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ON A THEOREM OF FROBENIUS

KÄTE FENCHEL

Let G be a finite group of order $|G|$. By Frobenius' famous theorem [2] the number of solutions of the equation $x^n = 1$, $x \in G$, $n \mid |G|$, is an integral multiplum of n. Frobenius conjectured that

$$L_n = L_n(G) = \{x \in G \mid x^n = 1\}$$

is a subgroup (of course characteristic), if $|L_n| = n$. In the following we consider the case $(n, |G|/n) \neq 1$. (This condition does not play a rôle in Lemma 1 — Corollary 4).

NOTATIONS. P_γ is a subgroup of order p^γ, $\mathfrak{N}_G(P_\gamma) = N_\gamma$, $|G|/|N_\gamma| = h_\gamma$, $\mathfrak{C}_G(P_\gamma) = C_\gamma$, $P_\alpha \in \mathrm{Syl}_p(G)$, $m_\gamma = (|N_\gamma|, n)$, $c_\gamma = (|C_\gamma|, n)$.

Let $n = n'p^\beta$, $(n', p) = 1$, $|L_n| = ns_\beta$, $|L_{n/p}| = (n/p)s_{\beta-1}$ with integral $s_\beta, s_{\beta-1}$. Then

$$L_n = L_{n/p} \cup L' , \tag{1}$$

where

$$L' = \{z = xy \mid |x| = p^\beta,\ y \in \mathfrak{C}_G(x) \cap L_{n'}\} .$$

Since $L_{n/p} \cap L' = \emptyset$, we get

$$ns_\beta = \frac{n}{p} s_{\beta-1} + |L'| . \tag{2}$$

Notice that for a subgroup $H \subset G$, $H \cap L_n(G) = L_{(n,|H|)}(H)$ so that

$$L' = \{z = xy \mid |x| = p^\beta,\ y \in L_{c'_\beta}(C_\beta)\} , \tag{3}$$

where

$$c_\beta = c'_\beta p^\beta, \quad (c'_\beta, p) = 1 .$$

LEMMA 1. *$\varphi(p^\beta)$ divides $|L'|$.*

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The proof is immediate, since by (3) every cyclic subgroup $\langle x\rangle$ of order p^β contributes $\varphi(p^\beta)$ elements to L'.

Hence we have

$$L' = \varphi(p^\beta)|\{\langle x\rangle y \mid y \in L_{c'_\beta}(C_\beta)\}|\,, \tag{4}$$

where $\langle x\rangle$ runs through the cyclic subgroups of order p^β in G. In the case $|L_n| = n$, that is, $s_\beta = 1$ we get by (2) and (4)

$$\frac{n'(p-s_{\beta-1})}{p-1} = |\{\langle x\rangle y \mid y \in L_{c'_\beta}(C_\beta)\}| \tag{5}$$

with the same range of $\langle x\rangle$ as in (4). The right hand side can be interpreted in two ways:

(A) It equals the number of cosets of the form $\langle x\rangle y$, where $|x| = p^\beta$, $y \in L_{c'_\beta}(C_\beta)$.

(B) It equals the number of cosets of the form $\langle x\rangle y$, where $y \in L_{n'}(G)$, $\langle x\rangle \subset \mathfrak{C}_G(y)$, $|x| = p^\beta$.

Frobenius' conjecture is valid for $n = p^\beta$ and for solvable groups [4, Theorem 9; 4,1]. Let $|G|+n$ be minimal so that the conjecture is false. It follows immediately that $n \nmid |G|$ and that L_n is not contained in a proper subgroup of G. We assume this minimal condition throughout the following.

LEMMA 2. *Suppose $|L_n| = n$, where L_n satisfies the minimal condition. Then no normal subgroup of G is contained in L_n.*

PROOF. Suppose $H \lhd G$, $H \subset L_n$. Let $n = n_1 d$, $|H| = h_1 d$, $(n_1, h_1) = 1$ so that $n_1 \mid |G/H|$. By Frobenius' theorem $\bar{x}^{n_1} = \bar{1}$ has $n_1 t$ solutions in $\bar{G} = G/H$ with integral t. If $x \to \bar{x}$ in the natural homomorphism $G \to \bar{G}$, then $x^{n_1} \in H$. For $g \in H \subset L_n$ we have $g^n = g^{|H|} = 1$, hence $g^d = 1$ and consequently $x^{n_1 d} = x^n = 1$ for $n_1 t|H|$ elements of G. Since $|L_n| = n$, $n_1 t|H| \leqq n$. Thus $|H| = d$ and $t = 1$. By the minimal condition $L_{n_1}(\bar{G}) = \bar{K}$ is a group of order n_1 and the inverse image K of K in G has order $n_1 d = n$. Then $K = L_n$ is a group, a contradiction.

