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AMENABILITY AND LOCALLY COMPACT SEMIGROUPS

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1. Introduction.

Much of the success of the theory of amenable locally compact groups stems from the fact that (roughly) the existence of a right invariant mean (RIM) on any "reasonable" space associated with a locally compact group G is equivalent to the existence of a RIM on any other reasonable space associated with G, i.e. amenability is independent of the choice of space (see [6, p. 26]).

In contrast, the theory of amenable locally compact topological semigroups seems to have made limited progress, despite the existence of a well-developed theory for discrete semigroups. (See [4].) Part of the reason for this is surely that it is *not* clear that amenability for such semigroups is (in the sense of the preceding paragraph) independent of the choice of space. This is reflected in the variety of definitions of amenability which have been suggested: left amenability has been defined as the existence of a left invariant mean (LIM) on (1) LUC(S) ([11], [9]), (2) MB(S) ([10]) and (3) M$(S)^*$ ([5]). (For the definition of these spaces, see section 2.) When S is a locally compact group, these three definitions are equivalent. (The equivalence of (1) and (2) is proved by a simple argument using [6, Theorem 2.2.1]; that (3) is equivalent to (1) is a corollary of [17, Theorem 5.2].) Lau ([10]) proves that (1) and (2) are equivalent when S is a non-locally-null measurable subsemigroup of a locally compact group. Other candidates for the choice of space are UC(S), RUC(S) and C(S).

It seems worthwhile investigating this theme for a class of semigroups which, while being much wider than the class of locally compact non-locally-null subsemigroups of locally compact groups, nonetheless preserves a vestige of structure which allows a chance of adapting the locally compact group approach resulting in [6, Theorem 2.2.1]. As $L_\infty(G)$ plays a pivotal role in this approach, it is natural to look for semigroups which have associated Banach spaces of measures analogous to the pre-dual $L_1(G)$ of $L_\infty(G)$. Now a measure $\mu \in L_1(G)$ if and only if the maps $g \to \delta_g * \mu$, $g \to \mu * \delta_g$ are continuous from G into $M(G)$, where $M(G)$ has a suitable (e.g. the weak) topology.

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In the sequence of papers [1], [2], [3], A. C. and J. W. Baker used measures with such continuity properties to construct for a locally compact semigroup S an analogue (called $L(S)$ in [14]) of $L_1(G)$, obtaining many of its basic properties. In his excellent thesis [14], G. L. G. Sleijpen proves a number of new results in this field, in particular giving an illuminating intrinsic characterisation of a large class of those semigroups which admit "many" such continuous measures ([14, p. 83]). Such semigroups are called *foundation semigroups.*

The class of foundation semigroups is extensive, and includes all discrete semigroups, all locally compact non-locally-null subsemigroups of locally compact groups and semigroups related to those considered by Rothman [13]. For many other examples, see [14, Appendix B].

Throughout the paper we shall be concerned with RIM-results: the corresponding LIM-results are proved similarly.

The two theorems of the paper are in section 6. The first theorem shows that the existence of one positive measure for which the maps $s \to \delta_s * \mu$, $s \to \mu * \delta_s$ are weakly continuous implies that the existence of a RIM on any one of the spaces C (S), MB (S), M (S)* is equivalent to the existence of a RIM on any of the others.

A complete generalisation of [6, Theorem 2.2.1] is given by the second theorem: in particular, this theorem applies to all foundation semigroups.

These theorems also show the equivalence of various types of topological RIM which occur in the literature. (In particular the type of mean (involving PM (S)) which has been used by Wong [16], [17] and Day [5] is discussed). In this connection, the present writer finds it helpful to unify these types of RIM by considering each as a mean invariant under a suitable semigroup action.

Section 2 introduces notations, while section 3 summarises the information used in the sequel concerning measures with continuous properties. Semigroup actions and convolution functions are studied in section 4, and the latter theme is continued in section 5, which builds up analogues of results in [6, pp. 22–24] and [8, § 20], without the technical complications involved in Haar measures, the modular function and the involution of M (G). We also show that the spaces on which we wish to study invariant means admit natural semigroup actions, and are connected by natural "morphisms".

An important task in the proving of our theorems is that of providing an analogue of what Greenleaf calls "the key lemma" ([6, p. 101]). Much of section 6 is devoted to this end.

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2. Notations.

Let S be a semigroup. If A,B are subsets of S, then

$$AB = \{ab : a \in A,\ b \in B\}, \qquad A^{-1}B = \{x \in S : ax \in B \text{ for some } a \in A\}$$

and

$$AB^{-1} = \{x \in S : xb \in A \text{ for some } b \in B\} .$$

A non-void subset I of S is called an *ideal* whenever $SI \cup IS \subset I$. A non-void subset T of S is a *subsemigroup* if $T^2 \subset T$. The set

$$\{c \in S : cs = sc \text{ for all } s \in S\}$$

is called the *centre* of S. $m(S)$ is the set of bounded complex-valued functions on S. $m(S)$ is a Banach algebra under pointwise operations and the sup-norm. For $f \in m(S)$ and $s \in S$, elements ${}_sf$ and f_s of $m(S)$ are defined by:

$${}_sf(t) = f(st), \quad f_s(t) = f(ts) \quad \text{for } t \in S .$$

Now let S be a locally compact topological (i.e. jointly continuous) Hausdorff semigroup.

We shall be concerned with means on the following closed subspace of $m(S)$:

$$\begin{aligned}
\mathrm{MB}(S) &= \{f \in m(S) : f \text{ is a Borel function}\}\\
\mathrm{C}(S) &= \{f \in m(S) : f \text{ is continuous}\}\\
\mathrm{LUC}(S) &= \{f \in \mathrm{C}(S) : \text{ the map } s \to {}_sf\ (s \in S) \text{ is (norm) continuous}\}\\
\mathrm{RUC}(S) &= \{f \in \mathrm{C}(S) : \text{ the map } s \to f_s\ (s \in S) \text{ is (norm) continuous}\}\\
\mathrm{UC}(S) &= \mathrm{LUC}(S) \cap \mathrm{RUC}(S) .
\end{aligned}$$

$\mathrm{LUC}(S)(\mathrm{RUC}(S))$ is the set of left (right) uniformly continuous bounded functions on S. $\mathrm{UC}(S)$ is the set of uniformly continuous bounded functions, while $\mathrm{C}(S)$ ($\mathrm{MB}(S)$) consists of the bounded continuous (Borel measurable) functions on S.

Each of these spaces is *translation invariant* in the sense that ${}_sf$ and f_s are in a space whenever f is.

