to understanding what a Hensel scheme looks like. The one other essential ingredient is the structure sheaf $\mathcal{O}_X^{\text{et}}$ of a Hensel scheme $(X, \mathcal{O}_X)$. This sheaf is defined as follows: for an étale map of Hensel schemes $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, we set $\mathcal{O}_X^{\text{et}}(Y) = \mathcal{O}_Y(Y)$. By Theorem 1, $\mathcal{O}_X^{\text{et}}$ is a sheaf on $(X_0)_{\text{et}}$, and in one case we can determine precisely which sheaf this is:

THEOREM 2. *Let i: $Y \to X$ be a closed immersion of schemes. Then in Y_{et} we have an isomorphism*:

$$\mathcal{O}_{X^h}^{\text{et}} \cong i^* \mathcal{O}_X^{\text{et}}$$

where i^* means "brutál restriction".

PROOF. The map of ringed spaces $(Y, \mathcal{O}_{X^h}) \to (X, \mathcal{O}_X)$ induces a natural map $i^*\mathcal{O}_X^{\text{et}} \to \mathcal{O}_{X^h}^{\text{et}}$. To show that this is an isomorphism, we need only show that the map$\mathcal{O}_{X,\xi}^{\text{et}} \to \mathcal{O}_{X^h,\xi}^{\text{et}}$ is an isomorphism for every geometric point ξ: Spec $(\Omega) \to Y$.

Let ξ lie over the point x in Y. We will compute $\mathcal{O}_{X^h,\xi}^{\text{et}}$ by working in the étale topology of X^h. Here,

$$\mathcal{O}_{X^h,\xi}^{\text{et}} = \text{lim-ind}\, \mathcal{O}_Z(Z),$$

where we range over étale maps of Hensel schemes $(Z, \mathcal{O}_Z) \to (Z, \mathcal{O}_{X^h})$ (and Z contains a point z lying over x where $k(x) \subseteq k(z) \subseteq \Omega$). For such a Z, $Z \times_{X^h} Y$ is a étale nbd of x in Y. By [8, IV.18.1.1], there is an affine étale nbd U of x in X so that $U \times_X Y$ is an open subset (containing the given point) of $Z \times_{X^h} Y$. By shrinking Z, we can assume $Z \times_{X^h} Y = U \times_X Y$, so that by Theorem 1, $Z = U^h$ (the henselization of U along $U \times_X Y$). By [10, 7.1.2.1], $\mathcal{O}_Z(Z)$ is the henselization of $\mathcal{O}_U(U)$ with respect to the ideal defining $U \times_X Y$ (we write this as $\mathcal{O}_U(U)^h$) so that

$$\mathcal{O}_{X^h,\xi}^{\text{et}} = \text{lim-ind}\, \mathcal{O}_U(U)^h$$

where we now range over étale nbds U of x in X. Henselization commutes with filtering direct limits, so we have

$$\mathcal{O}_{X^h,\xi}^{\text{et}} = (\text{lim-ind}\, \mathcal{O}_U(U))^h = (\mathcal{O}_{X,\xi}^{\text{et}})^h.$$

Since $\mathcal{O}_{X,\xi}^{\text{et}}$ is a Hensel local ring, it is Hensel with respect to any ideal, so $\mathcal{O}_{X^h,\xi}^{\text{et}} = \mathcal{O}_{X,\xi}^{\text{et}}$.

Thus, for any questions about the étale topology of a closed immersion i: $Y \to X$, we can forget about the henselization X^h and instead work with the sheaf $i^*\mathcal{O}_X^{\text{et}}$ on the site Y_{et}.

3. Other preliminaries.

There are lots of ways to detect étale maps:

PROPOSITION 3. *Let Y be a closed subscheme of a scheme X and let f: $W \to X$ be locally of finite presentation. Assume that f induces an isomorphism $W \times_X Y \xrightarrow{\sim} Y$. Then the following conditions are equivalent:*

1. *f: $W \to X$ is étale in a* nbd *of Y in X.*
2. *f^h: $W^h \to X^h$ is an isomorphism.*
3. *(If X is locally noetherian): $f^\wedge$: $W^\wedge \to X^\wedge$ is an isomorphism.*
4. *(If X is separated of finite type over* Spec (C)):

f^{an}: $W^{\text{an}} \to X^{\text{an}}$ induces a biholomorphism between complex nbds *of Y^{an} sitting in W^{an} and X^{an} respectively.*

PROOF. 1 $\Rightarrow$ 2. By Proposition 2, f^h is étale. Since f^h induces an isomorphism of Y, f^h is an isomorphism by Theorem 1.

