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JB ALGEBRAS WITH AN EXCEPTIONAL IDEAL

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Abstract.

Let $\mathscr{A}$ be an arbitrary JB algebras with exceptional ideal J. Then J is isomorphic to the set of all continuous cross sections vanishing at infinity of some fibre bundle $\{B, \Omega, \mathrm{pr}, M_3^8, E_6\}$, with locally compact base space Ω. Let $J^0 = \{a \in \mathscr{A} \mid a \circ b = 0 \text{ for all } b \in J\}$ be the annihilator of J. Then J^0 is a JB ideal and $\mathscr{A}/(J+J^0)$ is a special JB algebra of type $\mathrm{I}_{\leqq 3}$.

In [1] Alfsen, Shultz and Størmer have shown that every JB algebra $\mathscr{A}$ with identity is, modulo its unique exceptional ideal J, a JC algebra. These results extend also to the non-unital case, if one modifies the axioms for JB algebras slightly [7]. Thus we shall assume throughout that the JB algebras satisfy the axioms given in [7]. Then we may, if necessary, adjoin a unit and apply the methods of Alfsen, Shultz and Størmer.

If one interpretes a JB algebra as the set of observables of a physical system, one would consider J as an unphysical part. Hence it seems desirable to know how the exceptional ideal J lies in the JB algebra $\mathscr{A}$. Though one might expect J to be a direct summand, Alfsen, Shultz and Størmer have already constructed a counterexample for this conjecture [1, remark 9.8]. Here we shall show that in general J is almost a direct summand. More precisely we prove that $\mathscr{A}$ is an extension of $J \oplus J^0$ by a special JB algebra of type $\mathrm{I}_{\leqq 3}$. Here J^0 denotes the annihilator of J in $\mathscr{A}$. This is done by describing J as a M_3^8-fibre bundle over $\hat{J} = \mathrm{Prim}\, J$, its primitive ideal space. The proof for this follows the lines of the corresponding result for n-homogeneous C*-algebras [8].

Throughout $\mathscr{A}$ will denote an arbitrary JB algebra satisfying the axioms of [7] and J the exceptional ideal of $\mathscr{A}$. By an exceptional JB algebra J we understand a JB algebra, all of whose nontrivial factor representations are onto M_3^8. Equivalently an exceptional JB algebra can be defined via the s-identities [1, theorem 9.5].

LEMMA 1. *$\hat{J} = \mathrm{Prim}\, J$ with the Jacobson topology is a locally compact Hausdorff space and $\mathscr{C}_0(\hat{J})$, the algebra of all continuous real valued functions on $\hat{J}$ vanishing at infinity, is the center of J.*

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PROOF. The proof follows almost verbatim the proof of Kaplansky [5]. As an intermediate space one uses the set H of all J-homomorphisms of J into M_3^8 with the pointwise convergence topology. Using H we can define on $\hat{J}$ the weak topology as the quotient topology from H. With the aid of the functional calculus of JB algebras on shows then that J contains all weakly continuous real valued functions on $\hat{J}$ vanishing at infinity. The remainder is then almost trivial.

Since $\mathscr{C}_0(\hat{J})\subset J$, J clearly has a bounded (bound 1) central approximate unit.

Our next aim is to show that locally J looks like an algebra of continuous functions with values in M_3^8. To do this we have to lift the matrix units of M_3^8 locally.

LEMMA 2. *Let $a_1, a_2 \in \mathscr{C}$, the Cayley numbers, with*

$$a_i\circ a_j+a_j\circ a_i = -2\delta_{ij} \qquad i,j=1,2\ .$$

Then we have for every purely imaginary $a \in \mathscr{C}$

$$a_1\circ(a_2\circ a) = a_2\circ(a\circ a_1)\ .$$

This is shown by expanding a, a_1 and a_2 with respect to the canonical basis of $\mathscr{C}$ [2, p. 221].

LEMMA 3. *For each $P_0 \in \hat{J}$ there exists a neighborhood U of P_0 such that $J(U) \cong\mathscr{C}(\bar{U})\otimes M_3^8$. Here $J(U)$ is the quotient of J by the kernel of U.*

PROOF. Let $P_0 \in \hat{J}$ and let $\{e_{ij}\}_{i,j=1}^3$ be the usual matrix units of $M_3(\mathscr{C})$. Arguing as in [8, proof of the lemma in § 2] we can find elements e_1, e_2 and e_3 in J and a neighborhood U of P_0 such that $e_i(P)=e_{ii}$ and such that the $e_i(P)$ form a system of minimal orthogonal projections with sum 1 for all $P \in U$. Choose now $s \in J$ such that $s(P_0)=e_{12}+e_{21}$ and

$$[(e_1+e_2)\circ s](P) = [s\circ(e_1+e_2)](P) = s(P) \quad \text{for all } P\in U\ .$$

Replacing s by $s-U_{e_1}s-U_{e_2}s$ if necessary we may even assume $(U_{e_i}s)(P) = 0$ for all $P \in U$.

