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ALGEBRAIC TUBULAR NEIGHBORHOODS II

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Let Y be a closed subscheme of a Noetherian scheme X. This paper introduces a geometric model for an algebraic tubular neighborhood (nbd) of Y in X which has the same homotopy properties as the usual tubular nbds one encounters in differential topology. We replace X by hypercoverings $U.$ of X in the étale topology, take nbds $V.$ of $U. \times_X Y$ in $U.$, and then our tubular nbd is the system of these $V.$'s (see Definition 1.1 and the remarks following it). Since the inclusion $U. \times_X Y \subseteqq U.$ is "equivalent" to $Y \subseteqq X$, $V.$ really is a "nbd" of Y in X, and we show that these $V.$'s really closely approximate Y (see Theorems 1.3 and 2.2). We call this model geometric because it "sits" inside X (in that $V. \subseteqq U.$) and we can "remove" Y from it (by considering the system $V. \times_X Y$—see Definition 2.1). This last fact enables us to construct spherical fibrations similar to the ones in differential topology (see Definition 2.1, Proposition 3.1 and Theorem 3.2). We also compare this model to the henselization X^h defined in [4] (see Proposition 4.1). Finally, we give an application of our theory to the construction of an algebraic exponential map (Theorem 5.1).

This paper makes frequent use of the theory of simplicial schemes developed in [5]. One simplification is that all of our simplicial schemes $V.$ lie in $\mathrm{simpl}(X_{\mathrm{et}})$, so that we can work in the category $\mathrm{Et}(V.)$ rather than $C(V.)$ (see [5, Introduction]), and we need only consider sheaves on X_{et} rather than X_{fl} (see [5, § I.2]). All of the results of [5] apply in this case (see [5, Theorem III.10] and [7, Chapter 1]). We will also draw heavily on the proofs of the main theorems in [6] (also notation—see section 2).

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1.

Let Y be a closed subscheme of a Noetherian scheme X. These will be fixed for the rest of this section. The "tubular neighborhood" of Y in X, written $t_{X/Y}$, is defined as follows:

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DEFINITION 1.1. $t_{X/Y}$ is the full subcategory of simpl (X_{et}) whose objects are those simplicial schemes $V.$ which satisfy:

1. Each V_n is a Noetherian and separated over X.
2. $V.\times_X Y$ is a hypercovering of Y (see [2, § 8]).

As pointed out in [2, § 8], each $V.$ in $t_{X/Y}$ has a unique splitting.

Before discussing the properties of $t_{X/Y}$ let us give some motivation for Definition 1.1. Let $V.$ be a simplicial scheme satisfying the first condition of Definition 1.1. Then $V.$ is in $t_{X/Y}$ iff there is a simplicial map $\varphi.: V. \to U.$ for which the following conditions hold:

1. $\varphi.$ induces an isomorphism $V.\times_X Y \xrightarrow{\sim} U.\times_X Y$.
2. Each $\varphi_n: V_n \to U_n$ is an open immersion.
3. $U.$ is a hypercovering of X.

This is proved using the techniques of [2, § 8] (see [7, Chapter 1, Lemma III.1] for details). Thus, to get our tubular nbd, we replace X by hypercoverings $U.$, and then take "Zariski" nbds $V.$ of $U.\times_X Y$ (which is a hypercovering of Y).

$t_{X/Y}$ has the following properties:

LEMMA 1.2. 1. *Let I be the homotopy category of $t_{X/Y}$ (see [2, § 8]). Then I° is filtering.*

2. *Given $V.$ in $t_{X/Y}$ and a separated étale map $\varphi: W \to V_n$ of finite type such that $W\times_X Y \to V_n \times_X Y$ is onto, there is a map $\varphi.: W. \to Y.$ in $t_{X/Y}$ such that φ_n factors through φ.*

PROOF 1. We first note that if $V.$ and $W.$ are in $t_{X/Y}$, then so is $V.\times_X W.$ (for $(V.\times_X W.)\times_X Y = (V.\times_X Y.)\times_Y (W.\times_X Y.)$ is a hypercovering of Y by [1, V. 7.3.4.2]). And given $f, g: V. \to W.$ in $t_{X/Y}$, we want to find $h: Z. \to V.$ in $t_{X/Y}$ such that $f\circ h$ is homotopic to $g\circ h$.

To do this, set $Z. = \mathbf{Hom}(\Delta[1], W.) \times_{W.\times_X W.} V.$. Since each Z_n is calculated by finite inverse limits in X_{et} (see [1, V 7.3.7]), the first condition of Definition 1.1 is satisfied. The construction of $Z.$ commutes with $\times_X Y$, so that by [1, V 7.3.7], $Z.\times_X Y$ is a hypercovering of Y. Thus $Z.$ is in $t_{X/Y}$ and has the desired property (see [1, V 7.3.7]).

2. Given $V.$ in $t_{X/Y}$ and a separated étale map $\varphi: W \to V_n$, set $W. = \beta_n(\varphi)$ (β_n is the right adjoint to the functor $(W. \to V.) \to (W_n \to V_n)$ — see [5, Appendix B] for details). Adjointness gives us a map $\varphi.: W. \to V.$ where φ_n factors through φ. The formula for β_n ([5]) shows that condition 1 of Definition 1.1 is satisfied. This formula also shows that $W.\times_X Y = \beta_n(W\times_X Y \to V_n \times_X Y)$. Since

$W \times_X Y \to V_n \times_X Y$ is onto, the proof of Proposition III.7 in [5] shows that $W. \times_X Y$ is a hypercovering of Y.