2 $\Rightarrow$ 1 follows immediately from Proposition 2.

1 $\Leftrightarrow$ 3 is proved in [3, Lemma 4.2].

4 $\Rightarrow$ 1 is obvious, and to prove 1 $\Rightarrow$ 4, note that we can replace W by a nbd of Y which is of finite type over Spec (C). Then W^{an} and X^{an} are separable metric, and the conclusion follows from [12, Lemma 5.7].

We will also need:

LEMMA 1. *Let f: $U \to V$ be étale.*

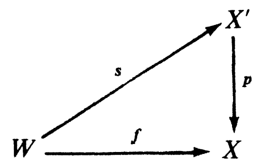
1. *If the family of irreducible components of V is locally finite, the same is true for U.*
2. *If V is normal and irreducible with generic point ξ, then $f^{-1}(\xi)$ is the set of generic points of the irreducible components U_α of U, and U is the disjoint union of the U_α (and U is normal).*

PROOF. Since f is étale, it is flat and its fibers are locally finite. So the first statement follows from [7, I 3.9.3 and 3.9.6 (ii)].

U is normal because f is étale, and is the sum of its irreducible components by 1 and [7, I 2.1.9]. $f^{-1}(\xi)$ certainly contains all the maximal points of U (see [7, I § 3.9]). Given any η in $f^{-1}(\xi)$, η is the specialization of a maximal point η'

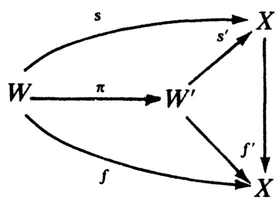
which also lies in $f^{-1}(\xi)$. Thus, the specialization takes place in $f^{-1}(\xi)$, which is discrete because f is étale. So $\eta = \eta'$.

Now, suppose that we have a commutative diagram:



where W is connected, X is normal and irreducible, f is étale and p is separated. Then W is normal and, by the second part of Lemma 1, irreducible.

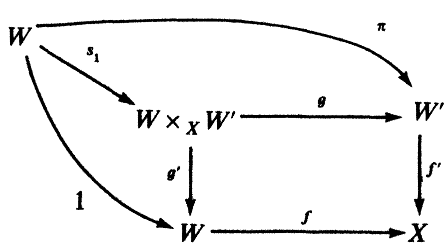
$\overline{s(W)}$, the closure of $s(W)$ in X', becomes a reduced closed subscheme of X', and is irreducible since W is. Since $s: W \to \overline{s(W)}$ is dominating and W is normal, s factors as $W \xrightarrow{\pi} W' \to \overline{s(W)}$, where W' is the normalization of $\overline{s(W)}$ (see [8, II 6.3.9]). We get a commutative diagram:



LEMMA 2. *With the above hypothesis, π is étale and f' is étale on the open set $\pi(W)$ of W'.*

PROOF. The map f': $W' \to X$ factors $W' \to \overline{s(W)} \to X' \xrightarrow{p} X$, so that f' is an affine map, followed by a closed immersion, followed by a separated map. Thus f' is separated.

We have a commutative diagram:



and the map s_1 exists by the universal property of $W \times_X W'$. Note that s_1 is a closed immersion since f' and hence g' are separated.

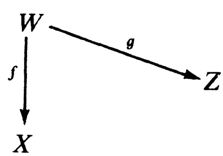
The generic point of W maps to the generic point of W' since π is dominating. But g is étale since f is, so by part 2 of Lemma 1, W is an irreducible component of $W \times_X W'$, which is open in $W \times_X W'$. Thus, π is étale, and then f' is étale on $\pi(W)$ by [8, IV 17.7.5].

4. Extending maps to a "nbd" of a closed subscheme.

Fix a base scheme S. Suppose that we have Y a closed subscheme of an S-scheme X, and let g_0: $Y \to Z$ be an S-morphism. There are several ways to formulate what it means for g_0 to "extend" to a nbd of Y in X:

DEFINITION 3. With the above notation, we say:

1. g_0 extends to an étale nbd of Y in X if there is a commutative diagram (over S):



where f: $W \to X$ is an étale nbd of Y in X with $g|_Y = g_0$.