Since $s^2(P_0)=(e_{11}+e_{22})$ we can replace s by an element $f(s^2)\circ s=s_{12}$, for at suitable continuous function f, such that

$$s_{12}^2(P) = (e_1+e_2)(P), \quad (U_{e_i}s_{12})(P) = 0 \quad i=1,2,3$$

holds in some neighborhood of P_0, which we denote by U again. Similarly determine an element $s_{13} \in J$ with analogous properties. Hence by the coordinatization theorem [4] we may choose coordinates of M_3^8 at each $P \in U$ such that

$$e_i(P) = e_{ii},\ s_{12}(P) = e_{12}+e_{21},\ s_{13}(P) = e_{13}+e_{31}$$

and

$$s_{23}(P) = 2(s_{12}\circ s_{13})(P) = e_{23}+e_{32} \qquad \text{for } i,j=1,2,3\ .$$

Now let $b_1, b_2, \ldots, b_8$ denote the canonical basis of $\mathscr{C}$. There exist elements $s_{12}^{(i)} \in J$, $i=2,3$, such that $s_{12}^{(i)}(P_0) = b_i e_{12} - b_i e_{21}$. Replacing $s_{12}^{(2)}$ by $s_{12}^{(2)} - s_{12}\circ(s_{12}\circ s_{12}^{(2)})$ if necessary and using again the functional calculus in J, we can achieve

$$s_{12}^{(i)}\circ s_{12}^{(j)}(P) = \delta_{ij}(e_1+e_2)(P), \qquad (U_{e_i}s_{12}^{(j)})(P) = 0, \qquad i,j=1,2\ ,$$

throughout a neighborhood of P_0, which we denote again by U. Above we have written $s_{ij}^{(1)}$ for s_{ij}. Arguing similarly with $s_{12}^{(3)} - \sum_{k=1}^{2} s_{12}^{(k)}\circ(s_{12}^{(k)}\circ s_{12}^{(3)})$ we construct a local symmetry $s_{12}^{(3)}$ such that

$$(1) \qquad s_{12}^{(i)}\circ s_{12}^{(j)}(P) = \delta_{ij}(e_1+e_2)(P), \qquad U_{e_i}s_{12}^{(j)}(P) = 0$$

holds for $P \in U$ and $i,j=1,2,3$.

Now let $s_{12}^{(4)} = 8(s_{13}\circ s_{12}^{(2)})\circ(s_{23}\circ s_{12}^{(3)})$. Then the local symmetries $s_{12}^{(i)}$, $i=1,\ldots,4$ will satisfy (1) throughout U. Now choose $s_{12}^{(5)} \in J$ with

$$s_{12}^{(5)}(P_0) = b_5 e_{12} - b_5 e_{21}\ .$$

Working with $s_{12}^{(5)} - \sum_{k=1}^{4} s_{12}^{(k)}\circ(s_{12}^{(k)}\circ s_{12}^{(5)})$ as above we can determine a local symmetry $s_{12}^{(5)} \in J$ such (1) will hold also for the larger system. Define now

$$s_{12}^{(5+i)} = 8(s_{13}\circ s_{12}^{(i)})\circ(s_{23}\circ s_{12}^{(5)}) \qquad i=1,2,3\ ,$$

then the system $(s_{12}^{(i)})$, $i=1,\ldots,8$ satisfies (1) in U_0. For the proof of this one needs lemma 2. The remaining local symmetries $s_{13}^{(i)}$ and $s_{23}^{(i)}$, $i=2,\ldots,8$ are then defined by $2s_{23}^{(1)}\circ s_{12}^{(i)} = s_{13}^{(i)}$ and $s_{23}^{(i)} = 2s_{12}^{(1)}\circ s_{13}^{(i)}$. This proves the lemma.