We next want to study $t_{X/Y}$ from a cohomological point of view. Let F be an abelian sheaf on X_{et}. Then we get a sheaf $\varepsilon^* F$ on each $V.$ in $t_{X/Y}$ ($\varepsilon^* F$ is in fact a local system on $V.$ — see [5, § I.6]), and cohomology groups $H^q(V., \varepsilon^* F)$ (see [5, § II.2]). Since homotopic maps in $t_{X/Y}$ induce the same map on cohomology (see [5, Corollary IV.7]), the functor $V. \mapsto H^q(V., \varepsilon^* F)$ factors through I, and so we can form $\varinjlim_{I^\circ} H^q(V., \varepsilon^* F)$. Also, each $V.$ in $t_{X/Y}$ gives us a cochain complex $F(V)$, and one sees easily that the functor $V. \mapsto H^q(F(V.))$ also factors through I. Thus, we can form $\varinjlim_{I^\circ} H^q(F(V.))$.

THEOREM 1.3. *For X, Y and F as above, there are canonical isomorphisms*:

$$\varinjlim_{I^\circ} H^q(V., \varepsilon^* F) \cong \varinjlim_{I^\circ} H^q(F(V.)) \cong H^q(Y, F|_Y)$$

(*where $F|_Y$ is the "brutal restriction" of F to Y*).

PROOF. We actually need a slightly more general result, which we formulate as follows. Let $F.$ be a local system (see [5, § I.6]) on a fixed $W.$ in $t_{X/Y}$. Let $J = I/W.$, the category of maps $V. \to W.$ in I. Then we have:

1. $\varinjlim_{J^\circ} H^q(V., F.)$
2. $\varinjlim_{J^\circ} H^q(F.(V.))$, where $F.(V.)$ is the obvious cochain complex one gets by regarding $F.$ as a $(V.)_{\text{et}}$ sheaf (see [5, § I.1 and § II.2]).
3. $F.|_{W. \times_X Y}$ is a local system on $W. \times_X Y$ and thus gives descent data by [5, § I.6]. Since $W. \times_X Y$ is a hypercovering of Y, we get a sheaf G on Y.

We will give canonical isomorphisms:

$$\varinjlim_{J^\circ} H^q(V., F.) \cong \varinjlim_{J^\circ} H^q(F.(V.)) \cong H^q(Y, G)\,.$$

The first isomorphism is easy: regarding $F.$ as a $(W.)_{\text{et}}$ sheaf (see [5, II.2]), we have a spectral sequence for α: $V. \to W.$ in $t_{X/Y}$:

$$E_1^{p,q}(\alpha.) = H^q(V_p, F_p) \Rightarrow H^{p+q}(V., F.)$$

which is functorial with respect to J from E_2 on. Thus we have a spectral sequence $\varinjlim_{J^\circ} E_2^{p,q}(\alpha.) \Rightarrow \varinjlim_{J^\circ} H^{p+q}(V., F.)$.

Let an element y in $E_2^{p,q}(\alpha.)$ be represented by a cocycle x in $E_1^{p,q}(\alpha.) = H^q(V_p, F_p)$, and assume that $q > 0$. Then we can find a surjective étale map φ: $Z \to V_p$, where φ is separated of finite type, such that x goes to zero in $H^q(Z, F_p)$. Applying Lemma 1.2, we get a map $\varphi.$: $Z. \to V.$ in $t_{X/Y}$ such that x goes to zero in $H^q(Z_p, F_p)$, and then y goes to zero in $E_2^{p,q}(\alpha. \circ \varphi)$. Thus $\varinjlim_{J^\circ} E_2^{p,q}(\alpha.) = 0$ for $q > 0$, so that

$$\varinjlim_{J^\circ} H^p(V., F.) \cong \varinjlim_{J^\circ} E_2^{p,0}(\alpha.) = \varinjlim_{J^\circ} H^p(F.(V.))\,.$$

Let $i: W. \times_X Y \to W.$ and $j: W. - W. \times_X Y \to W.$ be the natural inclusions. Then we have an exact sequence:

$$0 \to j_! j^* F. \to F. \to i_* i^* F. \to 0 .$$

Let $G. = j_! j^* F.$ and note that $i^* G. = 0$, that is for every p, $G_p|_{W_p \times_X Y} = 0$. Then we get the long exact sequence:

$$\ldots \to \varinjlim_{J^\circ} H^q(V., G.) \to \varinjlim_{J^\circ} H^q(V., F.) \to \varinjlim_{J^\circ} H^q(V., i_* i^* F.) \to \ldots$$

Note that $H^q(V., i_* i^* F.) \cong H^q(V. \times_X Y, i^* F.) \cong H^q(Y, G)$ since $V \times_X Y$ is a hypercovering of Y and $i^* F.$ determines G (see the proof of Theorem IV.2 of [5]). Thus, we need only show that $\varinjlim_{J^\circ} H^q(V., G.) = 0$ to prove the theorem.

By what we have already proved, this reduces to showing that $\varinjlim_{J^\circ} H^q(G.(V.)) = 0$ for all q. Let y in $H^q(G.(V.))$ be represented by a cocycle x in $G_q(V_q)$. Since $G_q|_{V_q \times_X Y} = 0$, in particular x vanishes in every fiber of G over $V_q \times_X Y$. Thus every point u of $V_q \times_X Y$ has an affine étale nbd $Z(u)$ of u in V_q where x goes to zero in $G_q(Z(u))$. Since $V_q \times_X Y$ is quasi-compact, we can find $Z(u_1), \ldots, Z(u_n)$ whose images cover $V_q \times_X Y$. If we set $Z = \bigcup_{i=1}^n Z(u_i)$, the map $\varphi: Z \to V_q$ satisfies the conditions of 2. of Lemma 1.2. Thus, we have a map $\varphi.: Z. \to V.$ in $t_{X/Y}$ so that x goes to zero in $G_q(Z_q)$, and hence y goes to zero in $H^q(G.(Z.))$. This shows that $\varinjlim_{J^\circ} H^q(G.(V.)) = 0$.

This theorem will be used in the proofs of Theorems 2.2 and 3.2 below, and it has already been used in [11, § 5].

2.

We now turn to the homotopy theory of our tubular neighborhood. We fix a closed connected subscheme Y of a connected noetherian scheme X. We also assume that $X - Y$ has a point ξ which specializes to a point η of Y. Then, by abuse of notation, we let $t_{X/Y}$ be the category whose objects are triples $(V., \xi', \eta')$, where $V.$ satisfies 1. and 2. of Definition 1.1, and ξ' and η' are points of V_0 lying above ξ and η respectively. An the morphism are point-preserving. Note that Theorem 1.3 still holds for $t_{X/Y}$.