2. g_0 extends to X^h if there is a S-map g^h: $X^h \to Z$ with $g^h|_Y = g_0$.

3. g_0 extends to $X^\wedge$ if there is a S-map $g^\wedge$: $X^h \to Z$ with $g^\wedge|_Y = g_0$.

We will consider the case of varieties over Spec ($\mathbb{C}$) at the end of this section.

From Proposition 3, we see that every extension of g_0 to an étale nbd of Y in X gives us an extension to X^h, and from 1 it is clear that an extension to X^h gives, via completion, an extension to $X^\wedge$. The next question is to ask under which conditions do these implications reverse?

With strong geometric assumptions on the embedding of Y in X, it is well known that the three notions coincide. For example, suppose that X and Z are locally of finite type over a locally noetherian base S. Assume also that X is regular, Z irreducible and Y is G-2 in X (see [9, V]). Then the proof of Theorem 4.3 of [3] (see also [4]) is easily adapted to show that any extension $g^\wedge: X^\wedge \to Z$ of g_0 is induced by an extension of g_0 to an étale nbd of Y in X. (the regularity of X enters in using Ex. V 4.12 of [9].)

What is more surprising is that notions 1 and 2 of Definition 3 coincide under very weak assumptions:

THEOREM 3. *Suppose that X is separated, normal and irreducible, and let Y be a closed connected subscheme of X.*

Then any extension of g_0 to g^h: $X^h \to Z$, where Z is locally of finite presentation and separated over S, comes from an extension of g_0 to an étale nbd *of Y in X.*

PROOF. Set $X' = X \times_S Z$. Then projection on the first factor gives us a map $p: X' \to X$ which is separated and locally of finite presentation, and g^h gives a section s^h: $X^h \to X'$ of p over X^h. If we can find a section s of p over an étale nbd W of Y in X extending s^h, then $g = \mathrm{pr}_2 \circ s$: $W \to Z$ is clearly the desired extension of g_0 to an étale nbd of Y in X. So we will work with s^h: $X^h \to X'$. The section we get over Y (induced by g_0) we will call s_0.

First, assume that X' and X are affine, say $X = \mathrm{Spec}\,(A)$. Then s^h: $X^h \to X'$ gives us a map

$$\Gamma(X', \mathcal{O}_{X'}) \to \Gamma(Y, \mathcal{O}_{X^h}) = A^h ,$$

which induces a map η: $\mathrm{Spec}\,(A^h) \to X'$ since X' is affine. But $\mathrm{Spec}\,(A^h)$ is an inverse limit of affine étale nbds of Y in X, and p is locally of finite presentation, so standard arguments from [8, IV § 8.8] show that η is induced by a map s: $W \to X'$, where W is an étale nbd of Y in X. Clearly s induces s^h. The general case is done in several steps. First, we show how to locally (in the Zariski topology around Y in X) find W. Let y be a point of Y and let U' be an affine nbd of $s_0(y)$ in X' (remember that we have s_0: $Y \to X'$). Then let U be an affine nbd of y in X so that $Y \cap U \subseteqq s_0^{-1}(U')$. Pulling everything back to U, we get

$$s^h|_{Y \cap U}: U^h \to p^{-1}(U) .$$

But by our choice of U and U', $U' \cap p^{-1}(U)$ is a nbd of $s_0(Y \cap U)$ in $p^{-1}(U)$, so $s^h|_{Y \cap U}$ clearly factors through $U' \cap p^{-1}(U)$, so that we have

$$s^h|_{Y \cap U}: U^h \to U' \cap p^{-1}(U) .$$

But $U' \cap p^{-1}(U) = U' \times_X U$ is affine since U' and U are affine and X is separated. Then we are in the affine case treated in the above paragraph. Thus, $s^h|_{Y \cap U}$ is induced by a map s: $W \to X'$ where W is an affine étale nbd of $Y \cap U$ in U.