If M is a complex normed space, then M^* is the continuous dual of M.

$\mathrm{C}_0(S)$ is the family of continuous functions in $\mathrm{C}(S)$ vanishing at infinity. $\mathrm{C}_0(S)^*$ can be identified with $\mathrm{M}(S)$, the set of all bounded, complex-valued, regular Borel measures on S. $\mathrm{M}(S)$ is a Banach algebra with convolution multiplication given by:

(1) $$(\mu * \nu)(f) = \iint f(st)\,d\mu(s)\,d\nu(t)$$

where $\mu, \nu \in \mathrm{M}(S)$ and $f \in \mathrm{C}_0(S)$. PM(S) is the convolution semigroup of probability measures on S, i.e.

$$\mathrm{PM}(S) = \{\mu \in \mathrm{M}(S) : \mu > 0, \|\mu\| = 1\}.$$

If $s \in S$, then δ_s is the point mass at s. For $\mu \in M(S)$, S_μ is the *support* of μ, i.e.

$$S_\mu = \{s \in S : |\mu|(U) > 0 \text{ whenever } U \text{ is an open neighbourhood of } s\}.$$

It is well-known that $S_{\mu * \nu} = \overline{S_\mu S_\nu}$. We shall often write $s\mu$ (μs) instead of $\delta_s * \mu$ ($\mu * \delta_s$).

Using (1), it is easily proved that if $f \in \mathrm{MB}(S)$, $s \in S$ and $\mu \in \mathrm{M}(S)$, then:

$$\text{(2)} \qquad \int f d(\mu s) = \int f(ts)\, d\mu(t), \qquad \int f d(s\mu) = \int f(st)\, d\mu(t).$$

Throughout the remainder of this paper, S will stand for a locally compact topological Hausdorff semigroup unless otherwise explicitly specified.

3. Properties of $L(S)$ and $L^n(S)$.

In general, we follow the notation of [14]. $L(S)$ *and* $L^n(S)$ are defined ([14], p. 36, 55):

$$L(S) = \{\mu \in \mathrm{M}(S) : \text{the maps } s \to s|\mu|,\ s \to |\mu|s\ (s \in S) \text{ are weakly continuous}\}$$

$$L^n(S) = \{\mu \in L(S) : \text{the maps } s \to s\mu,\ s \to \mu s\ (s \in S) \text{ are norm continuous}\}.$$

If $L(S) \neq (0)$, $\mathscr{F}(S)$, the *foundation* of S, is defined:

$$\mathscr{F}(S) = \overline{\bigcup \{S_\mu : \mu \in L(S)\}}.$$

If $\mathscr{F}(S) = S$, then S is called a *foundation semigroup.*

We now list a number of properties of these sets.

3.1. *$L(S)$ in an L-ideal of* $\mathrm{M}(S)$. *In particular, if $\nu \ll |\mu|$ where $\nu \in \mathrm{M}(S)$ and $\mu \in L(S)$, then $\nu \in L(S)$.* ([14], p. 37).

3.2. $[\mathrm{M}(\mathscr{F}(S)) * L(S)] \cup [L(S) * \mathrm{M}(\mathscr{F}(S))] \subset L^n(S) \subset L(S)$. ([14], p. 55).

3.3. *If $\mu \in L(S)$, then $s\mu, \mu s \ll |\mu|^2$ for all $s \in S_\mu$.* ([14, p. 36]. This is a very useful substitute for the lack (in general) of a measure μ_1 on S such that $L(S) = L_1(\mu_1)$.)

3.4. *If S is a foundation semigroup, $x \in S$ and O is a non-void open subset of S, then $O^{-1}(Ox)$ and $(xO)O^{-1}$ are neighbourhoods of x.* ([14, p. 76]. The assumption that S contains an identity is not required for this result.)

3.5. *$L^n(S)$ is a two-sided ideal of $M(\mathscr{I}(S))$ and so of $L(S)$.* (Use (3.2).)

3.6. $SL^n(S) \cup L^n(S)S \subset L^n(S)$. (Cf. [2], 3.6. If $s, t, t_1 \in S$ and $\mu \in L^n(S)$, then

$$\|(s\mu)t-(s\mu)t_1\| \leqq \|\mu t-\mu t_1\| \quad \text{and}$$

$$\|t(s\mu)-t_1(s\mu)\| = \|(ts)\mu-(t_1 s)\mu\| .$$

It follows that $s\mu \in L^n(S)$.)

3.7. *If $L(S) \neq (0)$, then $\mathscr{I}(S)$ is a closed two-sided ideal in S, and is a foundation semigroup.* ([14, p. 43].)

We introduce the convolution semigroups $P(S)$ and $P^n(S)$:

$$P(S) = \{\mu \in L(S) : \mu > 0, \|\mu\| = 1\}$$

$$P^n(S) = \{\mu \in L^n(S) : \mu > 0, \|\mu\| = 1\} .$$

3.8. *If $L(S) \neq (0)$, then $P(S)(P^n(S))$ is an ideal in* PM(S) $(P(S))$. (Use (3.1) and (3.5).)

4. Semigroup actions on normed spaces and convolutions.

If X is a non-void set, then $T(X)$ is the semigroup of transformations of X with composition multiplication. If S is a semigroup, then a homomorphism (anti-homomorphism) $\varphi: S \to T(X)$ is said to be a *left* (*right*) *action of S on X*. When the left (right) action φ involved is obvious, we shall write sx (xs) instead of $\varphi(s)x$ for $x \in X$, $s \in S$. Note that for a left action φ, $s(tx) = (st)x$ for all $s, t \in S$, $x \in X$. We shall refer to the left action φ as "given by the maps $x \to sx$". Similar considerations apply to right actions.

A pair (φ, ψ) where $\varphi(\psi)$ is a left (right) action of S on X is said to be a (*two-sided*) *action* if $(sx)t = s(xt)$ $(s, t \in S, x \in X)$ in an obvious notation. We say that *S acts on X* (through (φ, ψ)). We often write sxt rather than $(sx)t$.

A non-void subset Y of X is said to be *invariant* if $SY \cup YS \subset Y$.

Now let M be a normed space with typical elements μ, ν. *Throughout this paper, a left action of S on M will always have the properties*:

(1) *the map $\mu \to s\mu$ is linear for each s*
(2) $\|s\mu\| \leqq \|\mu\|$ *for all $s \in S$, $\mu \in M$* .

Right actions will have similar properties.

