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A FUNCTION PARAMETER IN CONNECTION WITH INTERPOLATION OF BANACH SPACES

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0. Introduction.

T. F. Kalugina has in [5] used the function norm

$$\Phi_{f,p}[u] = \left(\int_0^\infty \left[\frac{|u(t)|}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p}, \quad p \geqq 1 ,$$

and the K- and J-functionals due to J. Peetre [7]. Kalugina constructs interpolation spaces and proves the equivalence theorem (the K- and J-methods coincide) and the reiteration theorem. However, the two proofs are incomplete. Kalugina uses the inequality

$$K(t,a) \leqq c\left(\int_0^\infty \left[\min\left(1,\frac{1}{\tau}\right) J(\tau t, u(\tau t))\right]^p \frac{d\tau}{\tau}\right)^{1/p}$$

based on the inequality

$$K(t,a) \leqq c\left(\sum_{n=-\infty}^{\infty} K(t,u_n)^p\right)^{1/p}, \tag{1}$$

where $a=\sum_{n=-\infty}^{\infty} u_n$ with convergence in $\Sigma(\bar{A})$. However, this second inequality is false for $p>1$. In fact, if $a \in \Delta(\bar{A})$ we may choose

$$u_n = \begin{cases} \dfrac{a}{N}, & n=1,2,\ldots,N \\ 0 , & \text{otherwise} \end{cases}$$

Then the inequality (1) would imply

$$K(t,a) \leqq cK(t,a)N^{1/p-1} .$$

As $N \to \infty$ the inequality is false (if $p>1$).

We will show that the equivalence theorem follows from Peetre [7]. The reiteration theorem of Kalugina is more general than the corresponding

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theorem in [7]. We give a proof of a reiteration theorem with somewhat weaker assumptions than Kalugina.

We also study the function classes B_K and B_ψ introduced by Kalugina. We give a simplified definition of B_K and show that B_ψ and B_K essentially coincide.

In the last section we apply the machinery to interpolation between some Orlicz spaces.

1. The function classes B_K and B_ψ.

DEFINITION 1.1. The function f from $\mathbf{R}_+$ into $\mathbf{R}_+$ belongs to the function class B_K if and only if

(2) f is continuous and non-decreasing

$$(3) \qquad \bar{f}(s) = \sup_{t>0} \frac{f(st)}{f(t)} < \infty \quad \text{for every } s>0$$

$$(4) \qquad \int_0^\infty \min\left(1, \frac{1}{t}\right) \bar{f}(t) \frac{dt}{t} < \infty .$$

We give some properties of functions in B_K.

PROPOSITION 1.1. *For $f \in B_K$ holds*

$$(5) \qquad \underline{f}(s)\bar{f}\left(\frac{1}{s}\right) = 1, \quad \textit{where } \underline{f}(s) = \inf_{t>0} \frac{f(st)}{f(t)}$$

(6) $0 < \underline{f}(s)f(t) \leqq f(st) \leqq \bar{f}(s)f(t)$ (f *is quasi-homogeneous*)

(7) $\bar{f}$ *and* $\underline{f}$ *are non-decreasing and* $\bar{f}(1)=\underline{f}(1) = 1$

(8) $\bar{f}(st) \leqq \bar{f}(s)\bar{f}(t)$ ($\bar{f}$ *is submultiplicative*)

(9) $\bar{f}(s) = o(\max(1,s)), \quad s \to 0$ *or* ∞

(10) *for sufficiently small* $\varepsilon>0$

$\bar{f}(s) = o(\max(s^\varepsilon, s^{1-\varepsilon})), \quad s \to 0$ *or* ∞

PROOF. We start with (5).

$$\bar{f}\left(\frac{1}{s}\right) = \sup_{t>0} \frac{f\left(\frac{1}{s}t\right)}{f(t)} = \sup_{t>0} \frac{f(t)}{f(st)} = \frac{1}{\inf_{t>0} \frac{f(st)}{f(t)}} = \frac{1}{\underline{f}(s)} .$$

The assertions (6), (7) and (8) are quite trivial.

Now to the proof of (9). From (1), (4) and (7) we get

$$\frac{\bar{f}(s)}{s} \leqq \int_0^\infty \frac{\bar{f}(t)}{t}\frac{dt}{t} \to 0, \quad s \to \infty .$$

Moreover,

$$\bar{f}(s) \leqq \int_s^{es} \bar{f}(t)\frac{dt}{t} \to 0, \quad s \to 0 .$$

The proof of (9) is complete.

For the proof of (10) we can now refer to [4, p. 244] (cf. [3, p. 35]).