LEMMA 3. *Let $|L_n| = n$, $n = n'p^\beta$, $(n', p) = 1$ where L_n satisfies the minimal condition. If $(p-1, n') = 1$, L_n contains the Sylow p-subgroups of G.*

PROOF. Let $P_\alpha \in \mathrm{Syl}_p(G)$, $|P_\alpha| = p^\alpha$. Suppose $P_\alpha \not\subset L_n$. Then P_α must contain an element of order $p^{\beta+1}$, hence also an element of order p^β so that $|L'| > 0$. If $(p-1, n') = 1$, (5) implies $s_{\beta-1} = 1$, hence $|L_{n/p}| = n/p$. By the minimal condition $L_{n/p}$ is a normal subgroup of G, and it is contained in L_n, which contradicts Lemma 2.

Corollary 4. *If $|L_n|=n$, where L_n satisfies the minimal condition, the Sylow subgroups belonging to the minimal prime divisor of n are contained in L_n.*

Proof. For odd p, this is a direct consequence of Lemma 3. For $p=2$ and Syl_2 subgroups $P_\alpha \not\subset L_n$, we have $L_{n/2}=\frac{1}{2}ns_{\beta-1}$ and $L_{n/2} \nsubseteq L_n$ so that $s_{\beta-1}<2$. Hence $s_{\beta-1}=1$ which again contradicts Lemma 2.

Theorem 5. *Let again L_n satisfy the minimal condition. Suppose $(n,|G|/n)\neq 1$, $p \mid (n,|G|/n)$. Then the Sylow p-subgroups of G are cyclic.*

Proof. Let $n=p^\beta n'$, $(n',p)=1$, $P_\alpha \in \mathrm{Syl}_p(G)$, $p^\alpha > p^\beta$. Suppose exponent $P_\alpha \leqq p^\beta$. Then $P_\alpha \subset L_n$. But $np \mid |G|$ by $p^\alpha > p^\beta$ so that $|L_{np}|=nps_{\beta+1}>n$, $s_{\beta+1}$ integral, while $|L_{np}|=|L_n|=n$ for $P_\alpha \subset L_n$. Hence exponent $P_\alpha > p^\beta$, and there exists an element of order $p^{\beta+1}$ in G. Similarly to (5), we obtain

$$\frac{n'(ps_{\beta+1}-1)}{p-1} = |\{\langle u\rangle y \mid |u|=p^{\beta+1},\ y \in L_{c'_{\beta+1}}(C_{\beta+1})\}| \tag{6}$$

where $c_{\beta+1}=c'_{\beta+1}p^\beta=(|C_{\beta+1}|,n)$; $(c'_{\beta+1},p)=1$. Using interpretation (B) and observing that the number of cyclic subgroups of order $p^{\beta+1}$ in $\mathfrak{C}_G(y)$ is constant for the elements in the class (y) of conjugates to y, we get

$$\frac{n'(ps_{\beta+1}-1)}{p-1} = \sum_i h_i A(y_i)\,, \tag{7}$$

where $y_i \in L_{n'}$ is a representative of its class (y_i), $h_i=|(y_i)|$, and $A(y_i)$ the number of cyclic subgroups of order $p^{\beta+1}$ in $\mathfrak{C}_G(y_i)$. Since the left hand side of (7) is coprime with p, there exists an i such that $(h_iA(y_i),p)=1$. It follows from $(h_i,p)=1$ that $\mathfrak{C}_G(y_i)$ contains a Sylow p-subgroup P_α of G. By a result of P. Hall [5], the number of cyclic subgroups of order $p^{\beta+1}$ in a group $\mathfrak{C}_G(y_i)$ with Sylow p-subgroup P_α is for $\beta>0$ a multiplum of $p^{k_\beta-\beta}$. Here, according to P. Hall:

"(i) If P_α is regular, we may take $k_\beta=\varrho_\beta$ where ϱ_β is the order of $\Omega_\beta(P_\alpha)$.
(ii) If P_α is irregular and the elements of order less than or equal to p^β in P_α form a subgroup $\Omega_\beta(P_\alpha)$, we may take $k_\beta=\min\{\varrho_\beta,\beta(p-1)\}$.
(iii) If P_α is irregular and the elements of order less than or equal to p^β in P_α do not form a subgroup, we may take $k_\beta=\beta(p-1)$."