We will use the notations and conventions of [6] with one exception. If $\mathscr{H}_*$ is the pointed homotopy category, then we will work in the category $(\text{Pro-}\mathscr{H}_*)_\#$, obtained by inverting all $\#$ isomorphisms in Pro-$\mathscr{H}_*$ (see [2, § 4] and [7, Appendix C]).

From [5, § III.2 and § III.5] we have a functor which assigns to a pointed object $V.$ in simpl (X_{et}) it's étale homotopy type $\{V.\}_{\text{et}}$ in $(\text{Pro-}\mathscr{H}_*)_\#$. Note that by [5, Theorem IV.6], homotopic maps in simpl (X_{et}) go to the same map in $(\text{Pro-}\mathscr{H}_*)_\#$. Since I° is filtering (see Lemma 1.2) and $(\text{Pro-}\mathscr{H}_*)_\#$ has filtering inverse limits (see [2, § A.4]), we can define:

DEFINITION 2.1. 1. The homotopy of the tubular nbd is $\{T_{X/Y}\}_{\mathrm{et}} = \varprojlim_I \{V.\}_{\mathrm{et}}$.
2. The homotopy fiber of Y in X is the homotpy fiber of the map $\{T_{X/Y} - Y\}^{\wedge} \to \{T_{X/Y}\}^{\wedge}$, where $\wedge$ means profinite completion and $\{T_{X/Y} - Y\}_{\mathrm{et}} = \varprojlim_I \{V. - V. \times_X Y\}_{\mathrm{et}}$.

More precisely we define $\{T_{X/Y}\}_{\mathrm{et}}$ using either the points above ξ or the points above η. In any given context, it is clear which one we use. And, of course, there is no ambiguity for $\{T_{X/Y} - Y\}_{\mathrm{et}}$. We have the following natural maps:

Every $V.$ in $t_{X/Y}$ gives us a pointed map $\{V. \times_X Y\}_{\mathrm{et}} \to \{V.\}_{\mathrm{et}}$. Since $V. \times_X Y$ is a hypercovering of Y, $\{V. \times_X Y\}_{\mathrm{et}}$ is canonically isomorphic to $\{Y\}_{\mathrm{et}}$ (see [5, Theorem IV.2]). Thus, we have a map i: $\{Y\}_{\mathrm{et}} \to \{T_{X/Y}\}_{\mathrm{et}}$.

Every $V.$ in $t_{X/Y}$ gives us a pointed map $\{V. - V. \times_X Y\}_{\mathrm{et}} \to \{V.\}_{\mathrm{et}}$, which gives us the map j: $\{T_{X/Y} - Y\}_{\mathrm{et}} \to \{T_{X/Y}\}_{\mathrm{et}}$ referred to above.

And every $V.$ in $t_{X/Y}$ gives us a map $\{V.\}_{\mathrm{et}} \to \{X\}_{\mathrm{et}}$ (see [5, IV.1]), so we get a map u: $\{T_{X/Y}\}_{\mathrm{et}} \to \{X\}_{\mathrm{et}}$.

This constrution is quite functorial. Let φ: $X' \to X$ be a morphism, where X' is noetherian and has points ξ', η' lying above the points ξ, η of X. Set $Y' = Y \times_X X'$. Then we get a morphism of fibrations in $(\text{Pro-}\mathscr{H}_*)_\#$:

$$
(1) \qquad \begin{array}{ccccc}
F' & \to & \{T_{X'/Y'} - Y'\}^{\wedge} & \to & \{T_{X'/Y'}\}^{\wedge} \\
\downarrow & & \downarrow & & \downarrow \\
F & \to & \{T_{X/Y} - Y\}^{\wedge} & \longrightarrow & \{T_{X/Y}\}^{\wedge} .
\end{array}
$$

THEOREM 2.2. *The map* i: $\{Y\}_{\mathrm{et}} \to \{T_{X/Y}\}_{\mathrm{et}}$ *is an isomorphism in* $(\text{Pro-}\mathscr{H}_*)_\#$.

PROOF. First, note that $\{T_{X/Y}\}_{\mathrm{et}}$ is connected. For

$$H^0(\{T_{X/Y}\}_{\mathrm{et}}, Z) = \varinjlim_{I^\circ} H^0(\{V.\}_{\mathrm{et}}, Z) \cong \varinjlim_{I^\circ} H^0(V., Z)$$

(where Z is now a constant sheaf on $V.$ — see [2, Corollary 10.8]). By Theorem 1.3,

$$\varinjlim_{I^\circ} H^0(V., Z) \cong H^0(Y, Z|_Y) \cong Z$$

(because Y is connected).

We next show that we have an isomorphism on fundamental groups. So we must show that the map

$$\mathrm{Hom}\,(\pi_1(T_{X/Y}), G) \to \mathrm{Hom}\,(\pi_1(Y), G)$$

is an isomorphism for every group G. We know that $\mathrm{Hom}\,(\pi_1(Y), G) \cong H^1(Y, G)$, the set of pointed G torseurs (in the étale topology) over Y (see [6, Appendix 1]). We also have

$$\mathrm{Hom}\,(\pi_1(T_{X/Y}), G) = \varinjlim_{I^\circ} \mathrm{Hom}\,(\pi_1(V.), G) \cong \varinjlim_{I^\circ} H^1(V., G) .$$

The map $\varinjlim_{I^\circ} H^1(V_., G) \to H^1(Y, G)$ sends the G torseur $Z_.$ over $V_.$ in $t_{X/Y}$ to the G torseur $Z_. \times_X Y$ over $V_. \times_X Y$, which descends to a G torseur Z over Y (because $V_. \times_X Y$ is a hypercovering of Y).