REMARK 1.1. Combining

$$\int_0^\infty \left[\min\left(1, \frac{1}{t}\right)\bar{f}(t)\right]^p \frac{dt}{t} < \infty \quad \text{for any } p>0 , \tag{11}$$

$$\frac{1}{\sqrt[p]{p}} \leqq \frac{f(s)}{s}\left(\int_0^s \left[\frac{t}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p} \leqq \left(\int_1^\infty \left[\frac{\bar{f}(t)}{t}\right]^p \frac{dt}{t}\right)^{1/p} \tag{12}$$

and

$$\left(\int_1^\infty \left[\frac{1}{\bar{f}(t)}\right]^p \frac{dt}{t}\right)^{1/p} \leqq f(s)\left(\int_s^\infty \left[\frac{1}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p} \leqq \left(\int_0^1 [\bar{f}(t)]^p \frac{dt}{t}\right)^{1/p} \tag{13}$$

with Proposition 1.1 we obtain that our class B_K coincides with the B_K given by Kalugina. (11) is an immediate consequence of (10). The inequalities (12) and (13) follow from (2), (5) and (11). As an illustration, let us prove the second inequality in (12).

$$\frac{f(s)}{s}\left(\int_0^s \left[\frac{t}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p} = \frac{f(s)}{s}\left(\int_0^1 \left[\frac{st}{f(st)}\right]^p \frac{dt}{t}\right)^{1/p}$$

$$\leqq \left(\int_0^1 \left[\frac{t}{\underline{f}(t)}\right]^p \frac{dt}{t}\right)^{1/p} = \left(\int_1^\infty \left[\frac{1}{t\underline{f}(1/t)}\right]^p \frac{dt}{t}\right)^{1/p} = \left(\int_1^\infty \left[\frac{\bar{f}(t)}{t}\right]^p \frac{dt}{t}\right)^{1/p} .$$

EXAMPLE 1.1. (Kalugina). If $0<\alpha<1$, then $f(t)=t^\alpha$ belongs to B_K.

EXAMPLE 1.2. If $0<\alpha<\beta<1$, then

$$f(t) = \frac{t^\beta}{\log(1+t^\alpha)}$$

belongs to B_K. $\bar{f}(t)=\max(t^{\beta-\alpha}, t^\beta)$. (log denotes the natural logarithm function.)

Quite as Kalugina we make the following definition.

DEFINITION 1.2. The function class B_ψ consists of all continuously differentiable functions f such that

$$\sup_{t>0} \frac{tf'(t)}{f(t)} = \beta < 1 \tag{14}$$

$$\inf_{t>0} \frac{tf'(t)}{f(t)} = \alpha > 0 . \tag{15}$$

EXAMPLE 1.3. If $0<\alpha<1$, $\theta \in \mathbf{R}$, then

$$f(t) = t^\alpha(\log(1+t^\gamma))^\theta$$

belongs to B_ψ, if γ is small enough. In fact,

$$\sup_{t>0} \frac{tf'(t)}{f(t)} = \max(\alpha, \alpha+\theta\gamma)$$

$$\inf_{t>0} \frac{tf'(t)}{f(t)} = \min(\alpha, \alpha+\theta\gamma) .$$

PROPOSITION 1.2. $B_\psi \subset B_K$.

PROOF. We have to check (2)–(4). However, (2) is trivial. To prove (3) we define (following Kalugina)

$$g(s) = \frac{f(st)}{f(t)} .$$

Then

$$\alpha \leqq \frac{sg'(s)}{g(s)} \leqq \beta, \qquad g(1) = 1 .$$

Thus $g(s) \leqq s^\beta$, if $s \geqq 1$, and $g(s) \leqq s^\alpha$, if $0<s<1$. Consequently $\bar{f}(s) \leqq \max(s^\alpha, s^\beta)$ and (4) is also proved.

DEFINITION 1.3. Two positive functions f and g are called equivalent if there are two positive constants c_1 and c_2 such that

$$c_1 g(t) \leqq f(t) \leqq c_2 g(t), \qquad t>0 .$$

PROPOSITION 1.3. *If $f \in B_K$ then there is a function $g \in B_\psi$ such that f and g are equivalent.*

PROOF. Put

$$g(s) = \int_0^\infty \min\left(1, \frac{s}{t}\right) f(t) \frac{dt}{t}.$$

Then holds

$$f(s) \leqq \int_s^\infty \frac{s}{t} f(t) \frac{dt}{t} \leqq g(s) = \int_0^\infty \min\left(1, \frac{1}{t}\right) f(st) \frac{dt}{t}$$

$$\leqq f(s) \int_0^\infty \min\left(1, \frac{1}{t}\right) \bar{f}(t) \frac{dt}{t}.$$

The equivalence is proved. Since f is continuous and

$$g(s) = \int_0^s f(t) \frac{dt}{t} + s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t}$$

we get

$$g'(s) = \int_s^\infty \frac{f(t)}{t} \frac{dt}{t}$$

which is continuous. Consequently

$$\frac{sg'(s)}{g(s)} = s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t} \Big/ \left(\int_0^s f(t) \frac{dt}{t} + s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t} \right).$$