We next examine how compatible are these different local extensions of s^h. So suppose we have s_i: $W_i \to X'$, sections of p, where W_i is an étale nbd of $Y \cap U_i$ in U_i, and $s_i^h = s^h|_{Y \cap U_i}$ (all this for $i = 1, 2$). Then $W_1 \times_X W_2$ is an étale nbd of $Y \cap U_1 \cap U_2$, and we have maps

$$s_i \circ p_i: W_1 \times_X W_2 \to X', \quad i = 1, 2 ,$$

which are sections of p, and $(s_i \circ p_i)^h = s^h|_{Y \cap U_1 \cap U_2}$. We claim $s_1 \circ p_1 = s_2 \circ p_2$ in a nbd of $Y \cap U_1 \cap U_2$ in $W_1 \times_X W_2$.

To see this, first note that we can replace X with $U_1 \cap U_2$ and W_i, s_i, etc. by their restrictions to $U_1 \cap U_2$. Thus, we can assume that $U_1 = U_2 = X$. Then let W be the closed subscheme of $W_1 \times_X W_2$ where $p_1 \circ s_1 = p_2 \circ s_2$. Since $s_i|_Y = s_0$, W contains Y. Then we have a commutative diagram:

$$(2)\qquad \begin{array}{ccccc} W^h & \longrightarrow & W & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} \\ (W_1 \times_X W_2)^h & \longrightarrow & W_1 \times_X W_2 & \xrightarrow{(s_1, s_2)} & X' \times_X X' \end{array}$$

where the right square is cartesian by definition and the left square is cartesian by [10, 7.5.4]. Note that by [10, 7.1.1.3] and Proposition 3,

$$(W_1 \times_X W_2)^h = W_1^h \times_{X^h} W_2^h \cong X^h .$$

Since $s_1^h = s_2^h = s^h$, (2) gives us a cartesian diagram:

$$\begin{array}{ccc} W^h & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X^h & \longrightarrow & X' \times_X X' \end{array}$$

which shows that $W^h \to (W_1 \times_X W_2)^h$ is an isomorphism. Then Proposition 3 shows that W is étale in a nbd of Y, so that W contains a nbd of Y in $W_1 \times_X W_2$ as claimed.

So far we have made no use of the fact that X is normal. Thus one can always get the "incomplete" descent data described above.

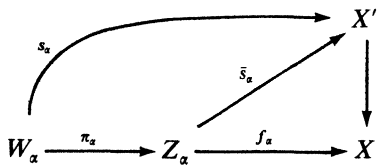
Next we refine this data slightly. Suppose we have $s\colon W \to X'$ where W is an étale nbd of $Y \cap U$ in U, for $U \subseteq X$ open. We will regard $Y \cap U$ as sitting in W. Let y be in $Y \cap U$, and let W_0 be the connected component of W containing y. Since X is irreducible, part 1 of Lemma 1 and [7, 0.2.1.5] show that W_0 is open in W. Set

$$U_0 = U - (Y \cap U - Y \cap U \cap W_0),$$

so that U_0 is a nbd of y in X. Then $W_0 \to X$ is an étale nbd of $U_0 \cap Y$ in U_0. Thus, in our local extensions $s\colon W \to X'$ of s^h, we can always assume that W is connected.

The above discussion gives us sections s_α: $W_\alpha \to X'$ of p such that W_α is a connected étale nbd of $Y \cap U_\alpha$ in U_α, where U_α is open in X, such that $s_\alpha^h = s^h|_{Y \cap U_\alpha}$ and $Y \subseteq \bigcup_\alpha U_\alpha$. From here on, we will freely refer to Lemma 2 (and the discussion preceding it) in 2. Let Z_α be the normalization of $\overline{s_\alpha(W_\alpha)}$. By Lemma 2, we get a commutative diagram:

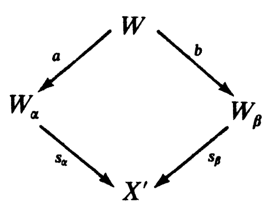
(3)



where π_α is étale and f_α is étale on $\pi_\alpha(W_\alpha)$. Since $Y \cap U_\alpha$ sits in W_α, we get a map φ_α: $Y \cap U_\alpha \to Z_\alpha$ which over U_α becomes a closed immersion $Y \cap U_\alpha \to f_\alpha^{-1}(U_\alpha)$ (we showed in Lemma 2 that f_α is separated). This allows us to form $\bar{s}_\alpha^h$, and clearly $\bar{s}_\alpha^h = s^h|_{Y \cap U_\alpha}$.