Since $(A(y_i),p)=1$, we must have $p^{k_\beta-\beta}=1$, hence $k_\beta=\beta$. By Corollary 4, $p\neq 2$, thus $\beta(p-1)>\beta$. Therefore only (i) and (ii) are possible, and we obtain $|\Omega_\beta(P_\alpha)|=p^\beta$. Since there exists an element of order p^β in P_α, $\Omega_\beta(P_\alpha)$ is the only subgroup of that order in P_α, and P_α is cyclic, since p is odd.

REMARK. Theorem 5 was proved by Richard Zemlin [7] in a different way in an unpublished PhD dissertation, Ohio State University 1954. He also proved that, under the condition $(n, |G|/n) \neq 1$, G must be simple, if the conjecture is false. We do not use this last fact in the following.

COROLLARY 6. *Let H be a subgroup of G, H in L_n (satisfying the minimal condition). Then $|H| \mid n$.*

PROOF. Let $|H| = h_1 q^\delta$, $(h_1, q) = 1$, $Q_\alpha \in \mathrm{Syl}_q(G)$. Either $(q, |G|/n) \neq 1$, so that Q_α is cyclic by Theorem 5, and there exists an element of order q^δ in $H \subset L_n$. Hence $q^\delta \mid n$. Or $(q, |G|/n) = 1$. Then certainly $q^\delta \mid n$.

Let now p be the minimal prime divisor of $(n, |G|/n)$. By Corollary 4, $p \neq 2$, and by Theorem 5, the Sylow p-subgroups P_α are cyclic. Hence all subgroups $P_\beta = \langle x \rangle$ of order p^β are conjugate and $|L_{c'_\beta}(C_\beta)|$ is constant (interpretation (A)). The number of cyclic subgroups of order p^β in G is $|G|/|N_\beta| = h_\beta$, where P_β is a fixed p^β-subgroup in L_n. By (5)

$$\frac{n'(p - s_{\beta-1})}{p-1} = h_\beta |L_{c'_\beta}(C_\beta)|, \qquad p > s_{\beta-1} > 1 \,. \tag{8}$$

Similarly to (8) we get with respect to N_β

$$\frac{m'_\beta(pt_\beta - t_{\beta-1})}{p-1} = |L_{c'_\beta}(C_\beta)| = c'_\beta T_\beta \,, \tag{9}$$

where $|L_{m_\beta}(N_\beta)| = m_\beta t_\beta$, $|L_{m_\beta/p}(N_\beta)| = t_{\beta-1} m_\beta/p, m_\beta = m'_\beta p^\beta$, $(m'_\beta, p) = 1$, t_β, $t_{\beta-1}$ and T_β integral $\geqq 1$ and

$$|L_{c_\beta}| = c_\beta T_\beta = p^\beta c'_\beta T_\beta = p^\beta |L_{c'_\beta}(C_\beta)| \,,$$

since $L_{c_\beta}(C_\beta) = P_\beta \times L_{c'_\beta}(P_\beta)$.

LEMMA 7. *$pt_\beta - t_{\beta-1}$ divides $p - s_{\beta-1}$.*

PROOF. By (8) and (9)

$$n'(p - s_{\beta-1}) = m'_\beta h_\beta (pt_\beta - t_{\beta-1}) \tag{10}$$

or

$$|N_\beta| = \frac{|G|}{n} m_\beta \frac{(pt_\beta - t_{\beta-1})}{p - s_{\beta-1}} \,. \tag{11}$$

Using $(|N_\beta|, n) = m_\beta$ as well as (10) and (11), we get

$$1 = (|N_\beta|/m_\beta, n/m_\beta) = \left(\frac{|G|}{n}\frac{pt_\beta - t_{\beta-1}}{p-s_{\beta-1}}, \frac{h_\beta(pt_\beta - t_{\beta-1}}{p-s_{\beta-1}}\right),$$

which proves the Lemma.