But

$$f(s) \leqq s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t} = \int_1^\infty \frac{f(st)}{t} \frac{dt}{t} \leqq f(s) \int_1^\infty \frac{\bar{f}(t)}{t} \frac{dt}{t} = f(s) \cdot c_1$$

and

$$f(s) c_2 = f(s) \int_0^1 \underline{f}(t) \frac{dt}{t} \leqq \int_0^s f(t) \frac{dt}{t} \leqq f(s) \int_0^1 \bar{f}(t) \frac{dt}{t} = f(s) \cdot c_3.$$

The convergences of the integrals follow from (4). Finally we obtain

$$0 < \frac{1}{c_3+1} = \frac{f(s)}{c_3 f(s) + f(s)} \leqq \frac{sg'(s)}{g(s)} \leqq \frac{c_1 f(s)}{c_2 f(s) + c_1 f(s)} = \frac{c_1}{c_2 + c_1} < 1.$$

The proof is complete.

REMARK 1.2. It is obvious that B_ψ is contained in the function class $\mathfrak{P}^{+-}(1)$ in [3, p. 37]. On the other hand, using the proof of Proposition 1.2 we achieve that every function in $\mathfrak{P}^{+-}(1)$ is equivalent to a function in B_K and thus equivalent to a function in B_ψ.

2. Construction of interpolation spaces.

Let A_0 and A_1 be Banach spaces continuously embedded in a Hausdorff topological linear space $\mathscr{A}$. Put $\Sigma(\bar{A})=A_0+A_1$ equipped with the norm $K(1,a)$, where

$$K(t,a) = K(t,a;\, A_0,A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0}+t\|a_1\|_{A_1}) .$$

Evidently

$$K(t,a;\, A_0,A_1) = tK(1/t,a;\, A_1,A_0) . \tag{16}$$

Furthermore, we put $\Delta(\bar{A})=A_0\cap A_1$ equipped with the norm $J(1,a)$, where

$$J(t,a) = J(t,a;\, A_0,A_1) = \max (\|a\|_{A_0}, t\|a\|_{A_1}) .$$

(Cf. [7, p. 8] and [2, p. 38].)

DEFINITION 2.1. Let $f\in B_K$ and let u be a non-negative function. Set

$$\Phi_{f,p}[u] = \left(\int_0^\infty \left[\frac{u(t)}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p},$$

where $p\geqq 1$. $\Phi_{f,p}$ will be a function norm. See [7, p. 10].

DEFINITION 2.2. We say that a function norm Φ is of genus $\leqq g$ if and only if

$$\Phi[u(\lambda t)] \leqq g(\lambda)\Phi[u(t)]$$

for all $\lambda>0$ and all non-negative functions u, which is measurable on $\mathbf{R}_+$ with respect to the measure dt/t.

PROPOSITION 2.1. *$\Phi_{f,p}$ is of genus $\leqq f$.*

PROOF. It follows immediately from (5) and (6).

DEFINITION 2.3. We denote by $(A_0,A_1)_{f,p;K}$ the set of elements $a\in\Sigma(\bar{A})$ such that the norm

$$\Phi_{f,p}[K(t,a)] < \infty .$$

REMARK 2.1. Kalugina uses the notation $S_{f,p;K}$ for $(A_0,A_1)_{f,p;K}$.

REMARK 2.2. If f is equivalent to g then $(A_0,A_1)_{f,p;K}=(A_0,A_1)_{g,p;K}$ with equivalent norms. From Proposition 1.3 follows that we may suppose that f is in B_ψ, which is done below.

PROPOSITION 2.2. $(A_0, A_1)_{f,p;K} = (A_1, A_0)_{g,p;K}$ *where* $g(t) = tf(1/t)$. *The norms are equal.*

PROOF. If we use (16) and make a simple substitution, then the proof is complete. Notice that g belongs to B_ψ.

The following lemma is a consequence of (2), (5), (11) and the elementary inequalities (17) and (18) below.

$$\min(1, s/t)K(t,a) \leqq K(s,a) \tag{17}$$

$$K(s,a) \leqq \min(1, s/t)J(t,a) \quad \text{if } a \in \Delta(\bar{A}) \tag{18}$$

(Cf. [7, pp. 9, 16].)

LEMMA 2.1.

$$K(t,a) \leqq cf(t)\|a\|_{(A_0,A_1)_{f,p;K}}$$

$$\|a\|_{(A_0,A_1)_{f,p;K}} \leqq \frac{c}{f(t)} J(t,a), \quad \text{if } a \in \Delta(\bar{A}).$$

The proof of the following proposition is standard. See e.g. [2, p. 47].