Let us show that for any α and β, $\overline{s_\alpha(W_\alpha)} = \overline{s_\beta(W_\beta)}$. Since Y is connected and $\{Y \cap U_\gamma\}$ is an open cover, for any α and β there is a chain of $Y \cap U_\gamma$'s, starting at $Y \cap U_\alpha$ and ending at $Y \cap U_\beta$, where two consecutive ones meet. Thus, we can assume that $Y \cap U_\alpha \cap U_\beta \neq \varnothing$. Replacing 1 and 2 by α and β in the fourth paragraph of the proof, we get a commutative diagram:

(4)



where W is an étale nbd of $Y \cap U_\alpha \cap U_\beta$ (so $W \neq \varnothing$) and a and b are étale. Then $a(W)$ is open in W_α, hence dense since W_α is irreducible. Then simple continuity shows that $\overline{s_\alpha(a(W))} = \overline{s_\alpha(W_\alpha)}$. The same is true for b and β, so the commutativity of (4) shows that $\overline{s_\alpha(W_\alpha)} = \overline{s_\beta(W_\beta)}$.

Thus $Z_\alpha = Z_\beta$ for any α and β, so we call this Z. Note that diagram (4) the factors through Z (see (3)). This shows that the map φ_α: $Y \cap U_\alpha \to Z$ defined above patch to give a map φ: $Y \to Z$, which is a section of f: $Z \to Y$. Set $U = \bigcup_\alpha \pi_\alpha(W_\alpha)$, and note that $\varphi(Y) \subseteq U$ and U is in fact an étale nbd of Y in X. We have the section $\bar{s}|_U$: $U \to X'$ of p, and $\bar{s}|_U$ clearly induces s^h. This proves the theorem.

We next turn our attention to notions 2 and 3 of Definition 3. These two concepts are obviously different. For example, let $Y=\{x\}$, where x is a closed point of X. The any $\varphi \in \hat{\mathcal{O}}_{X,x}$ gives a map $g^{\wedge}: X^{\wedge} \to A^1_Z$, and $g^{\wedge}$ is induced by a map $g^h: X^h \to A^1_Z$ iff φ lies in $\mathcal{O}^h_{X,x}$. Note, however, what is still true in this case: one can find g^h which approximates $g^{\wedge}$ arbitrarily closely (since $(\mathcal{O}^h_{X,x})^{\wedge} \cong \hat{\mathcal{O}}_{X,x}$). In fact, by the Artin approximation theorem, this is true for any S-map $g^{\wedge}: X^{\wedge}=\mathrm{Spf}\,(\hat{\mathcal{O}}_{X,x}) \to Z$, where Z is of finite type over S (see [16, III Example]).

Thus, for Y arbitrary, we can still ask if any $g^{\wedge}: X^{\wedge} \to Z$ extending $g_0: Y \to Z$ can be approximated by an extension $g^h: X^h \to Z$ of g_0. In section 6 we will see some special circumstances where this is possible. But in general the answer is no, as is shown by the following example due to Arthur Ogus (see [13] and [14, § 4]):

EXAMPLE 1. Let $Y \subseteq \boldsymbol{P}^2(\mathbf{C})$ be an elliptic curve, and let X be $\boldsymbol{P}^2(\mathbf{C})$ blown up at nine points of Y. Then Y sits inside X, and its normal bundle N has degree zero. If N is a non-torsion point of $\mathrm{Pic}^\circ(Y)$, then one easily shows that there is a unique map $g^{\wedge}: X^{\wedge} \to Y$ extending the identity map on Y. Ogus shows that if X is obtained by blowing up points in sufficiently general position, then the identity map on Y does not extend to *any* étale nbd of Y in X. From Theorem 3, we see that there is *no map* $g^h: X^h \to Y$ which induces the identity on Y.

We will return to this example several times.

Finally, let X and Z be separated schemes of finite type over $\mathbf{C}$. Then, in addition to Definition 3, we say that $g_0: Y \to Z$ extends to a nbd of Y^{an} if there is a complex nbd U of Y^{an} in X^{an}, and a map of analytic spaces $g^{\mathrm{an}}: U \to Z^{\mathrm{an}}$ such that $g^{\mathrm{an}}|_Y = g_0^{\mathrm{an}}$. We want to see how this notion compares to those introduced in Definition 3. We do not have a completely satisfactory answer (see below). But we can say some things.