Set $|L_{m_\beta p}(N_\beta)| = m_\beta p t_{\beta+1}$, $t_{\beta+1}$ integer, $t_{\beta+1} \geqq 1$. Then, corresponding to (8) and (9), we have (see (6)) for $c_{\beta+1} = c'_{\beta+1}p^\beta = (|C_{\beta+1}|, n)$, $(c'_{\beta+1}, p) = 1$ and $|L_{c'_{\beta+1}}(C_{\beta+1})| = c'_{\beta+1}T_{\beta+1}$, $T_{\beta+1} \geqq 1$ integral,

$$(12) \qquad \frac{n'(ps_{\beta+1}-1)}{p-1} = \frac{|G|}{|N_{\beta+1}|}|L_{c'_{\beta+1}}(C_{\beta+1})| = h_{\beta+1}c'_{\beta+1}T_{\beta+1} \text{ in } G\,,$$

$$(13) \qquad \frac{m'_\beta(pt_{\beta+1}-t_\beta)}{p-1} = \frac{|N_\beta|}{|N_{\beta+1}|}|L_{c'_{\beta+1}}(C_{\beta+1})| = \frac{h_{\beta+1}}{h_\beta}c'_{\beta+1}T_{\beta+1} \text{ in } N_\beta\,.$$

Combining (12) and (13), we get

$$(14) \qquad n'(ps_{\beta+1}-1) = h_\beta m'_\beta(pt_{\beta+1}-t_\beta)\,,$$

and by (10)

$$(15) \qquad \frac{ps_{\beta+1}-1}{pt_{\beta+1}-t_\beta} = \frac{p-s_{\beta-1}}{pt_\beta - t_{\beta-1}}\,.$$

Hence

$$(16) \qquad t_{\beta-1} \equiv s_{\beta-1}t_\beta \quad (\text{modulo}\, p)\,.$$

Let now $P_\alpha \subset N_\beta$ be fixed. Since P_α is cyclic and $p \neq 2$, N_α/C_α is a cyclic group of an order $r \mid p-1$ so that

$$N_\alpha = C_\alpha \cup C_\alpha y \cup \ldots \cup C_\alpha y^{r-1}, \quad y^r \in C_\alpha\,.$$

By Herzog [6]

$$|N_\alpha/C_\alpha| = |N_\beta/C_\beta|$$

and the coset representatives of N_α modulo C_α can also be taken as coset representatives of N_β modulo C_β so that also

$$(17) \qquad N_\beta = C_\beta \cup C_\beta y \cup \ldots \cup C_\beta y^{r-1}\,.$$

Further y^b has no fixed point on P_α, $b = 1, \ldots, r-1$.

Since p is the minimal divisor of $(n, |G|/n)$ and $r/p-1$, no prime divisor q of r divides $(n, |G|/n)$. Consequently the Sylow q-subgroups of G are either contained in L_n or in $L_{|G|/n}$. Thus for $r = r_1 r_2$, $r_1 = (n, r)$, we have $r_2 = (|G|/n, r)$ and $(r_1, r_2) = 1$.

LEMMA 8. $m_\beta/c_\beta = r_1$.

PROOF. By taking the greatest common divisor with m_β on both sides of $|N_\beta| = |C_\beta|r$ we get

$$m_\beta = c_\beta(m_\beta/c_\beta, r) = c_\beta(m_\beta/c_\beta, r_1),$$

since $(m_\beta/c_\beta, r_2) = 1$, such that $m_\beta/c_\beta \mid r_1$.

On the other hand, let q^δ be the highest power of the prime q dividing r_1, and let $Q_\alpha \in \mathrm{Syl}_q(N_\beta)$, $Q_\gamma = Q_\alpha \cap C_\beta \in \mathrm{Syl}_q(C_\beta)$. Then $q^\delta = |Q_\alpha|/|Q_\gamma|$. Let Q be a Sylow q-subgroup of G containing Q_α. By the remark above, $Q \subset L_n$, since $(q, |G|/n) = 1$ so that $Q_\alpha \subset L_n \cap N_\beta = L_{m_\beta}$. Then

$$|Q_\alpha| \mid (n, |N_\beta|), \qquad (n, |N_\beta|) = m_\beta,$$

by Corollary 6. In the same way we get $|Q_\gamma| \mid c_\beta$, and hence $|Q_\alpha|/|Q_\gamma| = q^\delta$ divides m_β/c_β. Since this is valid for all prime divisors of r_1, the Lemma is proved.