PROPOSITION 2.3. $(A_0, A_1)_{f,p;K}$ *is complete.*

THEOREM 2.1. (Interpolation theorem). *Let T be a bounded linear operator from A_i to B_i with norm M_i, $i=0,1$. Then T operates from $(A_0, A_1)_{f,p;K}$ to $(B_0, B_1)_{f,p;K}$ with norm M, where*

$$M \leqq M_0 \bar{f}\left(\frac{M_1}{M_0}\right).$$

PROOF. Since $\Phi_{f,p}$ is of genus $\leqq \bar{f}$ the theorem follows from [7, p. 17–18].

DEFINITION 2.4. We denote by $(A_0, A_1)_{f,p;J}$ the set of elements $a \in \Sigma(\bar{A})$ such that there is a piecewise-constant function $u(t) \in \Delta(\bar{A})$ with

$$a = \int_0^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } \Sigma(\bar{A}))$$

and

$$\Phi_{f,p}[J(t,u(t))] < \infty .$$

We equip $(A_0, A_1)_{f,p;J}$ with the norm $\inf_u \Phi_{f,p}[J(t,u(t))]$.

Remark 2.3. We call u a representation of a.

Lemma 2.2. *Let $a \in \Sigma(\bar{A})$ be such that*

$$\min(1, 1/t)K(t, a) \to 0, \quad t \to 0 \quad \text{or } \infty \tag{19}$$

Then there exists a representation u for a such that $J(t, u(t)) \leqq cK(t, a)$.

Proof. See [7, pp. 26–27].

We can now give the equivalence theorem.

Theorem 2.2. $(A_0, A_1)_{f,p;K} = (A_0, A_1)_{f,p;J}$ *with equivalence of norms.*

Proof. If we use Proposition 2.1, (4) and (9) we get a proof from [7, pp. 29, 13].

Remark 2.4. In the sequel we write $(A_0, A_1)_{f,p}$ instead of $(A_0, A_1)_{f,p;K}$ or $(A_0, A_1)_{f,p;J}$.

Before stating the reiteration theorem we need a definition.

Definition 2.5. Let X be a Banach space such that $\Delta(\bar{A}) \subset X \subset \Sigma(\bar{A})$. Then X is said to be of class $C_K(f, \bar{A})$, if

$$K(t, a; A_0, A_1) \leqq cf(t)\|a\|_X, \quad a \in X .$$

Moreover, X is said to be of class $C_J(f, \bar{A})$, if

$$\|a\|_X \leqq \frac{c}{f(t)} J(t, a; A_0, A_1), \quad a \in \Delta(\bar{A}) .$$

Remark 2.5. It follows from Lemma 2.1 and Theorem 2.2. that $(A_0, A_1)_{f,p}$ is of class $C_K(f, \bar{A}) \cap C_J(f, \bar{A})$.

Theorem 2.3. (The reiteration theorem). *Let A_0, A_1, X_0 and X_1 be Banach spaces with $\Delta(\bar{A}) \subset X_i \subset \Sigma(\bar{A})$, $i = 0, 1$. Suppose that f_0 and f_1 belong to B_ψ and that $\tau(t) = f_1(t)/f_0(t)$ fulfils the condition*

$$\left|\frac{t\tau'(t)}{\tau(t)}\right| \geqq \alpha > 0 . \tag{20}$$

Then the following statements hold for $\varphi \in B_\psi$

a) *if* X_i *is of class* $C_K(f_i, \bar{A})$, $i=0,1$, *then* $(X_0, X_1)_{\varphi,p} \subset (A_0, A_1)_{g,p}$, *where* $g(t) = f_0(t)\varphi(\tau(t))$

b) *if* X_i *is of class* $C_J(f_i, \bar{A})$, $i=0,1$, *then* $(X_0, X_1)_{\varphi,p} \supset (A_0, A_1)_{g,p}$, *with* g *as above.*

PROOF. It is sufficent to prove the theorem when

$$\frac{t\tau'(t)}{\tau(t)} \geqq \alpha > 0 . \tag{21}$$

In fact, the case when $t\tau(t)/\tau(t) \leqq -\alpha < 0$ then follows from (16) and Proposition 2.2.

Before proving a) we prove that $g \in B_\psi$. However, a simple derivation and usage of (14), (15) and (21) gives

$$\alpha_0 \leqq \frac{tg'(t)}{g(t)} \leqq \beta_1$$

where

$$\alpha_0 = \inf_{t>0} \frac{tf'_0(t)}{f_0(t)} \quad \text{and} \quad \beta_1 = \sup_{t>0} \frac{tf'_1(t)}{f_1(t)} .$$

Now to the proof of a). Let $a \in (X_0, X_1)_{\varphi,p}$. Furthermore, let $a = a_0 + a_1$ with $a_i \in X_i$, $i=0,1$. Since X_i is of class $C_K(f_i, \bar{A})$ we get

$$\begin{aligned} K(t,a;\, A_0, A_1) &\leqq K(t,a_0;\, A_0, A_1) + K(t,a_1;\, A_0, A_1) \\ &\leqq c(f_0(t)\|a_0\|_{X_0} + f_1(t)\|a_1\|_{X_1}) . \end{aligned}$$