A left (right) action of S on M induces as follows a dual right (left) action on M^*:

$$(\alpha s)(\mu) = \alpha(s\mu) \qquad (s\alpha)(\mu) = \alpha(\mu s) \tag{3}$$

where $\alpha \in M^*$, $\mu \in M$ and $s \in S$.

A (two-sided) action of S on M induces the obvious (two-sided) action on M^*. In order to bring out a useful analogy between M^* and spaces of functions on S, we define: $\alpha_s = s\alpha$, ${}_s\alpha = \alpha s$ for such an action. (Note that the natural action of S on $m(S)$ is given by: $sf = f_s, fs = {}_sf$ ($f \in m(S)$, $s \in S$), where f_s and ${}_sf$ are as in section 2.)

Given an action on M and $\alpha \in M^*$, $\mu \in M$, the functions $\alpha * \mu$ and $\mu * \alpha$ in $m(S)$ are given by the well-known convolution formulae (cf. [12], p. 431, [14], p. 23):

$$\alpha * \mu(s) = \alpha(\mu s) \qquad \mu * \alpha(s) = \alpha(s\mu)\,. \tag{4}$$

We list the following formulae which are well-known in one form or another and whose verifications are trivial.

4.1. PROPOSITION. *Let S act on M and $\alpha \in M^*$, $\mu \in M$ and $s \in S$. Then*:

$$\begin{aligned} {}_s(\alpha * \mu) &= \alpha * \mu s \qquad (\alpha * \mu)_s = \alpha_s * \mu \qquad {}_s\alpha * \mu = \alpha * s\mu \\ {}_s(\mu * \alpha) &= \mu * {}_s\alpha \qquad (\mu * \alpha)_s = s\mu * \alpha \qquad \mu s * \alpha = \mu * \alpha_s\,. \end{aligned}$$

(As a useful mnemonic, each of these can be "deduced" using a simple associative law, e.g. ${}_s(\alpha * \mu) = (\alpha * \mu)s = \alpha * \mu s$.)

Now suppose that S is a locally compact semigroup. An action of S on the normed space M is said to be *weakly* (*norm*) *continuous* if, for each $\mu \in M$, the maps $s \to s\mu$ and $s \to \mu s$ are continuous from S into M when M has the weak (norm) topology.

4.2. PROPOSITION. *Let S be a topological semigroup acting on a normed space M. Let $\alpha \in M^*$ and $\mu \in M$.*

(i) $\alpha * \mu(\mu * \alpha) \in m(S)$ *and* $\|\alpha * \mu\|(\|\mu * \alpha\|) \leqq \|\alpha\|\,\|\mu\|$.
(ii) *If the action is weakly continuous, then* $\alpha * \mu(\mu * \alpha)$ *belongs to* $C(S)$.
(iii) *If the action is norm continuous, then* $\alpha * \mu \in \mathrm{LUC}(S)$ *and* $\mu * \alpha \in \mathrm{RUC}(S)$.

PROOF. (i) ([3, p. 686]) Let $t \in S$. Then

$$|(\alpha * \mu)(t)| = |\alpha(\mu t)| \leqq \|\alpha\|\,\|\mu t\| \leqq \|\alpha\|\,\|\mu\|\,.$$

So $\|\alpha * \mu\| \leqq \|\alpha\| \, \|\mu\|$.

(ii) ([3, 2.2, p. 686]) If $s_\delta \to s$ in S, then

$$(\alpha * \mu)(s_\delta) = \alpha(\mu s_\delta) \to \alpha(\mu s) = (\alpha * \mu)(s)$$

since $\mu s_\delta \to \mu s$ weakly. So using (i), $\alpha * \mu \in C(S)$.

(iii) (cf. [11, p. 68]) If $s_\delta \to s$ in S, then

$$\|_{s_\delta}(\alpha * \mu) - {}_s(\alpha * \mu)\| = \|\alpha * (\mu s_\delta - \mu s)\| \leqq \|\alpha\| \, \|\mu s_\delta - \mu s\| \to 0$$

(using (4.1)). So using (ii), $\alpha * \mu \in \mathrm{LUC}(S)$. Similarly, $\mu * \alpha \in \mathrm{RUC}(S)$.

If S is a semigroup with a left action on M, then $\mu_0 \in M$ is said to be an *identity* for the action if $\|\mu_0\| = 1$ and $s\mu_0 = \mu_0$ for all $s \in S$. An element $m \in M^*$ is said to be an *S-right invariant mean* (S-RIM) (with respect to μ_0) if $\|m\| = 1 = m(\mu_0)$ and $m(s\mu) = m(\mu)$ for all $\mu \in M$, $s \in S$. In most normal circumstances, there is a natural identity for the action.

If S has a left action on the normed space M (N) with identity μ_0 (ν_0), then a continuous linear map $\varphi \colon M \to N$ is said to be a (left) *S-morphism* if $\|\varphi\| = 1$, $\varphi(\mu_0) = \nu_0$ and $\varphi(s\mu) = s\varphi(\mu)$ for all $\mu \in M$, $s \in S$.

The obvious analogues for right and two-sided actions apply.

The following three simple results have often been used (implicitly) in the literature. The writer has been unable to find explicit statements.

4.3. PROPOSITION. *In the above notation let $\varphi \colon M \to N$ be a left S-morphism. If m is a* RIM *on N, then $m \circ \varphi$ is an* RIM *on M.*

PROOF. Trivial.

4.4. PROPOSITION. *Let S have a left action on M. Suppose T_1 and T_2 are subsemigroups of S and that $T_1 T_2 \subset T_1$. Then the existence of a T_1-*RIM *on M implies the existence of a T_2-*RIM *on M.*

PROOF. If m is a T_1-RIM on M, $\mu \in M$ and $t_1 \in T_1$, $t_2 \in T_2$, then $m(t_2\mu) = m((t_1 t_2)\mu) = m(\mu)$, so that m is a T_2-RIM on M.

4.5. COROLLARY. *If T is a right ideal in S, then there exists an S-*RIM *on M if and only if there exists a T-*RIM *on M.*

PROOF. Since $TS \subset T$, $ST \subset S$, the result follows by applying (4.4), firstly with $T_1 = T$, $T_2 = S$ and secondly with $T_1 = S$, $T_2 = T$.

5. Convolutions, actions and morphisms.

We specialise the considerations of section 4 by studying natural actions and morphisms on the subspaces of $m(S)$ listed in section 2. We shall also be concerned with the duals of certain spaces of measures. The semigroups involved will be S, $P^n(S)$, $P(S)$ and $\mathrm{PM}\,(S)$.