THEOREM 9. *Suppose* $|L_n| = n \mid |G|$, $(n, |G|/n) \neq 1$. *Then the assumption that the Frobenius conjecture is valid for all pairs* $\{n_1, |G_1|\}$ *with* $n_1 + |G_1| < n + |G|$ *and is false for the pair* $\{n, |G|\}$ *leads to a contradiction.*

PROOF. (i) Suppose first $m_\beta/c_\beta = m'_\beta/c'_\beta = r_1 = 1$. Then we get by (9) and Lemma 8

$$\frac{r_1(pt_\beta - t_{\beta-1})}{p-1} = \frac{pt_\beta - t_{\beta-1}}{p-1} = T_\beta,$$

where T_β is an integer $\geqq 1$. This is impossible, since by Lemma 7, $pt_\beta - t_{\beta-1} \leqq p - s_{\beta-1}$ which is less than $p-1$, if the Frobenius conjecture is false for the pair $\{n, |G|\}$. For, by the same argument as in Lemma 3, $p - s_{\beta-1} = p-1$ gives a contradiction to Lemma 2.

(ii) Hence we must assume $r_1 \neq 1$. By (17)

$$L_{m_\beta} = N_\beta \cap L_{m_\beta} = (C_\beta \cap L_{m_\beta}) \cup (C_\beta y \cap L_{m_\beta}) \cup \ldots \cup (C_\beta y^{r-1} \cap L_{m_\beta}) \tag{18}$$

such that

$$m_\beta t_\beta = |L_{c_\beta}| + |C_\beta y \cap L_{m_\beta}| + \ldots + |C_\beta y^{r-1} \cap L_{m_\beta}|. \tag{19}$$

Since N_β/C_β is cyclic, $[N_\beta, N_\beta] \subseteqq C_\beta$, and the cosets $C_\beta y^b$ consist of whole classes of (in N_β) conjugate elements as does L_{m_β} such that also $C_\beta y^b \cap L_{m_\beta}$ consists of whole classes of N_β. The number of elements in a class (xy^b), $x \in C_\beta$, is $|N_\beta : \mathfrak{C}_{N_\beta}(xy^b)|$. Now no subgroup of p-power order can be centralized by xy^b. For if this were the case, xy^b would also centralize $\Omega_1(P_\gamma)$ which equals $\Omega_1(P_\beta)$

for all $P_\gamma \subset N_\beta$. But xy^b is a p'-element, since all p-irregular elements of L_{m_β} are contained in L_{c_β}. Since xy^b normalizes P_β, it would consequently centralize P_β [3, Theorem 5; 3,10] which is not the case for $b<r$. Hence $\mathfrak{C}_{N_\beta}(xy^b)$ contains no p-element so that

$$(|\mathfrak{C}_{N_\beta}(xy^b)|, p) = 1 \qquad \text{and} \qquad |N_\beta : \mathfrak{C}_{N_\beta}(xy^b)| \equiv 0 \pmod{p^\alpha}.$$

Since this is valid for all classes in $C_\beta y^b \cap L_{m_\beta}$

$$|C_\beta y^b \cap L_{m_\beta}| \equiv 0 \pmod{p^\alpha} \qquad \text{for all } b=1,\ldots,r-1 .$$