Consequently

$$K(t,a;\, A_0, A_1) \leqq cf_0(t)K(\tau(t), a;\, X_0, X_1) .$$

Then

$$\Phi_{g,p}[K(t,a;\, A_0, A_1)] \leqq c\left(\int_0^\infty \left[\frac{K(\tau(t), a;\, X_0, X_1)}{\varphi(\tau(t))}\right]^p \frac{dt}{t}\right)^{1/p} .$$

From (21) follows that τ has an inverse function η and that $\tau(t) \to 0$, as $t \to 0$ and that $\tau(t) \to \infty$ as $t \to \infty$. If we make the substitution $t = \eta(s)$, we get

$$\begin{aligned} \Phi_{g,p}[K(t,a;\, A_0, A_1)] &\leqq c\left(\int_0^\infty \left[\frac{K(s,a;\, X_0, X_1)}{\varphi(s)}\right]^p \frac{s\eta'(s)}{\eta(s)} \frac{ds}{s}\right)^{1/p} \\ &\leqq c_1\left(\int_0^\infty \left[\frac{K(s,a;\, X_0, X_1)}{\varphi(s)}\right]^p \frac{ds}{s}\right)^{1/p} . \end{aligned}$$

The last inequality depends on the inequality

$$\frac{s\eta'(s)}{\eta(s)} < \frac{1}{\alpha}. \tag{22}$$

We have proved a).

b) Let $a \in (A_0, A_1)_{g,p}$. According to Lemma 2.1 we have

$$K(t,a;\, A_0, A_1) \leqq c f_0(t)\varphi(\tau(t))\|a\|_{(A_0,A_1)_{g,p}}. \tag{23}$$

Using (9) and (21) we see that

$$K(t,a;\, A_0, A_1) \to 0, \quad t \to 0.$$

If we write (23) as

$$\frac{1}{t}K(t,a;\, A_0, A_1) \leqq c\frac{f_1(t)}{t}\cdot\frac{\varphi(\tau(t))}{\tau(t)}\|a\|_{(A_0,A_1)_{g,p}}$$

and use (9) and (21) again we get that

$$\frac{1}{t}K(t,a;\, A_0, A_1) \to 0, \quad t \to \infty.$$

Summing up we have proved that

$$\min\left(1,\frac{1}{t}\right)K(t,a;\, A_0, A_1) \to 0, \quad t \to 0 \text{ or } \infty.$$

Then Lemma 2.2 gives us a representation $u(t)$ with

$$J(t,u(t);\, A_0, A_1) \leqq cK(t,a;\, A_0, A_1). \tag{24}$$

Put

$$v(s) = u(\eta(s))\frac{s\eta'(s)}{\eta(s)}$$

where η is the inverse function of τ. Then holds

$$a = \int_0^\infty v(s)\frac{ds}{s}.$$

We get

$$\begin{aligned} K(\xi,a;\, X_0, X_1) &\leqq \int_0^\infty K(\xi,v(s);\, X_0, X_1)\frac{ds}{s} \\ &\leqq \int_0^\infty \min\left(1,\frac{\xi}{s}\right)J(s,v(s);\, X_0, X_1)\frac{ds}{s} \\ &= \int_0^\infty \min\left(1,\frac{\xi}{\tau(t)}\right)J(\tau(t),u(t);\, X_0, X_1)\frac{dt}{t}. \end{aligned}$$

But X_i is of class $C_J(f_i, \bar{A})$, $i=0,1$. Thus

$$K(\xi,a;\,X_0,X_1) \leqq c\int_0^\infty \min\left(1,\frac{\xi}{\tau(t)}\right)\frac{J(t,u(t);\,A_0,A_1)}{f_0(t)}\frac{dt}{t}$$
$$\leqq c_1\int_0^\infty \min\left(1,\frac{\xi}{\tau(t)}\right)\frac{K(t,a;\,A_0,A_1)}{f_0(t)}\frac{dt}{t},$$

where the last inequality follows from (24). We get

$$\Phi_{\varphi,p}[K(\xi,a;\,X_0,X_1)]$$
$$\leqq c_2\left(\int_0^\infty\left[\int_0^\infty \min\left(1,\frac{\xi}{\tau(t)}\right)\frac{K(t,a;\,A_0,A_1)}{f_0(t)\varphi(\xi)}\frac{dt}{t}\right]^p\frac{d\xi}{\xi}\right)^{1/p}$$
$$\leqq c_3\left(\int_0^\infty\left[\int_0^\infty \min\left(1,\frac{1}{\sigma}\right)\frac{K(\eta(\xi\sigma),a;\,A_0,A_1)}{f_0(\eta(\xi\sigma))\varphi(\xi)}\frac{d\sigma}{\sigma}\right]^p\frac{d\xi}{\xi}\right)^{1/p},$$

where we have made the substitution $t=\eta(\xi\sigma)$ and used the inequality (22).