The spaces we are concerned with are linked up as follows:

$$\begin{tikzcd}
 & \mathrm{LUC}\,(S) \arrow[dr] & & & \mathrm{M}\,(S)^* \arrow[d, "R"] \\
\mathrm{UC}\,(S) \arrow[ur] \arrow[dr] & & \mathrm{C}\,(S) \arrow[r] & \mathrm{MB}\,(S) \arrow[ur, "Q"] \arrow[r, "Q"'] \arrow[dr, "Q"'] & L(S)^* \arrow[d, "R"] \\
 & \mathrm{RUC}\,(S) \arrow[ur] & & & L^n(S)^*
\end{tikzcd} \tag{5}$$

Each of the unnamed arrows signifies the obvious inclusion map. The map $Q\colon \mathrm{MB}\,(S) \to \mathrm{M}\,(S)^*$ is defined by:

$$(Qf)(\mu) = \int_S f(t)\,d\mu(t) \qquad (\mu \in \mathrm{M}\,(S)) . \tag{6}$$

The other maps Q are defined similarly. (Cf. [16, p. 618].)

Each R is the obvious restriction map. It is trivial that (5) commutes.

We shall show that these maps have pleasant morphism properties. For all morphisms concerned, the identity will be the constant function 1 for the function spaces in (5) and the linear functional 1 given by $1(\mu)=\mu(S)$ for the dual spaces in (5).

If $f \in \mathrm{MB}\,(S)$ and $\mu \in \mathrm{M}\,(S)$, then $\mu * f(f*\mu)$ is defined to be $\mu * Qf\,(Qf*\mu)$. Using (4) and (2),

$$(\mu * f)(s) = \int_S f(st)\,d\mu(t) \qquad (f*\mu)(s) = \int_S f(ts)\,d\mu(t) . \tag{7}$$

(See [12, p. 434].) By (5.1), $\mu * f$ and $f * \mu$ belong to $\mathrm{MB}\,(S)$.

If $\alpha \in \mathrm{M}\,(S)^*$, the functions $\mu * \alpha$ and $\alpha * \mu$ are of little use since there is no guarantee that they are measurable. We therefore resort to an "Arens-type" definition: the elements $\mu * \alpha$ and $\alpha * \mu$ of $\mathrm{M}\,(S)^*$ are defined by:

$$(\alpha * \mu)(\nu) = \alpha(\mu * \nu) \qquad (\mu * \alpha)(\nu) = \alpha(\nu * \mu) \qquad (\nu \in \mathrm{M}\,(S)) . \tag{8}$$

Similar definitions apply to convolutions for $L(S)^*$ and $L^n(S)^*$.

When $\alpha = Qf$ for some $f \in \mathrm{MB}(S)$, there is no danger in the two interpretations of, for example, $Qf * \mu$, since $Q(Qf * \mu)$ (according to (4)) is the same as $Qf * \mu$ according to (8). (For interpreting $Qf * \mu$ as a function, if $v \in M(S)$,

$$\begin{aligned} Q[Qf * \mu](v) &= \int (Qf * \mu)(t)\, dv(t) = \int dv(t) \int f(st)\, d\mu(s) \\ &= \iint f(st)\, d\mu(s)\, dv(t) \\ &= Qf(\mu * v) \\ &= (Qf * \mu)(v) \end{aligned}$$

where $Qf * \mu$ at the end is interpreted as a member of $\mathrm{M}(S)^*$.) The context will always make clear whether $\alpha * \mu$ is to be regarded as a function or a linear functional.

5.1. PROPOSITION. PM (S) *has a two-sided action given by the maps* $f \to \mu * f$ *and* $f \to f * \mu$ *on each of the spaces in* (9) *and the connecting maps are* PM (S)-*morphisms.*

(9)
$$\begin{tikzcd} & & \mathrm{M}(S)^* \arrow[d, "R"] \\ \mathrm{C}(S) \arrow[r] & \mathrm{MB}(S) \arrow[ur, "Q"] \arrow[r] & L(S)^* \end{tikzcd}$$

PROOF. Using (3.1), the maps $v \to \mu * v$, $v \to v * \mu$ define a two-sided action of PM (S) on each of M (S) and $L(S)$. As the maps given in the Proposition are the duals of these maps, PM (S) acts on M $(S)^*$ and $L(S)^*$. Trivially, R is a PM (S)-morphism.

If $f \in \mathrm{C}(S)$ and $\mu \in \mathrm{PM}(S)$, the fact that $\mu * f (f * \mu) \in \mathrm{C}(S)$ follows easily (as Sleijpen observes ([14, p. 22])) from [1, Lemma (2.1)]. (See also [12, 3.2].) The verifications that we do have an action on $\mathrm{C}(S)$ are easy; for example, that $\mu * (v * f) = (\mu * v) * f$ ($f \in \mathrm{C}(S)$, $\mu, v \in \mathrm{PM}(S)$) follows by a familiar Fubini-type argument.

Similar considerations apply to MB (S) once we have shown that $f * \mu$ and $\mu * f$ are in MB (S) for $f \in \mathrm{MB}(S)$, $\mu \in \mathrm{PM}(S)$. Extending [2, Lemma 3.1], Sleijpen [14, p. 24] shows that $\chi_A * \mu$ is upper semi-continuous (and so Borel measurable) for every closed subset A of S. Now if $\{A_n\}$ is an increasing (decreasing) sequence of Borel sets with $\chi_{A_n} * \mu$ measurable for each n, then for each s,

$$\chi_{A_n} * \mu(s) = \int \chi_{A_n}(ts)\,d\mu(t) \to \chi_A * \mu(s), \quad \text{where } A = \bigcup A_n (\bigcap A_n).$$

So $\chi_A * \mu \in \mathrm{MB}(S)$. Using the monotone class lemma and the fact that the closed subsets of S generate the Borel sets, $\chi_A * \mu \in \mathrm{MB}(S)$ for every Borel set A. Finally if $f \in \mathrm{MB}(S)$ and $\{f_n\}$ is a bounded sequence of simple Borel functions converging pointwise to f, $f_n * \mu \to f * \mu$ pointwise, so that $f * \mu \in \mathrm{MB}(S)$. A similar argument applies to $\mu * f$.

It is easily verified that the connecting maps in (9) are PM (S)-morphisms.

The following result is easily proved.

5.2. Proposition. *$R: L(S)^* \to L^n(S)^*$ is a $P(S)$-morphism.*

Proof. Trivial using (3.5).