We will construct an inverse to this map. Let Z be a G torseur over Y. For every geometric point γ of Y, we have an inclusion of strict henselizations $Y_\gamma \subseteq X_\gamma$. Z gives us a G torsuer Z_γ over Y_γ, which extends uniquely to a G torseur over X_γ. Then usual descent (see [9, IV. § 8]) gives us an affine étale nbd $V(\gamma)$ of γ in X and a G torseur $Z(\gamma)$ over $V(\gamma)$ such that $Z(\gamma) \times_X Y \cong Z \times_Y (V(\gamma) \times_X Y)$. We can find $V(\gamma_1), \ldots, V(\gamma_n)$ whose images cover Y, and then we set $V_0 = \bigcup_{i=1}^n V(\gamma_i)$. The map $V_0 \to X$ is separated, étale, of finite type, and induces a surjection $V_0 \times_X Y \to Y$ (so that $V_. = \operatorname{cosk}_0 V_0$ is in $t_{X/Y}$). And we have a G torseur Z_0 over V_0 such that $Z_0 \times_X Y \cong Z \times_Y (V_0 \times_X Y)$.

Then we have G torseurs $p_1^* Z_0$ and $p_2^* Z_0$ over $V_1 = V_0 \times_X V_0$. Restricting to $V_1 \times_X Y$, we also have an isomorphism $\varphi \colon p_1^* Z_0 \times_X Y \xrightarrow{\sim} p_2^* Z_0 \times_X Y$. Arguing as we did in the above paragraph gives us a separated, étale map of finite type $W \to V_1$, which induces a surjection $W \times_X Y \to V_1 \times_X Y$, and an isomorphism

$$\bar{\varphi} \colon p_1^* Z_0 \times_{V_1} W \xrightarrow{\sim} p_2^* Z_0 \times_{V_1} W$$

compatible with φ. Then Lemma 1.2 gives us a map $W_. \to V_.$ in $t_{X/Y}$ with the following property:

(*) There is a G-torseur Z_0 on W_0 (with $Z_0 \times_X Y \cong Z \times_Y (W_0 \times_X Y)$) and an isomorphism $\bar{\varphi} \colon d_0^* Z_0 \xrightarrow{\sim} d_1^* Z_0$ on W_1 compatible with the obvious isomorphism $\varphi \colon d_0^* Z_0 \times_X Y \cong d_1^* Z_0 \times_X Y$ on $W_1 \times_X Y$.

Finally, the cocycle condition $d_1^* \bar{\varphi} = d_0^* \bar{\varphi} \circ d_2^* \bar{\varphi}$ holds on $W_2 \times_X Y$, and hence on some nbd W' of $W_2 \times_X Y$ in W_2. Applying Lemma 1.2 to $W' \to W_2$, we get $W'_.$ in $t_{X/Y}$ which satisfies both (*) and the cocycle condition. But then, by Proposition I.8 of [5], this gives us a G torseur $Z'_.$ on $W'_.$.

This gives us a map $H^1(Y, G) \to \varinjlim_{I^\circ} H^1(V_., G)$ which is easily checked to be well defined. The fact that it is the desired inverse is then immediate.

We next compare cohomology. If Γ is a twisted coefficient system on $\{T_{X/Y}\}_{\text{et}}$, we want to show that the map on cohomology

$$H^q(\{T_{X/Y}\}_{\text{et}}, \Gamma) \to H^q(\{Y\}_{\text{et}}, i^* \Gamma)$$

is an isomorphism. Γ is determined by a locally constant abelian sheaf $\Gamma_.$ on some $W_.$ in $t_{X/Y}$ (see [2, § 10]), and by [5, Proposition I.9], $\Gamma_.$ is a local system on $W_.$. The locally constant sheaf $\Gamma_.|_{W_. \times_X Y}$ on $W_. \times_X Y$ descends to a locally constant sheaf G on Y (since $W_. \times_X Y$ is a hypercovering of Y) and this corresponds to $i^* \Gamma$. And the map on cohomology becomes the map $\varinjlim_J H^q(V_., \Gamma_.) \to H^q(Y, G)$, where $J = I/W_.$. By Theorem 1.3, this map is an isomorphism.

Then, by the Artin–Mazur–Whitehead Theorem [2, § 4], the map $i: \{Y\}_{\text{et}} \to \{T_{X/Y}\}_{\text{et}}$ is an isomorphism in $(\text{Pro-}\mathscr{H}_*)_\#$.

3.

We next want to study the homotopy fiber F of Y in X (see Definition 2.1). We will use the techniques of [6], but here we are dealing with $\{T_{X/Y}-Y\}_{\text{et}} \to \{T_{X/Y}\}_{\text{et}}$ rather than $\{X-Y\}_{\text{et}} \to \{X\}_{\text{et}}$. The arguments of [6] were completely formal (and hence will apply immediately to the case at hand) once we knew certain facts about the local cohomology sheaves $\underline{H}^q_Y(X,F)$ for all F a locally constant sheaf on X (see [6, Appendix 2]). But the corresponding facts for $\underline{H}^q_{V.\times_X Y}(V.,F.)$ are easy to derive (the main tool is Proposition A.1 of the Appendix).

We now turn to the theorems. As usual, Y is a closed subscheme of a noetherian scheme X. They both are connected and have points as explained in section 2. F is the homotopy fiber of the map $\{T_{X/Y}-Y\}^\wedge \to \{T_{X/Y}\}^\wedge$.

THEOREM 3.1. *Let X be smooth over a scheme S and assume that Y has relative codimension $\geqq c$ (over S).*

If L is a set of primes invertible on X, then $\pi_q((F)^\wedge_L)=0$ for $q \leqq 2c-2$. Furthermore, if $c>1$, then $\pi_{2c-1}((F)^\wedge_L) \cong \hat{Z}_L(c)$.

PROOF. First, let's show that Theorems 2.3 and 2.4 of [6] apply to the fibration $F \to \{T_{X/Y}-Y\}^\wedge \to \{T_{X/Y}\}^\wedge$.