First, if g_0 extends to $g^h: X^h \to Z$, then g_0 does extend to a nbd of Y^{an} in X^{an}. For recall from the proof of Theorem 3 that we do get some incomplete patching data. Applying Proposition 3 to this, we can find U_i open in X^{an}, covering Y^{an}, and maps $g_i^{\mathrm{an}}: U_i \to Z^{\mathrm{an}}$ extending $g_0^{\mathrm{an}}|_{Y^{\mathrm{an}} \cap U_i}$, such that $g_i^{\mathrm{an}} = g_j^{\mathrm{an}}$ in a nbd of $Y^{\mathrm{an}} \cap U_i \cap U_j$. We want to find U open in X^{an} with $Y^{\mathrm{an}} \subseteq U$ and a map $g^{\mathrm{an}}: U \to Z^{\mathrm{an}}$ such that $g^{\mathrm{an}}|_{U \cap U_i} = g_i^{\mathrm{an}}|_{U \cap U_i}$. We can assume that $\{U_i\}$ is a finite covering, and then by induction reduce to the case $X^{\mathrm{an}} = U_1 \cap U_2$, and $g_1^{\mathrm{an}} = g_2^{\mathrm{an}}$ on a nbd W of $Y^{\mathrm{an}} \cap U_1 \cap U_2$. X^{an} is a normal topological space, so we can find disjoint open sets V_i with

$$X^{\mathrm{an}} - U_1 \subseteq V_2, \ X^{\mathrm{an}} - U_2 \subseteq V_1 .$$

Then $V_i \subseteq U_i$ and $Y \subseteq V_1 \cup V_2 \cup W$ (we will call this open set U). Then define $g^{\mathrm{an}}: U \to Z^{\mathrm{an}}$ by

$$g^{\mathrm{an}}|_{V_i} = g_i^{\mathrm{an}}|_{V_i}, \quad g^{\mathrm{an}}|_W = g_i^{\mathrm{an}}|_W .$$

Second, if we are given $g^{an}: U \to Z^{an}$ extending g_0, and Y is complete, it follows easily from GAGA that g^{an} induces a map $g^\wedge: X^\wedge \to Z$. However, if Y is not complete, this need not happen:

EXAMPLE 2. Let Y be a variety with a non-trivial line bundle L such that L^{an} is trivial. (For example, let Y be the universal extension of an abelian variety A, and let L be the pull back of a non-zero element of $\mathrm{Pic}^\circ(A)$.) Let $X = V(L)$, and let $Y \subseteq X$ be the inclusion of the zero section.

Then the inclusion g_0: $Y = Y \times \{0\} \subseteq Y \times A^1 = Z$ has an analytic extension $g^{an}: X^{an} \to Z^{an}$, which is an isomorphism since L^{an} is trivial. But $(g^{an})^\wedge: (X^{an})^\wedge \to Z^{an}$ is *not* induced by any map $g^\wedge: X^\wedge \to Z$. For such a $\hat{g}$ would have to induce an isomorphism $X^\wedge \cong Z^\wedge$, which would imply that L is trivial.

But even if Y is complete, our knowledge is still incomplete. In Example 1, it is unknown whether or not the identity map on Y extends to a nbd of Y^{an} in X^{an}. This is the major undecided question in this area, and as Ogus shows in [14, § 4] its answer has many interesting implications.

5. The indeterminacy of étale nbds.

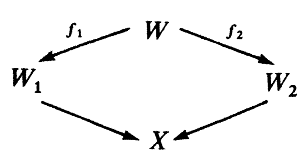
Assume that we still have X and Z over S, where Z is locally of finite presentation, and that we have an S-morphism g_0: $Y \to Z$, Y is closed in X.

While X^h and $X^\wedge$ are functorial notions of a "nbd" of Y in X, there are lots of étale nbds of Y in X. So one can first ask what it means for extensions to two different étale nbds to be the same. Also, one can ask if there is an extension to a "best" or "biggest" étale nbd.