In the next proposition, $\mu * \alpha * \nu$ is to be interpreted as a linear functional (as in (8)). $\pi(\mu * \alpha * \nu)$ is that element of $m(S)$ defined by: $\pi(\mu * \alpha * \nu)(s) = (\mu * \alpha * \nu)(\delta_s)$ $(s \in S)$.

5.3. Proposition. *Let $L(S) \neq (0)$. If $\alpha \in L^n(S)^*$ and $\nu \in P^n(S)$ then the map $\mu \to \pi(\mu * \alpha * \nu)$ from $L^n(S)$ into UC (S) is a linear, continuous mapping which preserves the left S-action and has norm $\leqq \|\alpha\|$.*

Proof. If $\beta = \alpha * \nu \in L^n(S)^*$, then $\pi(\mu * \alpha * \nu) = \mu * \beta$ where the latter is interpreted according to (4). By (4.2), $\mu * \beta \in \mathrm{RUC}(S)$. Applying a similar argument to $\gamma * \nu$ where $\gamma = \mu * \alpha \in L^n(S)^*$, $\pi(\mu * \alpha * \nu) \in \mathrm{UC}(S)$. The other verifications are routine.

5.4. Proposition. *Suppose $L(S) \neq (0)$. Under the maps $f \to \mu * f$, $f \to f * \mu$, $P^n(S)(S)$ has a two-sided action on each of the spaces in* (5), *and the connecting maps are $P^n(S)(S)$-morphisms.*

Proof. Note that $P^n(S) \neq \varnothing$ by (3.8). Since $P^n(S)$ and S can be regarded as subsemigroups of PM (S) and $P^n(S) \subset P(S)$, application of (5.1) and (5.2) shows that we need only consider that part of (5) to the left of MB (S). The only non-trivial verification is that $P^n(S)$ acts on LUC (S) and RUC (S).

If $f \in \mathrm{LUC}(S)$, $\mu \in P^n(S)$, then $f * \mu \in \mathrm{LUC}(S)$ by (4.2) (iii), while for $s, t \in S$,

$$\|{}_s(\mu * f) - {}_t(\mu * f)\| = \|\mu * ({}_sf - {}_tf)\| \leqq \|\mu\|\,\|{}_sf - {}_tf\| \to 0$$

as $s \to t$ using (4.1) and (4.2) (i). Since $\mu * f \in C(S)$ ((4.2)(ii)), $\mu * f \in \mathrm{LUC}(S)$. The fact that the given maps give rise to an action on $\mathrm{LUC}(S)$ follows from (5.1) since $\mathrm{LUC}(S) \subset C(S)$. The corresponding result for $\mathrm{RUC}(S)$ is proved similarly, and the result for $\mathrm{UC}(S)$ follows using $\mathrm{UC}(S) = \mathrm{LUC}(S) \cap \mathrm{RUC}(S)$.

6. The main results.

In the proof of [6, Lemma 2.2.2], a crucial step involves the inferring (by a simple argument involving Haar measure) that if $\alpha \in L_1(G)^*$ is left invariant, then α is constant on $P(G)$. We require an analogous result ((6.4)).

Let S be a locally compact semigroup and $t \in S$. Define recursively a sequence $\{S_n(t)\}$ of subsets of S as follows:

$$S_1(t) = \{t\}, \quad S_{n+1}(t) = S^{-1}(S(S_n(t))) \quad (n=1,2,\ldots).$$

We set $S_\infty(t) = \bigcup_1^\infty S_n(t)$. (Cf. [7, pp. 585–586].)

6.1. LEMMA. *Let $s, t \in S$.*

(i) *$S_\infty(t)$ is a left ideal of S and $S^{-1}(S_\infty(t)) \subset S_\infty(t)$.*
(ii) *Either $S_\infty(t) = S_\infty(s)$ or $S_\infty(t) \cap S_\infty(s) = \varnothing$.*
(iii) *S is a pairwise-disjoint union of sets of the form $S_\infty(t)$.*
(iv) *If S is a foundation semigroup, then $S_\infty(t)$ is clopen in S.*

PROOF. (i) If $x \in S_n(t)$ for some n, then

$$sx \in x^{-1}((xs)S_n(t)) \subset S_{n+1}(t) \subset S_\infty(t),$$

so that $S_\infty(t)$ is a left ideal. Since

$$S^{-1}(S_n(t)) \subset (tS)^{-1}(tS_n(t)) \subset S_{n+1}(t),$$

$$S^{-1}(S_\infty(t)) \subset S_\infty(t).$$

(ii) Suppose $x \in S_\infty(t) \cap S_\infty(s)$. Then $x \in S_n(t) \cap S_m(s)$ for some n, m. Since $x \in S_n(t)$, we can find s_r, t_r, u_r in S such that

$$(10) \quad s_{n-1}x = t_{n-1}u_{n-1}, s_{n-2}u_{n-1} = t_{n-2}u_{n-2}, \ldots, s_2u_3 = t_2u_2, s_1u_2 = t_1t.$$

We can rewrite this as $t \in t_1^{-1}s_1t_2^{-1}s_2t_3^{-1}s_3 \ldots t_{n-1}^{-1}s_{n-1}x$ where brackets are omitted without risk of confusion. Similarly, since $x \in S_m(s)$, we can find $\sigma_r, \tau_r \in S$ such that

$$(11) \quad x \in \sigma_{m-1}^{-1}\tau_{m-1}\sigma_{m-2}^{-1}\tau_{m-2} \ldots \sigma_1^{-1}\tau_1 s.$$

Combining (11) with the above expression for t,

$$t \in t_1^{-1}s_1 \ldots t_{n-1}^{-1}s_{n-1}\sigma_{m-1}^{-1}\tau_{m-1} \ldots \sigma_1^{-1}\tau_1 s \in S_{n+m-1}(s) \subset S_\infty(s).$$

It follows that $S_\infty(t) \subset S_\infty(s)$ and conversely. So $S_\infty(s) = S_\infty(t)$. This proves the result.

(iii) This is a trivial consequence of (ii) and the fact that $t \in t^{-1}(tt) \subset S_\infty(t)$ for all $t \in S$.

(*iv*) *If* $x \in S_\infty(t)$, then $S^{-1}(Sx) \subset S_\infty(t)$ which is therefore a neighbourhood of x by (3.4). So $S_\infty(t)$ is open. Using (iii), $S_\infty(t)$ is also closed.