The proofs of these theorems use Propositions 2.1 and 2.2 of [6] and the vanishing of local cohomology. But Proposition 2.1 of [6] generalizes easily to the inclusion $V.-V.\times_X Y \to V.$ for $V.$ in $\text{simpl}(X_{\text{et}})$ (one studies the map $H^1(V.,G) \to H^1(V.-V.\times_X Y,G)$, using [1, XVI 3.2.1 and 3.3] and the fact that a G torseur $F.$ on $V.$ is determined by the G torseurs F_n on V_n). Proposition 2.2 of [6] certainly generalizes. Finally, we have to show that

$$H^q_{V.\times_X Y}(V.,Z/nZ) = 0 \quad \text{for } q<2c .$$

This is equivalent to showing $\underline{H}^q_{V.\times_X Y}(V.,Z/nZ)=0$ for $q<2c$, which by Proposition A.1 is equivalent to $\underline{H}^q_{V_n\times_X Y}(V_n,Z/nZ)=0$ for all $n \geqq 0$ and $q<2c$. Since V_n is étale over X, we are done by Proposition A4 of [6].

Then $\pi_q((F)^\wedge_L)=0$, $q<2c-2$, follows from the proof of Theorem 2.3 of [6] (where $K=\{\text{all primes}\}$), and the isomorphism $\pi_{2c-1}((F)^\wedge_L) \cong \hat{Z}_L(c)$ follows from the proof of $2 \Rightarrow 1$ of Theorem 2.4 of [6] (and the fact that $\pi_1(Y) \to \pi_1(T_{X/Y})$ is a isomorphism — see Theorem 2.2).

When $c=1$, it is also true that $\pi_1((F)\hat{_L})\cong\hat{Z}_L(1)$, but the proof is more difficult (it uses Theorem 1.3 and the techniques of [6, Theorem 2.10]) and therefore is omitted.

And now, the main theorem of the paper:

THEOREM 3.2. *Let X and Y be smooth over a base scheme S, and assume that Y has relative codimension c in X.*

If L is a set of primes invertible on X, then $(F)\hat{_L}\cong(S^{2c-1})\hat{_L}$ in $(\text{Pro-}\mathscr{H}_*)_\natural$.

PROOF. Let $V.$ be in simpl (X_{et}). Then $V_n\times_X Y\subseteqq V_n$ are both smooth over S and the relative codimension is c, so that

$$\underline{H}^{2c}_{V_n\times_X Y}(V_n, Z/nZ(c)) \cong Z/nZ|_{V_n\times_X Y}$$

and $\underline{H}^q_{V_n\times_X Y}(V_n, Z/nZ(c))=0$ for $q\neq 2c$ and n invertible on X (see [6, Proposition A.5]). Then Proposition A.1 and the functoriality of Proposition A.5 of [6] gives us a canonical isomorphism

$$\underline{H}^{2c}_{V.\times_X Y}(V, Z/nZ(c)) \cong Z/nZ|_{V.\times_X Y}$$

(and $\underline{H}^q_{V.\times_X Y}(V., Z/nZ(c))=0$ for $q\neq 2c$). Then, as in [6, Appendix 2], we get various Thom isomorphism theorems.

Thus, when $c>1$, the proof of $2\Rightarrow 1$ of Theorem 2.5 of [6] can be applied to the fibration $F\to\{T_{X/Y}-Y\}\hat{}\to\{T_{X/Y}\}\hat{}$ (with $K=\{\text{all primes}\}$). The isomorphism $\{Y\}_{\text{et}}\xrightarrow{\sim}\{T_{X/Y}\}_{\text{et}}$ of Theorem 2.2 shows that condition 2. of Theorem 2.5 of [6] is satisfied.

When $c=1$, we must show that the results used in Theorem 2.10 of [6] apply to $V.$ in simpl (X_{et}). We first need some notation. Take $V.$ in simpl (X_{et}), and set $V'_.=V.\times_X Y$. A geometric point γ: Spec $(\Omega)\to V'_n$ gives us a map Spec $(\Omega)\times\Delta[n]\to V'_.$. γ is called a "point" of $V'_.$, and $V.^\gamma$ (respectively $V.'^\gamma$) is the henselization of $V.$ (respectively $V'_.$) at Spec $(\Omega)\times\Delta[n]$ (i.e. V^γ_m is the henselization of V_m at the points (Spec $(\Omega)\times\Delta[n])_m$). A "specialization" $\gamma\to\gamma'$ of "points" of $V'_.$ means one of the following:

1. γ and γ' are geometric points of V'_n and γ specializes to γ' in the usual sense.
2. γ' is a geometric point of V'_n and there is a map f: $[m]\to[n]$ in Δ such that $\gamma=V'(f)(\gamma')$.

Note that a "specialization" $\gamma\to\gamma'$ gives us a map $V.^\gamma\to V.^{\gamma'}$. And $V'_.$ is connected iff any two "points" lying in V'_0 can be connected by a sequence of specializations lying in V'_0 or V'_1.

The construction of Lemma 2.6 of [6] is quite functorial, so that a G torseur $Z.$ over $V.-V'_.$ extends to $Z.$ over $V.$, where each Z_n is flat and finite over V_n, and smooth over S (assuming that $\# G$ is invertible on X). Now, suppose that

$$\pi_1(V_.^\eta - V_.'^\eta) \to \pi_1(V_. - V_.') \to G$$

is onto, where η is our chosen point of V_0' (see section 2). Since $V_.^\eta - V_.'^\eta$ consists of $V_n^\eta - V_n'^\eta \cong X_\eta - Y_\eta$. in dimension n,

$$\pi_1(V_.^\eta - V_.'^\eta) \cong \pi_1(X_\eta - Y_\eta) \cong \pi_1(V_n^\eta - V_n'^\eta) .$$

Thus, for each n,

$$\pi_1(V_n^\eta - V_n'^\eta) \to \pi_1(V_n - V_n') \to G$$

is also onto, so that $Z_n \times_X Y \to V_n'$ is radicial and has a section (which is easily seen to be canonical). Thus $Z_. \times_X Y \to V'_.$ has a section. So the analogue of Lemma 2.6 of [6] is true for $V_.$.