(For $b \not\equiv 0$ (modulo r_2) even $C_\beta y^b \cap L_{m_\beta} = \emptyset$, as is easily seen). With

$$|L_{c_\beta}| = c_\beta T_\beta = \frac{m_\beta(pt_\beta - t_{\beta-1})}{p-1} \tag{20}$$

we then have by (19)

$$m_\beta t_\beta \equiv \frac{m_\beta(pt_\beta - t_{\beta-1})}{p-1} \pmod{p^\alpha}$$

or

$$m'_\beta t_\beta \equiv \frac{m'_\beta(pt_\beta - t_{\beta-1})}{p-1} \pmod{p^{\alpha-\beta}},$$

where $\alpha-\beta \geqq 1$

which implies

$$t_{\beta-1} \equiv t_\beta \pmod{p^{\alpha-\beta}}, \tag{21}$$

since $(m'_\beta, p)=1$. By (16)

$$t_{\beta-1} \equiv s_{\beta-1} t_\beta \pmod{p}. \tag{22}$$

Combining (21) and (22), we get

$$t_{\beta-1}(s_{\beta-1}-1) \equiv 0 \pmod{p}.$$

Now $s_{\beta-1}<p$, because $|L_{n/p}| = (n/p)s_{\beta-1} < |L_n| = n$ so that $s_{\beta-1}-1\equiv 0$ (modulo p) implies $s_{\beta-1}=1$ which contradicts the minimal condition, as we saw above. Hence $t_{\beta-1} \equiv t_\beta \equiv 0$ (modulo p) by (21) and thus

$$T_\beta = \frac{r_1(pt_\beta - t_{\beta-1})}{p-1} \equiv 0 \pmod{p}.$$

But $T_\beta < r_1 \leqq p-1$, since $pt_\beta - t_{\beta-1} < p-1$ so that we would have $T_\beta = 0$ which is absurd. Hence the Frobenius conjecture is proved in the case $(n, |G|/n) \neq 1$.

ADDENDUM. *If* $|L_n| = n = p^\beta n'$ *and* $p \mid (n, |G|/n)$, *all elements of* L_n, *that normalize* P_β, *centralize* P_β, *i.e.* $r_1 = 1$.

PROOF. By R. Brauer's definition [1], N_β is a group of metacyclic type which means the following: N_β has a cyclic Sylow p-subgroup P_α and a normal subgroup $K(=O_{p'}(N_\beta))$ of index $p^\alpha r$ such that $\bar{N}_\beta = N_\beta/K$ is a metacyclic group of order $p^\alpha r$ defined by

$$\bar{N}_\beta = \langle \bar{x}_\alpha, \bar{y}, \bar{x}_\alpha^{p^\alpha} = \bar{1}, \bar{y}^r = \bar{1}, \bar{y}^{-1}\bar{x}_\alpha\bar{y} = \bar{x}_\alpha^j \rangle ,$$

where j belongs to exponent r modulo p^α. Here $O_{p'}(N_\beta) \subset C_\beta$, N_β/C_β is cyclic of order r and $C_\beta = O_{p'}(N_\beta)P_\alpha$ by Burnside's theorem, since $P_\alpha \subset C_\beta$ and P_α is contained in the center of its normalizer $\mathfrak{N}_{C_\beta}(P_\alpha) = N_\alpha \cap C_\beta$ which equals C_α by Herzog [6].

By Theorem 9, L_n is a group. Then also $L_n \cap N_\beta$ and $\bar{L}_{m_\beta} = L_{m_\beta}O_{p'}(N_\beta)/O_{p'}(N_\beta)$ are groups. $\bar{g} \in \bar{L}_{m_\beta}$ satisfies the conditions $\bar{g}^{m_\beta} = \bar{1}$ and $\bar{g}^{p^\alpha r} = \bar{1}$, hence

$$\bar{L}_{m_\beta} = \{\bar{g} \in \bar{N}_\beta \mid \bar{g}^{r_1 p^\beta} = \bar{1}\} .$$

If $r_1 \neq 1$, $\bar{L}_{m_\beta}$ contains an element $\bar{y}_1$ of an order dividing r_1, and consequently $\bar{L}_{m_\beta}$ contains the whole class $(\bar{y}_1)$ of conjugate elements in N_β which consists of all elements $\bar{x}_\alpha^b \bar{y}_1$, $b = 0, \ldots, p^\alpha - 1$. Since $\bar{L}_{m_\beta}$ is a group and $\bar{y}_1$ and $\bar{x}_\alpha \bar{y}_1$ are contained in $\bar{L}_{m_\beta}$, also $\bar{x}_\alpha$ would be contained in $\bar{L}_{m_\beta}$ which is not the case, since $|\bar{x}_\alpha| = p^\alpha > p^\beta$. Hence $r_1 = 1$.

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SØNDERENGEN 110
2860 SØBORG
DENMARK