We can define an equivalence relation "$\sim$" on S as follows: $t \sim s$ means $S_\infty(t) = S_\infty(s)$. (This relation is vaguely akin to the "left Γ-equivalence relation" in [4, (7A)].) S is defined to be *left connected* if $t \sim s$ for all $t, s \in S$, i.e. $S_\infty(t) = S$ for all $t \in S$. The next lemma gives information about when S is left connected.

6.2. LEMMA. *Let S be a locally compact semigroup. Then S is left connected if any one of the conditions* (a), (b), (c) *holds*:

(a) *S is a foundation semigroup and* UC(S) *has an* S-RIM.
(b) *S is a foundation semigroup and the clopen left ideals of S have the finite intersection property.*
(c) *The centre of S is not empty.*

PROOF. (a) (cf. [4, 3L″].) Suppose $t \in S$ and $S_\infty(t) \neq S$. Let $E = S_\infty(t)$, $F = S \setminus E$. By (6.1), (i), (iii), (iv), E and F are clopen left ideals of S and $S^{-1}E \subset E$, $S^{-1}F \subset F$. So $\chi_E \in C(S)$; we show $\chi_E \in UC(S)$.

If $s \in S$, it is readily verified that:

$$(\chi_E)_s = \begin{cases} 1 & (s \in E) \\ 0 & (s \notin E) \end{cases} \qquad {}_s(\chi_E) = \chi_E \,. \tag{12}$$

If $s_\delta \to s$, then $\{s_\delta\}$ is eventually in $E(F)$ if $s \in E(F)$. Using (12), $\chi_E \in UC(S)$. Finally, if m is an S-RIM on UC(S), a contradiction results by applying m to $(\chi_E)_s$ for some $s \in E$ and some $s \in F$.

So $S_\infty(t) = S$ for all $t \in S$.

(b) This is a trivial consequence of (6.1).

(c) If $t \in S$, $c \in t^{-1}(ct) \subset S_\infty(t)$ for any c in the centre of S. The result now follows from (6.1), (iii).

EXAMPLE. If S is the discrete free semigroup on two generators x, y, then S is the disjoint union of $S_\infty(x)$ and $S_\infty(y)$.

6.3. LEMMA. *Let S be a foundation semigroup and M be an L-subalgebra of $L(S)$ i.e. M is a closed subalgebra of $L(S)$ and if $\nu \ll |\mu|$ with $\mu \in M$, $\nu \in L(S)$ then $\nu \in M$. Suppose also that*

$$S = \bigcup \{S_\mu : \mu \in M\}\,.$$

Let $\theta \in M^*$ be such that $\theta(s\mu)=\theta(\mu)$ for all $s \in S$, $\mu \in M$. Suppose further that $S = \bigcup \{\Sigma_\beta : \beta \in B\}$ where $\{\Sigma_\beta\}$ is a pairwise-disjoint family of subsets of S each of the form $S_\infty(x)$ ((6.1), (iii)). Then for each β there exists $c_\beta \in \mathbf{C}$ such that for all $\mu \in M$,

$$\theta(\mu) = \sum_\beta c_\beta \mu(\Sigma_\beta) .$$

Proof. Let

$$P(M) = \{\mu \in M : \mu > 0, \|\mu\| = 1\} .$$

It is clearly enough to prove the result for the semigroup $P(M)$. The proof falls into three stages. We show, firstly, that for each $\mu \in P(M)$, $\theta|_{L_1(\mu)}$ can be represented as an element of $L_\infty(\mu)$ by the (continuous, bounded) function $\theta * \mu$. We then prove that this function is the same for each $\mu \in P(M)$ and finally deduce the result by showing that $\theta * \mu$ is constant on each Σ_β.

(i) Let $\mu \in P(M)$ and choose $\xi \in P(M)$ such that μ, $s\mu$ and μs are absolutely continuous with respect to ξ for all $s \in S_\mu$. (For example, using (3.3), we could take $\xi = \frac{1}{2}(\mu + \mu^2)$.) For $\varphi \in L_1(\mu)$, μ_φ is the measure associated with φ. Clearly, $s\mu_\varphi \ll \xi$, $\mu_\varphi s \ll \xi$ for all $s \in S_\mu$. Note that $L_1(\xi) \subset M$ using (3.1). Let $\theta|_{L_1(\xi)}$ be given by a bounded Borel function $\theta_\xi \in L_\infty(\xi)$. Let $s \in S_\mu$. Then for $\varphi \in L_1(\mu)$,

$$\int \theta_\xi(t)\varphi(t)\, d\mu(t) = \theta(\mu_\varphi) = \theta(s\mu_\varphi) = \int \theta_\xi(st)\varphi(t)\, d\mu(t)$$

so that given $s \in S_\mu$,

$$\theta_\xi(st) = \theta_\xi(t) \quad \text{a.e. } \mu . \tag{13}$$

By (4.2)(ii), $\theta * \mu \in C(S)$. Now for $\varphi \in L_1(\mu)$,

$$\begin{aligned}
\int_{S_\mu} \varphi(t)(\theta * \mu)(t)\, d\mu(t) &= \int_{S_\mu} \varphi(t)\theta(\mu t)\, d\mu(t) \\
&= \int_{S_\mu} \varphi(t)\, d\mu(t) \int_{S_\mu} \theta_\xi(st)\, d\mu(s) \\
&\qquad (\mu t \ll \xi \text{ for } t \in S_\mu) \\
&= \int_{S_\mu} d\mu(s) \int_{S_\mu} \theta_\xi(st)\varphi(t)\, d\mu(t) \\
&\qquad \text{(by Fubini's theorem)} \\
&= \int_{S_\mu} \theta_\xi(t)\varphi(t)\, d\mu(t) \\
&\qquad \text{(use (13) and } \mu(S)=1) .
\end{aligned}$$

So

(14) $$\theta * \mu = \theta_\xi \text{ a.e. } \mu. \text{ So } \theta|_{L_1(\mu)} \text{ is given by } \theta * \mu .$$

(ii) Using (14) with ξ in place of μ, (13) is true with $\theta * \xi$ in place of θ_ξ. Since $\theta * \xi$ is continuous, (13) simplifies to:

(15) $$(\theta * \xi)(st) = (\theta * \xi)(t) \qquad (s, t \in S_\mu) .$$

Let $\mu_1, \mu_2 \in P(M)$ and take $\mu \gg \frac{1}{2}(\mu_1 + \mu_2)$, $\mu \in$ P(M). If $t \in S_\mu$, we have, by (15),

$$\theta * \mu_1(t) = \theta(\mu_1 t) = \int_{S_{\mu^1}} (\theta * \xi)(st)\, d\mu_1(s)$$

$$= \int (\theta * \xi)(t)\, d\mu_1(s) = (\theta * \xi)(t) .$$

So for $\mu \gg \frac{1}{2}(\mu_1 + \mu_2)$, $\theta * \mu_1 = \theta * \mu_2$ on S_μ. It now follows that $\theta * \mu_1 = \theta * \mu_2$ on $\bigcup \{S_\mu : \mu \in P(M)\}$ and, by continuity and a hypothesis of the lemma, $\theta * \mu_1 = \theta * \mu_2$ on S.