Our discussion of "points" and "specialization" shows that Lemma 2.7 of [6] generalizes easily to $V_.$, and Lemma 2.9 of [6] also follows easily. Note also that if $F_.$ is a local system on $V_. - V_.'$, then $j_* F_.$ is a local system on $V_.$. Thus, the second condition of Lemma 2.9 of [6] is true for $\{Y\}_{\text{et}} \to \{T_{X/Y}\}_{\text{et}}$ by Theorem 1.3.

The proof of Theorem 2.10 of [6] applies only to the homotopy fiber of $\{T_{X/Y} - Y\}\hat{}_L \to \{T_{X/Y}\}\hat{}_L$, which is *not* what we're dealing with. However, a close scrutiny of that proof shows that there is only one extra ingredient needed: we must show that any finite covering space of $\{T_{X/Y}\}\hat{}$ has the property of Lemma 2.9 of [6] (this is used twice; first, to show $\pi_1(E)$ is abelian, and then again in the second paragraph after (9) of [6]).

A finite covering space of $\{T_{X/Y}\}\hat{}$ is described by a G-torseur (G finite) $f_.: Z_. \to W_.$ where $W_.$ is in $t_{X/Y}$ ($\varprojlim_J \{V_. \times_{W_.} Z_.\}\hat{}$ is the desired covering space, where J is the category of maps $V_. \to W_.$ in I). If $F_.$ is a local system on $Z_. - Z_. \times_X Y$, then we must show that the map

$$\varinjlim_{J^\circ} H^q(V_. \times_{W_.} Z_., j_* F_.) \to \varinjlim_{J^\circ} H^q((V_. \times_{W_.} Z_.) \times_X Y, i^* j_* F_.)$$

is an isomorphism. Since each f_n is finite, this map reduces to the map

$$\varinjlim_{J^\circ} H^q(V_., f_{.*} j_* F_.) \to \varinjlim_{J^\circ} (V_. \times_X Y, i^* f_{.*} j_* F_.)$$

(see [6, Lemma 2.9]) which is an isomorphism by Theorem 1.3.

Thus, we can apply the proof of Theorem 2.10 of [6] to $F \to \{T_{X/Y} - Y\}\hat{} \to \{T_{X/Y}\}\hat{}$, and we see that $(F)\hat{}_L \cong (S^1)\hat{}_L$ because $\pi_1(Y) \cong \pi_1(T_{X/Y})$.

Let us mention (without proof) two further results which have a bearing on Theorem 3.2.

1. There is a very explicit description of the action of $\pi_1(T_{X/Y}) \cong \pi_1(Y)$ on $\pi_{2c-1}((F)\hat{}_L)$ involving $\mu_L(Y, c)$ — see [6, Theorem 2.5].

2. If L is a set of primes invertible on X then the homotopy fiber of $\{T_{X/Y}-Y\}\hat{}_L \to \{T_{X/Y}\}\hat{}_L$ is a $(S^{2c-1})\hat{}_L$ iff $\mu_L(Y,c)$ is C_L complete (see [6, Theorem 2.5 and Theorem 2.10]).

We next study naturality. Given X, Y and S as in Theorem 3.2, let $\varphi: S' \to S$. be any morphism and assume that S' is noetherian. Let $X' = X \times_X S'$, $Y' = Y \times_X S'$. Then $Y' \subseteqq X'$ are both smooth over S' and Y' has relative codimension c in X'. We will assume that Y' is connected and that X' has points ξ', η' lying over the points ξ, η of X (see section 2). Then (1) of section 2 gives us a natural map $F' \to F$.

PROPOSITION 3.3. *The map $F' \to F$ induces an isomorphism $(F')\hat{}_L \to (F)\hat{}_L$ for any set L of primes invertible on X.*

PROOF. In (8) of [6] we have a canonical isomorphism

$$\operatorname{Hom}(\pi_{2c-1}((F)\hat{}_L), Z/nZ) \cong \operatorname{Hom}(\hat{Z}_L(c), Z/nZ)$$

which comes from the isomorphism (7) of [6]. To see if this is functorial under base change, we need to study the isomorphism

$$H^{2c}_{V.\times_X Y}(V., Z/nZ) \cong H^0(V. \times_X Y, Z/nZ(-c))$$

(which follows from the proof of Theorem 3.2) for $V.$ in simpl (X_{et}). Let $W. = V. \times_X X'$, and let $\varphi: W. \to V.$ be the projection map. Then, using Proposition A.1 and the functoriality of [6, Proposition A.5], we see that

$$\underline{H}^{2c}_{W.\times_{X'} Y'}(W., Z/nZ) \cong \varphi^* \underline{H}^{2c}_{V.\times_X Y}(V., Z/nZ),$$

and this gives us the desired functoriality.

Thus, for n invertible on X,

$$\operatorname{Hom}(\pi_{2c-1}((F)\hat{}_L), Z/nZ) \xrightarrow{\sim} \operatorname{Hom}(\pi_{2c-1}((F')\hat{}_L), Z/nZ),$$

which implies $\pi_{2c-1}((F')\hat{}_L) \xrightarrow{\sim} \pi_{2c-1}((F)\hat{}_L)$. Since $(F')\hat{}_L$ and $(F)\hat{}_L$ are isomorphic to $(S^{2c-1})\hat{}_L$, this shows $(F')\hat{}_L \xrightarrow{\sim} (F)\hat{}_L$.

We will use this in section 5.

4.

To see how our tubular nbd relates to global henselization [4] we have:

PROPOSITION 4.1. *Let S be a noetherian scheme, and let $Y \subseteqq X_1$, $Y \subseteqq X_2$ be two closed embeddings (over S), where X_1 and X_2 are of finite type over S. Assume that Y is connected and the henselizations X_1^h and X_2^h (see [4]) are isomorphic over S. Then:*

1. *There is a simplicial scheme* $V.$ *which lies in both* $t_{X_1/Y}$ *and* $t_{X_2/Y}$ *such that* $V.\times_{X_1} Y = V.\times_{X_2} Y$. *Thus, there is an equivalence of categories* $(t_{X_1/Y})/V. \cong (t_{X_2/Y})/V.$.