According to Minkowski's inequality we have

$$\Phi_{\varphi,p}[K(\xi,a;\,X_0,X_1)]$$
$$\leqq c_3\int_0^\infty\left(\int_0^\infty\left[\min\left(1,\frac{1}{\sigma}\right)\frac{K(\eta(\xi\sigma),a;\,A_0,A_1)}{f_0(\eta(\xi\sigma))\varphi(\xi)}\right]^p\frac{d\xi}{\xi}\right)^{1/p}\frac{d\sigma}{\sigma}$$
$$= c_3\int_0^\infty \min\left(1,\frac{1}{\sigma}\right)\left(\int_0^\infty\left[\frac{K(\eta(x),a;\,A_0,A_1)}{f_0(\eta(x))\varphi(x/\sigma)}\right]^p\frac{dx}{x}\right)^{1/p}\frac{d\sigma}{\sigma}.$$

But using (4)–(6) we achieve

$$\Phi_{\varphi,p}[K(\xi,a;\,X_0,X_1)]$$
$$\leqq c_3\int_0^\infty \min\left(1,\frac{1}{\sigma}\right)\bar{\varphi}(\sigma)\frac{d\sigma}{\sigma}\left(\int_0^\infty\left[\frac{K(\eta(x),a;\,A_0,A_1)}{f_0(\eta(x))\varphi(x)}\right]^p\frac{dx}{x}\right)^{1/p}.$$

Again, making the substitution $x=\tau(y)$ and observing that $y\tau'(y)/\tau(y)<1$, we get

$$\Phi_{\varphi,p}[K(\xi,a;\,X_0,X_1)] \leqq c_4\left(\int_0^\infty\left[\frac{K(y,a;\,A_0,A_1)}{f_0(y)\varphi(\tau(y))}\right]^p\frac{dy}{y}\right)^{1/p}$$
$$= c_4\Phi_{g,p}[K(y,a;\,A_0,A_1)].$$

The proof is complete.

3. Interpolation between some Orlicz spaces.

In this section we shall interpolate between Orlicz spaces of the type $L^p(\log^+ L)^s$.

We denote by $L^p = L^p(I, d\mu, \mathrm{R})$ the space of all μ-measurable real-valued functions such that

$$\|f\|_{L^p} = \left(\int_I |f(s)|^p \, d\mu\right)^{1/p} < \infty .$$

Before stating the following lemma we refer to [2, pp. 6–7] for a definition of the decreasing rearrangement function a^* of a.

LEMMA 3.1. *Let $f \in B_\psi$. Then the function a belongs to $(L^1, L^\infty)_{f,p}$ if and only if*

$$\int_0^\infty \left[\frac{ta^*(t)}{f(t)}\right]^p \frac{dt}{t} < \infty . \tag{25}$$

PROOF. First we observe that

$$K(t,a;\, L^1, L^\infty) = \int_0^t a^*(s)\, ds = \int_0^1 ta^*(ts)\, ds . \tag{26}$$

See [2, p. 109]. Since obviously $K(t,a;\, L^1, L^\infty) \geqq ta^*(t)$ we achieve that (25) is true if $a \in (L^1, L^\infty)_{f,p}$. Conversely, we now assume that a fulfils (25). In view of (4), (5), (6), (26) and Minkowski's inequality, we obtain

$$\left(\int_0^\infty \left[\frac{K(t,a)}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p} = \left(\int_0^\infty \left[\int_0^1 \frac{ta^*(ts)}{f(t)}\, ds\right]^p \frac{dt}{t}\right)^{1/p}$$

$$\leqq \int_0^1 \left(\int_0^\infty \left[\frac{ta^*(ts)}{f(t)}\right]^p \frac{dt}{t}\right)^{1/p} ds = \int_0^1 \left(\int_0^\infty \left[\frac{\sigma a^*(\sigma)}{sf(\sigma/s)}\right]^p \frac{d\sigma}{\sigma}\right)^{1/p} ds$$

$$\leqq \int_0^1 \bar{f}(s) \frac{ds}{s} \left(\int_0^\infty \left[\frac{\sigma a^*(\sigma)}{f(\sigma)}\right]^p \frac{d\sigma}{\sigma}\right)^{1/p} .$$

The proof is complete.

DEFINITION 3.1. A Banach space X of real-valued Lebesque measurable functions on an interval I is said to be a Banach function space if

(27) $|g| \leqq |f|$ a.e. and $f \in X$ implies that $g \in X$ and $\|g\|_X \leqq \|f\|_X$

(28) $f_n \in X$, $\|f_n\|_X \leqq M$ and $0 \leqq f_n \nearrow f$ implies that $f \in X$ and $\|f\|_X \leqq M$.

EXAMPLE 3.1. The Lebesgue L^p spaces, Lorentz spaces and Orlicz spaces are Banach function spaces.