(iii) Let $g = \theta * \mu$ for any $\mu \in P(M)$. If $s, t \in \bigcup \{S_\mu : \mu \in P(M)\}$, choose μ such that $s, t \in S_\mu$. By (15), $g(st) = g(t)$, and using the continuity of g,

(16) $$g(st) = g(t) \qquad (s, t \in S) .$$

Let $\beta \in B$, $\Sigma_\beta = S_\infty(t)$ and $s \in S_\infty(t)$. Using (10), there exist s_r, t_r and u_r in S such that

$$s_{n-1}s = t_{n-1}u_{n-1}, s_{n-2}u_{n-1} = t_{n-2}u_{n-2}, \ldots, s_2 u_3 = t_2 u_2, s_1 u_2 = t_1 t$$

for some n. Applying (16)

$$g(s) = g(u_{n-1}) = g(u_{n-2}) = \ldots = g(u_2) = g(t)$$

so that g is constant on Σ_β. As $\theta(\mu) = \int g(t)\, d\mu(t)$ for all $\mu \in P(M)$, we have, using (6.1),

$$\theta(\mu) = \sum c_\beta \mu(\Sigma_\beta) \qquad (\mu \in P(M))$$

where $\{c_\beta\} = g\{\Sigma_\beta\}$ for each β.

6.4. COROLLARY. *If S is left connected, then θ is constant on $P(M)$.*

6.5. LEMMA. *Let S be a locally compact semigroup with $L(S) \neq (0)$.*

(i) *If* UC (S) *has an* S-RIM *and* $\mathscr{I}(S)$ *is left connected then* $L(S)^*$ *has a* $P(S)$-RIM.
(ii) *If* C (S) *has an* S-RIM *then* $L(S)^*$ *has a* $P(S)$-RIM.

PROOF. (i) Suppose $\mathrm{UC}(S)$ has an S-RIM m and that $I=\mathscr{I}(S)$ is left connected. Let $\nu \in P^n(S)$, $\mu \in L(S)$ and $\alpha \in L^n(S)^*$. If $s \in I$, then $s\mu \in L^n(S)$ by (3.2). Using (5.3), define $h\colon I \to \mathbf{C}$ by:

$$h(s) = m(\pi(s\mu * \alpha * \nu)) \qquad (s \in I)\,.$$

By (5.4) and the invariance of m, $h(st)=h(t)$ $(s, t \in I)$. Since I is left connected, h is constant on I by an argument similar to that used in (6.3) to show that g was constant.

Now fix α and ν and define $\theta_\alpha \in L(S)^*$ by:

$$\theta_\alpha(\mu) = m(\pi(t\mu * \alpha * \nu)) \qquad (\mu \in L(S))$$

for any $t \in I$. (By the above, θ_α is independent of the choice of t.) Then $\theta_\alpha(s\mu) = \theta_\alpha(\mu)$ for all $s \in I$, $\mu \in L(S)$.

All the conditions of (6.3) and (6.4) are satisfied with I, $L(S)$ and θ_α in place of S, M, θ. So θ_α has a constant value $m'(\alpha)$ on $P(S)$. Using (5.3) and the fact that $\theta_1 = 1$, it readily follows that $m' \in L(S)^*$ and $\|m'\| = 1 = m(1)$. Now if $\mu, \mu_1 \in P(S)$ and $\alpha \in L^n(S)^*$, then

$$m'(\mu_1 * \alpha) = m(\pi(t(\mu * \mu_1) * \alpha * \nu)) = m'(\alpha)$$

since $\mu * \mu_1 \in P(S)$. So m' is a $P(S)$-RIM on $L^n(S)^*$.

Using the natural $P(S)$-morphism from $L(S)^*$ into $L^n(S)^*$ ((5.2)) and (4.3), there is a $P(S)$-RIM on $L(S)^*$.

(ii) Let $C(S)$ have an S-RIM. Since $\mathrm{UC}(S) \subset C(S)$, $\mathrm{UC}(S)$ has an S-RIM. Using (i), it suffices to show that $I=\mathscr{I}(S)$ is left connected. Fix $t \in I$. Define $\varphi\colon \mathrm{UC}(I) \to \mathrm{C}(S)$ by: $\varphi(f) = {}_tf$ where ${}_tf(s) = f(ts)$ for $f \in \mathrm{UC}(I)$, $s \in S$. (This definition makes sense by (3.7).) φ is a left I-morphism, and so by (4.3), $\mathrm{UC}(I)$ has an I-RIM. I is therefore left connected by (6.2)(a).

6.6. THEOREM. *Suppose $L(S) \neq (0)$. If P, P_1 belong to $\{S, P^n(S), P(S), \mathrm{PM}(S)\}$ and M, M_1 belong to $\{\mathrm{M}(S)^*, L(S)^*, \mathrm{MB}(S), \mathrm{C}(S)\}$, then the existence of a P-RIM on M is equivalent to the existence of a P_1-RIM on M_1.*

PROOF. The phrase "P-RIM on M" will mean: "the existence of a P-RIM on M". For any $P \in \{S, P^n(S), P(S), \mathrm{PM}(S)\}$,

(17)
$$\begin{tikzcd} P\text{-RIM on } L(S)^* \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & P\text{-RIM on } \mathrm{MB}(S) \arrow[r, Rightarrow] & P\text{-RIM on } \mathrm{C}(S) \\ & P\text{-RIM on } \mathrm{M}(S)^* \arrow[ur, Rightarrow] & & \end{tikzcd}$$

using (5.1) and (4.3). (Note that $P \subset \mathrm{PM}(S)$.)

Also, for any $M \in \{\mathrm{M}(S)^*, L(S)^*, \mathrm{MB}(S), \mathrm{C}(S)\}$,

(18) $\quad \mathrm{PM}(S)\text{-RIM on } M \Leftrightarrow P(S)\text{-RIM on } M \Leftrightarrow P^n(S)\text{-RIM on } M \Rightarrow S\text{-RIM on } M$

using (4.5) and (4.4). (Note that $P(S)$ ($P^n(S)$) is a right ideal in $\mathrm{PM}(S)$ ($P(S)$) and that $P^n(S)S \subset P^n(S)$ by (3.8) and (3.6).)