The answer to the first question is contained in:

PROPOSITION 4. *Let* g_i: $W_i \to Z$, $i = 1, 2$, *be extensions of* g_0 *to étale* nbds *of* Y *in* X. *The following statements are equivalent*:

1. *There is a commutative diagram of étale* nbds *of* Y *in* X:



inducing the identity over Y, *such that* $g_1 \circ f_1 = g_2 \circ f_2$.

2. $g_1^h = g_2^h: X^h \to Z$.
3. (*If* X *and* S *are locally noetherian*) $\hat{g_1} = \hat{g_2}$: $X^\wedge \to Z$.

4. (*If X and S are separated of finite type over* Spec (**C**)) $g_1^{an} = g_2^{an}$ *in some* nbd *of* Y^{an} *in* X^{an}.

Proof. The statements $1 \Rightarrow 2$, $1 \Rightarrow 3$ and $1 \Rightarrow 4$ all follow immediately from Proposition 3. The proof of Theorem 3 shows that $2 \Rightarrow 1$ (see the arguments centering around diagram (2)). To show $3 \Rightarrow 1$, we proceed as in $2 \Rightarrow 1$, using [7, I 10.9.9] (respectively [7, I 10.9.7]) instead of [10, 7.5.4] (respectively [10, 7.1.1.3]). And to prove $4 \Rightarrow 1$, note that if $g_1^{an} = g_2^{an}$ in some nbd of Y^{an}, then

$$(g_1^{an})\hat{} = (g_2^{an})\hat{}: (X^{an})\hat{} \to Z^{an}.$$

Then, by faithful flatness, $\hat{g_1} = \hat{g_2}$, so that we are done by $3 \Rightarrow 1$.

This leads naturally to the following:

Definition 4. Two extensions $g_i\colon W_i \to Z$, $i = 1,2$, of g_0 to an étale nbd of Y in X are said to be equivalent, written $g_1 \sim g_2$, if any of the conditions of Proposition 4 are fulfilled.

We next turn to the question of finding a "best" member of each equivalence class:

Proposition 5. *Assume that X is normal and irreducible, with Y a closed connected subscheme, and that Z is locally of finite presentation and separated over S.*

Then, in any equivalence class $\mathscr{C}$ of extensions of g_0 to étale nbds *of Y in X, there is an extension $g\colon W \to Z$, unique up to isomorphism, with the following property:*

For any $g'\colon W' \to Z$ in the equivalence class $\mathscr{C}$, where W' is connected, there is a morphism $\eta\colon W' \to W$ of étale nbds *of Y in X such that $g' = g \circ \eta$. Furthermore, if W' is separated over X, η is unique.*

Proof. Any extension $g'\colon W' \to Z$ of g_0 gives us a section $s'\colon W' \to X \times_X Z = X'$ of $p = p_1\colon X' \to X$ over W'. It follows from the proof of Theorem 3 that $\overline{s'(W')}$ is independent of which $g'\colon W' \to Z$ in $\mathscr{C}$ that we use (as long as W' is connected). Let W_0 be the normalization of $\overline{s'(W')}$. Then, for any $g'\colon W' \to Z$ in $\mathscr{C}$ with W' connected, we get an X-morphism $\pi'\colon W' \to W_0$ (see (3)). Let W be the union of the open sets $\pi'(W')$ in W_0. As we saw in the proof of Theorem 3, W is an étale nbd of Y in X, and gives us an extension $g\colon W \to Z$ of g_0, which has the desired properties.

6. Applications to the study of embeddings.

Here we give an informal discussion of what happens when we apply the concepts and results of sections 4 and 5 to study closed embeddings of a given scheme Y in various schemes X.

Fix a base scheme S, and let X_1 and X_2 be two schemes which are locally of finite presentation and separated over S. Suppose that we are given two closed immersions $Y \hookrightarrow X_1$ and $Y \hookrightarrow X_2$. As in Definition 3, there are three notions of what "equivalent" embeddings should mean:

1. There is a commutative diagram over S:

$$(5) \qquad \begin{array}{ccc} & W & \\ {}^{f_1}\swarrow & & \searrow{}^{f_2} \\ X_1 & & X_2 \end{array}$$

making W an étale nbd of both Y in X_1 and Y in X_2 (with respect to a fixed embedding $Y \to W$). We say that W is a common étale nbd of the embeddings.