REMARK 3.1. When we omit condition (28) in Definition 3.1, then we talk about X as a quasi-Banach function space.

LEMMA 3.2. *If g_ν, $\nu=1,2,\ldots$, are non-negative functions in a quasi-Banach function space X and $\sum_{\nu=1}^{\infty} g_\nu$ is convergent in X with the sum g, then*

$$g_n(x) \leqq g(x) \quad \text{a.e.,} \quad n=1,2,\ldots . \tag{29}$$

PROOF. (We owe the argument to J. Bergh.) Since

$$0 \leqq g_n(x) \leqq \sum_{\nu=1}^{n} g_\nu(x)$$

(29) follows if we prove $\sum_{\nu=1}^{n} g_\nu(x) \leqq g(x)$ a.e. Consider therefore the set

$$E_n = \left\{x : \sum_{\nu=1}^{n} g_\nu(x) > g(x)\right\}.$$

Suppose now that E_{n_0} has positive measure. Put

$$r_\nu(x) = \begin{cases} g_\nu(x), & \text{if } x \in E_{n_0} \\ 0 \quad , & \text{otherwise} \end{cases}$$

and

$$r(x) = \begin{cases} g(x), & \text{if } x \in E_{n_0} \\ 0 \quad , & \text{otherwise} \end{cases}$$

Then

$$\left|\sum_{\nu=1}^{n} r_\nu(x) - r(x)\right| \leqq \left|\sum_{\nu=1}^{n} g_\nu(x) - g(x)\right|. \tag{30}$$

Combining (27) and (30) we get that $\sum_{\nu=1}^{\infty} r_\nu(x)$ is convergent in X with the sum $r(x)$. But for $n \geqq n_0$

$$0 < \sum_{\nu=1}^{n_0} r_\nu(x) - r(x) \leqq \sum_{\nu=1}^{n} r_\nu(x) - r(x), \quad \text{if } x \in E_{n_0}.$$

Then (27) gives that

$$\left\|\sum_{\nu=1}^{n_0} r_\nu - r\right\|_X \leqq \left\|\sum_{\nu=1}^{n} r_\nu - r\right\|_X \to 0, \quad n \to \infty$$

which contradicts that E_{n_0} has positive measure. The proof is complete.

PROPOSITION 3.1. *If X and Y are two quasi-Banach function spaces, which coincide algebraically, then they have equivalent norms.*

PROOF. Since $\||f|\| = |f|$, (27) yields that $\||f|\|_X = \|f\|_X$. Consequently, it is sufficient to prove the equivalence of norms for non-negative functions. Let us assume that we cannot find a positive constant C such that $\|f\|_X \geqq C\|f\|_Y$ for all non-negative functions in $X = Y$. Then we could find non-negative functions f_n such that

$$(31) \qquad \|f_n\|_X < n^{-3}\|f_n\|_Y, \quad n = 1, 2, \ldots .$$

Put

$$g_n = \frac{nf_n}{\|f_n\|_Y} .$$

From (31) follows that $\|g_n\|_X \leqq n^{-2}$. Since X is complete we get that

$$g = \sum_{n=1}^{\infty} g_n$$

is convergent in X. But $g_n \geqq 0$. Thus Lemma 3.2 gives

$$(32) \qquad g_n(x) \leqq g(x) \quad \text{a.e.}$$

But $X = Y$ algebraically. Combining (27) and (32) we get

$$n = \|g_n\|_Y \leqq \|g\|_Y, \quad n = 1, 2, \ldots$$

which is a contradiction. The proof is complete.

REMARK 3.2. In [6, p. 128] we have found ideas similar to the proof above.

In the sequel we shall only use $I = (0, 1)$ and the Lebesgue-measure dx. Then e.g. condition (25) reads

$$(33) \qquad \int_0^1 \left[\frac{ta^*(t)}{f(t)}\right]^p \frac{dt}{t} < \infty .$$

The following lemma may be compared to [1, p. 218].

LEMMA 3.3. *Let $\theta \geqq 0$, $\gamma > 0$ and $p \geqq 1$. Then the following statements are equivalent.*

(a) $a \in L^p(\log^+ L)^{\theta p}$

(b) $\int_0^1 [|a(t)|(\log^+ |a(t)|)^\theta]^p \, dt < \infty$

(c) $\int_0^1 [a^*(t)(\log^+ a^*(t))^\theta]^p \, dt < \infty$

(d) $\int_0^1 [a^*(t)(\log (1+a^*(t)^\gamma))^\theta]^p \, dt < \infty$

(e) $\int_0^1 [a^*(t)(\log (1+t^{-\gamma}))^\theta]^p \, dt < \infty .$

PROOF. That (a) and (b) are equivalent follows from the definition. Since a and a^* are equimeasurable (See [2, p. 7]), then (b) and (c) are equivalent. The equivalence between (c) and (d) depends on the limit