2. *In* $(\text{Pro-}\mathscr{H}_*)_\#$, *the two fibrations* $F_i \to \{T_{X_i/Y} - Y\}^\wedge \to \{T_{X_i/Y}\}^\wedge$, $i=1,2$, *are isomorphic.*

PROOF. Note that 2 follows immediately from 1. Thus, all we have to do is find $V.$.

From the isomorphism φ: $X_1^h \cong X_2^h$ (which is assumed to be the identity on Y), we get a map φ^h: $X_1^h \to X_2$. Then the proof of Theorem 3 of [4] gives us the following "incomplete" descent data:

1. Affine opens U_α in X_1 (which cover Y), affine étale nbds V_α of $U_\alpha \cap Y$ in U_α, and maps φ_α: $V_\alpha \to X_2$ such that φ_α^h: $V_\alpha^h = U_\alpha^h \to X_2$ is just $\varphi^h|_{U_\alpha \cap Y}$ (it follows that φ_α is étale — see [4, Proposition 3]). We can assume that the number of U_α's is finite.

2. For each α and β, there is a nbd $V_{\alpha\beta}$ of $(V_\alpha \times_{X_1} V_\beta) \times_X Y = U_\alpha \cap U_\beta \cap Y$ such that $p_1 \circ \varphi_\alpha = p_2 \circ \varphi_\beta$ (where the p_i are the projection maps). Note that $V_{\alpha\alpha} = V_\alpha$.

Then set $V_0 = \coprod_\alpha V_\alpha$ and $V_1 = \coprod_{\alpha,\beta} V_{\alpha\beta}$. We have obvious maps s_0: $V_0 \to V_1$ and d_i: $V_1 \to V_0$ for $i=0,1$. So V_0 and V_1 form a truncated simplicial scheme $V./1$. Set $V. = \text{cosk}_1\,(V./1)$. Then one checks easily that $V.$ is in $t_{X_1/Y}$.

We also have an étale map φ_0: $V_0 \to X_2$ such that $d_0 \circ \varphi_0 = d_1 \circ \varphi_0$, which shows that $V.$ lies in $\text{simpl}\,((X_2)_{\text{et}})$. By construction $V.\times_{X_2} Y = V.\times_{X_1} Y$, so that $V.$ lies in $t_{X_2/Y}$.

5.

If we have X, Y and S as in Theorem 3.2, there is another spherical fibration to consider. Let $\alpha: N \to Y$ be the normal bundle of the embedding $Y \subseteq X$, a vector bundle of rank c, where c is the codimension of Y in X. We will use Y to denote the zero section of N.

Let $\bar{F}$ be the homotopy fiber of the map $\{N - Y\}^\wedge \to \{N\}^\wedge$ in $(\text{Pro-}\mathscr{H}_*)_\#$. Then Theorem 2.14 of [6] shows that $(\bar{F})^\wedge_L \cong (S^{2c-1})^\wedge_L$ where L is any set of primes invertible on X. So we can ask how this spherical fibration compares to the one of Theorem 3.2. Here is one case when we know the answer:

THEOREM 5.1. *Let* $Y \subseteq X$ *be connected schemes which are smooth over an algebraically closed field k. If L is a set of primes invertible on* X, *then there is an isomorphism of fibrations in* $(\text{Pro-}\mathscr{H}_*)_\#$:

$$(2)\qquad \begin{array}{ccccc} (\bar{F})^\wedge_L & \to & \{N-Y\}^\wedge_L & \to & \{N\}^\wedge_L \\ \Big|\cong & & \Big|\cong & & \Big|\cong \\ (F)^\wedge_L & \to & \{T_{X/Y}-Y\}^\wedge_L & \to & \{T_{X/Y}\}^\wedge_L . \end{array}$$

PROOF. Because k is algebraically closed, $\mu_L(Z,c)$ is trivial for any connected scheme Z over k (see [6, Proposition A 14]). Thus, by 2. of [6, Theorem 2.14], the first line of (2) is a fibration, and the second line is a fibration by 2. of the remarks following the proof of Theorem 3.2.

The first thing is to note that we have a map of fibrations:

$$\begin{array}{ccccc} (S^{2c-1})^\wedge_L & \to & \{T_{N/Y}-Y\}^\wedge_L & \to & \{T_{N/Y}\}^\wedge_L \\ \downarrow & & \downarrow & & \downarrow \\ (S^{2c-1})^\wedge_L & \to & \{N-Y\}^\wedge_L & \to & \{N\}^\wedge_L . \end{array}$$

The map $\{T_{N/Y}\}^\wedge_L \to \{N\}^\wedge_L$ is an isomorphism because both are isomorphic to $\{Y\}^\wedge_L$. And the vertical map on the left is also an isomorphism (this is similar to the proof of Proposition 3.3). Then the long exact sequence of homotopy groups and the 5-lemma show that the arrow in the middle is an isomorphism in $(\text{Pro-}\mathscr{H}_*)_\#$.

We next recall a construction used in [3, Chapter 1, Proposition 5.1]. There we find a smooth variety D over k which has an embedding $Y\times A^1 \to D$ and a smooth map $\pi\colon D \to A^1$ such that the diagram:

$$\begin{array}{ccc} Y\times A^1 & \to & D \\ {\scriptstyle\text{projection}}\searrow & & \swarrow{\scriptstyle\pi} \\ & A^1 & \end{array}$$

commutes. Furthermore, if D_s is the fiber of π over the point $s\in A^1$, then:

1. If $s=0$, the embedding $Y\subseteq D_0$ we get from (2) is isomorphic to the embedding $Y\subseteq N$.
2. If $s\neq 0$, the embedding $Y\subseteq D_s$ is isomorphic to the embedding $Y\subseteq X$.