2. There is an S-isomorphism $\varphi^h\colon X_1^h \xrightarrow{\sim} X_2^h$ with $\varphi^h|_Y = 1_Y$.
3. There is an S-isomorphism $\varphi^\wedge\colon X_1^\wedge \xrightarrow{\sim} X_2^\wedge$ with $\varphi^\wedge|_Y = 1_Y$.

A common étale nbd induces an isomorphism $\varphi^h\colon X_1^h \xrightarrow{\sim} X_2^h$ by setting $\varphi^h = f_2^h \circ (f_1^h)^{-1}$ (see Proposition 3). An given isomorphism φ^h, completion gives us $\varphi^\wedge$. Do these implications ever reverse themselves?

To compare notions 1 and 2, we have the following theorem which is an immediate corollary of Theorems 3 and Proposition 3:

THEOREM 4. *If X_1 and X_2 are separated, normal and irreducible, and if Y is connected, then any S-isomorphism $\varphi^h\colon X_1^h \xrightarrow{\sim} X_2^h$ is induced by a common étale* nbd (5) *of the two embeddings.*

Based on what we found in 4, the correct way to compare notion 3 with 1 and 2 is as follows: given two embeddings which are formally equivalent (which means an S-isomorphism $\varphi^\wedge\colon X_1^\wedge \to X_2^\wedge$), is there an isomorphism $\varphi^h\colon X_1^h \xrightarrow{\sim} X_2^h$ (or even a common étale nbd inducing φ^h) which approximates $\varphi^\wedge$?

Under strong geometric conditions, the answer to this question is yes. For example if S is locally noetherian and X_1 and X_2 are regular and irreducible, and each embedding $Y \subseteq X_i$ is G-2, then any formal equivalence is actually *induced* by a common étale nbd (see [3] and [4]). Or if $S = \mathrm{Spec}(k)$, k an algebraically closed field, Y is proper over k and each embedding $Y \subseteq X_i$ has negative normal bundle, then using [2, Theorem 6.32 and [1, Corollary 2.6], it is easy to show that $\varphi^{\wedge}$ can be approximated by a common étale nbd of the embeddings.

But in general, the answer is no:

EXAMPLE 1 (Continued). Let $V = V(N)$, and let $Y \to V$ be the zero section. Then one easily shows that there is a unique isomorphism $\varphi^{\wedge}: X^{\wedge} \to V^{\wedge}$ inducing the identity on Y (see [13]). But there is *no map* $\varphi^h: X^h \to V^h$ inducing the identity on Y: for then the composition $X^h \to V^h \to V \to Y$ would extend the identity map on Y. We have already seen that no such map exists.

If $S = \mathrm{Spec}\,(\mathbf{C})$ and X_1 and X_2 are of finite type, then, along with the above three notions, one can ask whether two embeddings $Y \hookrightarrow X_i$ $(i=1,2)$ have biholomorphic nbds in the complex topology, an isomorphism $\varphi^{an}: U_1 \xrightarrow{\sim} U_2$, where $Y^{an} \subseteqq U_i \subseteqq X^{an}$ is open.

If there is an isomorphism $\varphi^h: X_1^h \to X_2^h$, then the argument of 4 shows that φ^h induces a biholomorphism $\varphi^{an}: U_1 \xrightarrow{\sim} U_2$. And given φ^{an}, it induces an isomorphism $\varphi^{\wedge}: X_1^{\wedge} \to X_2^{\wedge}$ if Y is complete. If Y is not complete, this might not be so: in Example 2, we had $\varphi^{an}: X^{an} \xrightarrow{\sim} Z^{an}$, an biholomorphism of the embeddings $Y \subseteqq X$ and $Y \subseteqq Z$, but as we saw, there was *no* isomorphism $\varphi^{\wedge}$: $X^{\wedge} \xrightarrow{\sim} Z^{\wedge}$.

Even when Y is complete, the situation is not clear. In Example 1, it is not known whether the two embeddings $Y \subseteqq X$ and $Y \subseteqq V$ have biholomorphic nbds.

Finally, as in 5, one can determine when two common étale nbds give the "same" equivalence of two embeddings $Y \subseteqq X_1$ and $Y \subseteqq X_2$. What is more interesting is that, under the hypothesis of Theorem 4, if the two embeddings have a common étale nbd, then one can find a "biggest" one (which is essentially the normalization of its graph in $X_1 \times_S X_2$).

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