$$\lim_{s\to\infty} \frac{\log^+ s}{\log (1+s^\gamma)} = \frac{1}{\gamma} > 0$$

and elementary comparison tests for integrals. It remains to prove that (d) is equivalent to (e). Let us assume that (e) is true. Then $a \in L^1$ and we get that $ta^*(t)$ is bounded. Remembering (e) we get the convergence of (d). To prove that (d) implies (e) we define

$$E = \{t : a^*(t) > t^{-\delta}\}$$
$$F = \{t : a^*(t) \leqq t^{-\delta}\}$$

where $0<\delta p<1$. Then holds

$$\int_0^1 [a^*(t)(\log (1+t^{-\gamma}))^\theta]^p \, dt \leqq \int_F \left[\frac{\log (1+t^{-\gamma})}{t^\delta}\right]^p dt + $$
$$+ \int_E [a^*(t)(\log (1+a^*(t)^{\gamma/\delta}))^\theta]^p \, dt .$$

But $\delta p<1$. Then the integral over F is convergent. The convergence of (d) implies the convergence of the last integral. The proof is complete.

PROPOSITION 3.2. *Let $p>1$ and $f(t)=t^\alpha(\log (1+t^{-\gamma}))^{-\theta}$ with $\alpha=1-1/p$, $\theta\geqq 0$ and $0<\gamma<(1-\alpha)/\theta$. Then holds*

$$(L^1, L^\infty)_{f,p} = L^p(\log^+ L)^{\theta p}$$

with equivalent norms.

PROOF. First we notice that $f \in B_\psi$. See Example 1.3. From (33) follows that $a \in (L^1, L^\infty)_{f,p}$ if and only if

$$\int_0^1 \left[\frac{ta^*(t)}{f(t)}\right]^p \frac{dt}{t} = \int_0^1 t^{(1-\alpha)p-1}[a^*(t)(\log(1+t^{-\gamma}))^\theta]^p\,dt$$
$$= \int_0^1 [a^*(t)(\log(1+t^{-\gamma}))^\theta]^p\,dt < \infty .$$

But according to Lemma 3.3 this is equivalent to $a \in L^p(\log^+ L)^{\theta p}$. We have proved that the interpolation space and the Orlicz space coincide algebraically. If we can prove that $(L^1, L^\infty)_{f,p}$ is a quasi-Banach function space, then the equivalence of norms follows from Proposition 3.1. However, Proposition 2.3 yields the completeness. Since $|g| \leqq |f|$ implies that $g^*(t) \leqq f^*(t)$, then (26) implies that condition (27) is fulfilled. The proof is complete.

Finally we shall use the reiteration theorem to prove the following proposition.

PROPOSITION 3.3. *Up to equivalence of norms we have*

$$(L^{p_0}(\log^+ L)^{s_0}, L^{p_1}(\log^+ L)^{s_1})_{\varphi,p} = L^p(\log^+ L)^s$$

with $\varphi(t) = t^\alpha$, $0 < \alpha < 1$, $p_0 \neq p_1$, $p_i > 1$ *for* $i = 0, 1$. *Furthermore*

$$\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1} \quad \textit{and} \quad \frac{s}{p} = \frac{1-\alpha}{p_0}s_0 + \frac{\alpha}{p_1}s_1 .$$

PROOF. Put

$$f_i(t) = t^{\alpha_i}(\log(1+t^{-\gamma}))^{-\theta_i} ,$$

where γ is positive and sufficiently small, $\alpha_i = 1 - 1/p_i$ and $s_i = \theta_i p_i$, $i = 0, 1$. From Proposition 3.2 we obtain that

$$(L^1, L^\infty)_{f_i, p_i} = L^{p_i}(\log^+ L)^{s_i}, \quad i = 0, 1 .$$

If we choose γ small enough, then $f_1(t)/f_0(t)$ fulfils condition (20). (Cf. Example 1.3). In view of the reiteration theorem we achieve

$$(L^{p_0}(\log^+ L)^{s_0}, L^{p_1}(\log^+ L)^{s_1})_{\varphi,p} = (L^1, L^\infty)_{g,p}$$

where

$$g(t) = f_0(t)\varphi\left(\frac{f_1(t)}{f_0(t)}\right) = \frac{t^{(1-\alpha)\alpha_0 + \alpha\alpha_1}}{(\log(1+t^{-\gamma}))^{(1-\alpha)\theta_0 + \alpha\theta_1}} .$$

If we use Proposition 3.2 again we get

$$(L^1, L^\infty)_{g,p} = L^p(\log^+ L)^s$$

with

$$\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1} \quad \text{and} \quad \frac{s}{p} = \frac{1-\alpha}{p_1} s_0 + \frac{\alpha}{p_1} s_1 .$$

The proof is complete.

REMARK 3.3. The same result can be found in [3, p. 49]. However, the restrictions $p_0 \neq p_1$, $p_i > 1$ for $i = 0, 1$ do not appear in [3].

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