From (17) it is enough to prove:

(19) $\quad P\text{-RIM on } \mathrm{C}(S) \Rightarrow P\text{-RIM on } L(S)^* \quad$ (for all P)

and:

(20) $\quad$ the conditions "P-RIM on $\mathrm{C}(S)$" are all equivalent .

Using the two equivalences in (18) with $M=\mathrm{C}(S)$ and $M=L(S)^*$, (19) will follow from:

(21) $\quad P(S)\text{-RIM on } \mathrm{C}(S) \Rightarrow P(S)\text{-RIM on } L(S)^*$

and

(22) $\quad S\text{-RIM on } \mathrm{C}(S) \Rightarrow S\text{-RIM on } L(S)^*$

while using (18) again with $M=\mathrm{C}(S)$, (20) will follow from

(23) $\quad S\text{-RIM on } \mathrm{C}(S) \Rightarrow P(S)\text{-RIM on } \mathrm{C}(S)$

(21), (22) and (23) will follow from:

(24)
$$\begin{tikzcd}
 & & S\text{-RIM on } L(S)^* \\
P(S)\text{-RIM on } \mathrm{C}(S) \arrow[dr, Rightarrow, "(a)"'] & & P(S)\text{-RIM on } L(S)^* \arrow[ll, Rightarrow, "(c)"'] \arrow[u, Rightarrow, "(d)"'] \\
 & S\text{-RIM on } \mathrm{C}(S) \arrow[ur, Rightarrow, "(b)"'] &
\end{tikzcd}$$

(18) implies (a) and (d); (17) implies (c). (b) follows from (6.5)(ii).

6.7. THEOREM. *Suppose* $L(S) \neq (0)$. *Let* $P \in \{S, P^n(S)\}$ *and* $M \in \{\mathrm{M}(S)^*, L(S)^*, \mathrm{MB}(S), \mathrm{C}(S), \mathrm{LUC}(S)\}$. *Then the following are equivalent*:

(A) *there exists a* P-RIM *on* M;

(B) *there exists an* S-RIM *on* $\mathrm{UC}(S)$ *and* $\mathscr{I}(S)$ *is left connected*;

(C) *there exists a* $P^n(S)$-RIM *on* $\mathrm{UC}(S)$ *and* $\mathscr{I}(S)$ *is left connected.*

PROOF. By (5.4), each M admits the canonical P-action. Suppose $\mathrm{LUC}(S)$ has a P-RIM. Since $PS \subset P$ (by (3.6)), $\mathrm{LUC}(S)$ has an S-RIM. Fix $t \in I = \mathscr{I}(S)$, $\mu \in P^n(S)$. Define α: $\mathrm{UC}(I) \to \mathrm{C}(S)$ by: $\alpha(f)(s) = f(ts)$ $(s \in S)$. Using (3.7), (5.4), (4.2)(iii), $f \to \alpha(f) * \mu$ is a left I-morphism into $\mathrm{LUC}(S)$. By (4.3), $\mathrm{UC}(I)$ has an I-RIM and so I is left connected by (6.2)(a). Restricting the relevant RIM's on $\mathrm{LUC}(S)$ to $\mathrm{UC}(S)$, we deduce:

$$S\text{-RIM } (P^n(S)\text{-RIM}) \text{ on } \mathrm{LUC}(S) \Rightarrow (\mathrm{B})\ ((\mathrm{C}))\,. \tag{25}$$

Using (6.6), a simple argument shows that the theorem will be proved once we have shown:

$$\begin{tikzcd}
P^n(S)\text{-RIM on } \mathrm{LUC}(S) \arrow[dr, Rightarrow, "(\mathrm{f})"'] & & P^n(S)\text{-RIM on } L(S)^* \arrow[ll, Rightarrow, "(\mathrm{e})"'] \arrow[dr, Rightarrow, "(\mathrm{d})"] & \\
& (\mathrm{C}) \arrow[dr, Rightarrow, "(\mathrm{a})"'] & & S\text{-RIM on } L(S)^* \arrow[d, Rightarrow, "(\mathrm{c})"] \\
& & (\mathrm{B}) \arrow[uu, Rightarrow, "(\mathrm{g})"] & S\text{-RIM on } \mathrm{LUC}(S) \arrow[l, Rightarrow, "(\mathrm{b})"]
\end{tikzcd} \tag{26}$$

(a) and (d) follow from (4.4). (c) and (e) result by applying (4.3) to the morphism Q: $\mathrm{LUC}(S) \to L(S)^*$ ((6)). (b) and (f) follow from (25). (g) follows from (6.5)(i) since $P^n(S) \subset P(S)$.

6.8. COROLLARY. *Let* $L(S) \neq (0)$ *and* $P, P_1 \in \{S, P^n(S)\}$,

$$M, M_1 \in \{M(S)^*, L(S)^*, \mathrm{MB}(S), \mathrm{C}(S), \mathrm{LUC}(S), \mathrm{RUC}(S), \mathrm{UC}(S)\}\,.$$

Then the existence of a P-RIM *on* M *is equivalent to the existence of a* P_1-RIM *on* M *if* S *satisfies either of the two conditions*:

(i) *S is a foundation semigroup.*
(ii) *the centre of $\mathscr{I}(S)$ is not empty.*

PROOF. If either (i) or (ii) is satisfied, the condition "$\mathscr{I}(S)$ is left connected" can be deleted in (B) and (C) of (6.7) using (6.2). The result follows from (6.7), an easy argument coping with $\mathrm{RUC}(S)$.

6.9. CONCLUSION. (6.8) says roughly that for a semigroup S satisfying (i) or (ii), the right invariant means associated with S behave as well as could possibly be expected. The class of such semi-groups is wide and includes the main class of semigroups studied in [14], viz. foundation semigroups with identity.

I do not know if the conclusion of (6.7) is true for a locally compact semigroup for which $L(S) \neq (0)$. Using (6.2)(a), this would follow if we could show that when $L(S) \neq (0)$, there is a left $\mathscr{I}(S)$-morphism from UC $(\mathscr{I}(S))$ into UC (S).

What happens if $L(S) = (0)$ remains obscure. It may be possible to make some progress by replacing $L(S)$ by a normed space admitting a weakly (norm) continuous action as in (4.2).

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