Thus, if we start with $Y\times A^1\subseteq D$ smooth over the base A^1 and base change to the point 0, we get $Y\subseteq N$. And if we base change to a point $\neq 0$, we get $Y\subseteq X$. Thus, we get morphisms of fibrations (see (1) and Proposition 3.3):

$$\begin{array}{ccccc} (S^{2c-1})^\wedge_L & \to & \{T_{N/Y}-Y\}^\wedge_L & \to & \{T_{N/Y}\}^\wedge_L \\ \downarrow & & \downarrow & & \searrow \\ (S^{2c-1})^\wedge_L & \to & \{T_{D/Y\times A^1}-Y\times A^1\}^\wedge_L & \to & \{T_{D/Y\times A^1}\}^\wedge_L \\ \uparrow & & \uparrow & & \nearrow \\ (S^{2c-1})^\wedge_L & \to & \{T_{X/Y}-Y\}^\wedge_L & \to & \{T_{X/Y}\}^\wedge_L \end{array}$$

where the vertical maps on the left are isomorphisms by Proposition 3.3. And the vertical maps on the right are the same as the maps $\{Y\}^\wedge_L \to$

$\{Y \times A^1\}\hat{}_L \leftarrow \{Y\}\hat{}_L$ by Theorem 2.2 and some obvious functoriality. These are isomorphisms too, so that by the argument we used above, the vertical maps in the middle are isomorphism in $(\text{Pro-}\mathcal{H}_*)_\#$.

There is good reason to compare these two fibrations. Working with the fiber of $\{N-Y\}\hat{}_L \to \{N\}\hat{}_L$ is easier, in some ways it is more natural (it mimics what one usually does in differential geometry), and it suffices for many applications (see [8, §§ 5–6]). However, it lacks a natural map to X, i.e. a map $\{N-Y\}\hat{}_L \to \{X-Y\}\hat{}_L$ (which in differential geometry is provided by the exponential map). However, there is a completely natural map $\{T_{X/Y}-Y\}_{\text{et}} \to \{X-Y\}_{\text{et}}$ (see section 2). Thus, Theorem 5.1 gives us a map in $(\text{Pro-}\mathcal{H}_*)_\#$

$$\{N-Y\}\hat{}_L \cong \{T_{X/Y}-Y\}\hat{}_L \to \{X-Y\}\hat{}_L$$

which we call an "algebraic exponential map".

Theorem 5.1 also shows that two embeddings of a scheme can have the same spherical fibrations yet have distinct henselizations (see Proposition 4.1). For in Example 1 of [4, § 4 and § 6], we have smooth varieties $Y \subseteq X$ over C where X^h and N^h (N is the normal bundle) are *not* isomorphic, yet the spherical fibrations given by the tubular neighborhoods of $Y \subseteq X$ and $Y \subseteq N$ are isomorphic by the above theorem.

Appendix. Local cohomology of simplical schemes.

Let Y be a closed subscheme of a scheme X, and let $V_.$ be in simpl (X_{et}). We set $V'_. = V_. \times_X Y$. For an abelian $\text{Et}(V_.)$ sheaf $F_.$ (see [5, § I.5]), the inclusion $V_. - V'_. \to V_.$ gives us local cohomology sheaves $\underline{H}^q_{V_. \times_X Y}(V_., F_.)$ on $\text{Et}(V_.)$ (see [1, V. 6.3]).

PROPOSITION A.1. *For every n, $\underline{H}^q_{V'_.}(V_., F_.)_n \cong \underline{H}^q_{V'_.}(V_n, F_n)$, and for $f\colon [n] \to [m]$ in Δ, the map*

$$\underline{H}^q_{V'_m}(V_m, F_m) \underset{\substack{\uparrow\\ \text{induced by}}}{\to} \underline{H}^q_{V'_m}(V_m, V(f)^*F_n) \underset{\substack{\uparrow\\ \text{base change}}}{\xleftarrow{\sim}} V(f)^*\underline{H}^q_{V'_m}(V_n, F_n) \tag{3}$$

$$F(f)\colon F_m \to V(f)^*F_n$$

is the transition map for $\underline{H}^q_{V'_.}(V_., F_.)$ (see [5, § I.5]—and the base change map is an isomorphism because $V(f)$ is étale).

PROOF. Define functors S^q from $\text{Et}(V_.)$ to itself by setting $S^q(F_.)_n = \underline{H}^q_{V'_n}(V_n, F_n)$ and using (3) to define the transition maps $S^q(F_.)(f)$. This gives us a connected sequence of functors $\{S^q\}_{q \geq 0}$.

By [10, VI 6.14, 3rd paragraph] every Et $(V.)$ sheaf $G.$ has an injective resolution $I.^*$ in Et $(V.)$ such that each I_n^* is an injective resolution of G_n for every n. Since $S^q(I.^p)=0$ for $q>0$, it follows easily that the S^q are the derived functors of S^0. So we need only show $S^0(F.)=\underline{H}^0_{V}(V.,F.)$.

Let $U \to V_n$ be étale. By 1 of Corollary I.6 of [5], it follows that

$$\begin{aligned}\underline{H}^0_{V'.}(V.,F.)_n(U) &= \underline{H}^0_{V'.}(V.,F.)(U\times\Delta[n]) \\ &= \ker\left(H^0(U\times\Delta[n],F.)\to H^0((U-U\times_X Y)\times\Delta[n],F.)\right),\end{aligned}$$

and this, by the same corollary, is

$$\ker\left(H^0(U,F_n)\to H^0(U-U\times_X Y,F_n)\right)\cong \underline{H}^0_{V'_n}(V_n,F_n)(U)\,.$$

The transition maps are easy to check, and then we have $\underline{H}^0_{V'.}(V.,F)\cong S^0(F.)$.

Note that if $F.$ is a local system, then $\underline{H}^q_{V'.}(V.,F.)$ is also a local system. In this case, one can regard $F.$ as a $(V.)_{\text{et}}$ sheaf, or one gets the same local cohomology sheaves (see [7, Chapter 1, Theorem I.17]).

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