An immediate consequence of this lemma is

THEOREM 1. a) *J defines a fibre bundle $\mathscr{B}(J) = \{B, \mathrm{pr}, \hat{J}, M_3^8, E_6\}$, the structure bundle, where $B = \bigcup_{P\in J} J(P)$ and where* pr *is defined by* $\mathrm{pr}\,(a(P)) = P \in \hat{J}$.

b) *Conversely, the system $J(\mathscr{B})$ of all continuous cross sections vanishing at infinity of a fibre bundle $\mathscr{B} = \{B, \mathrm{pr}, \Omega, M_3^8, E_6\}$ with Ω locally compact is a* JB *algebra.*

c) *J is isomorphic to the Jordan algebra of all continuous cross sections of its structure bundle.*

d) *Two exceptional* JB *algebras are isomorphic iff their structure bundles are isomorphic.*

This theorem is shown almost like the corresponding results for n-homogeneous C*-algebras in [8, § 2] once one knows that $E_6 = \mathrm{Aut}\, M_3^8$ [3]. Obviously the norm of the JB algebra in (b) is given by the sup norm of the fibres. In order to prove (c) one defines for each $a \in J$ the map $f_a: J \ni P \to a(P)$. Clearly f_a is a continuous cross section vanishing at infinity and $a \to f_a$ is a J-isomorphism of J onto $f(J)$. It remains to show that $f(J) = J(\mathscr{B}(J))$, i.e. that f is onto. For this it suffices that every continuous cross section with compact support is in $f(J)$. However since $\mathscr{C}_0(\hat{J}) \subset J$ and since J is locally trivial a partition of unity argument reduces this problem to the case of a trivial bundle.

Now let $\mathscr{A}$ be an arbitrary JB algebra with exceptional ideal J. For each $y \in \mathscr{A}$ let $J(y)$ denote the JB subalgebra of $\mathscr{A}$ generated by y. If $\mathscr{A}$ has a unit we choose the subalgebra generated by y and 1. Then we can define the centralizer $\mathscr{Z}(y)$ by

$$\mathscr{Z}(y) = \{a \in \mathscr{A} \mid a \circ (y^n \circ y^m) = (a \circ y^n) \circ y^m \text{ for all } n, m \geqq 0\} .$$

Clearly

$$\mathscr{Z}(y) = \{a \in \mathscr{A} \mid a \circ (y_1 \circ y_2) = (a \circ y_1) \circ y_2 \text{ for all } y_i \in J(y)\} .$$

It is not too hard to see that this definition coincides with that of [1]. For $S \subset \mathscr{A}$ let $\mathscr{Z}(S) = \bigcap_{y \in S} \mathscr{Z}(y)$ denote the centralizer of S.

Lemma 4. $J \cap \mathscr{Z}(J) = \mathscr{C}_0(\hat{J}) \subset \mathscr{Z}(\mathscr{A})$

Proof. Let $a \in \mathscr{Z}(J) \cap J$ and $b \in \mathscr{A}$. Consider $c = a \circ (b^n \circ b^m) - (a \circ b^n) \circ b^m$ and let π be an arbitrary factor representation of $\mathscr{A}$. Then we have $\pi(c) = 0$ if J is annihilated and $\pi(a) = \alpha \cdot 1$ or $\pi(c) = 0$ if J is not annihilated. Since all factor representations are faithful, we get $c = 0$.

Lemma 5. *The annihilator J^0 of J, defined by*

$$J^0 = \{a \in \mathscr{A} \mid a \circ b = 0 \textit{ for all } b \in J\}$$

satisfies

$$J^0 = \{a \in \mathscr{A} \mid a \circ b = 0 \textit{ for all } b \in \mathscr{C}_0(\hat{J})\} .$$

J^0 is a closed special JB *ideal of $\mathscr{A}$.*

This lemma follows immediately from the fact that J contains a bounded central approximate unit.

Definition. A JB algebra $\mathscr{A}$ is called essentially exceptional if $J^0 = (0)$.

THEOREM 2. *Let $\mathscr{A}$ be an arbitrary* JB *algebra with exceptional ideal J. Then $J \oplus J^0$ is c closed* JB *ideal of $\mathscr{A}$ and $\mathscr{A}/(J \oplus J^0)$ is a special* JB *algebra of type* $\mathrm{I}_{\leqq 3}$, *i.e. all factors representations of $\mathscr{A}(J \oplus J^0)$ are special Jordan factors of type* $\mathrm{I}_{\leqq 3}$.

PROOF. Since the set of factor representations of type $\mathrm{I}_{\leqq 3}$ is closed in $\hat{\mathscr{A}}$ and since $\hat{J}$ is dense in $(\mathscr{A}/J^0)\hat{}$, $(\mathscr{A}/J^0)$ is a JB algebra of type $\mathrm{I}_{\leqq 3}$. Hence $\mathscr{A}/(J \oplus J^0)$ is a special JB algebra of type $\mathrm{I}_{\leqq 3}$.

LEMMA 6. *Let $\mathscr{A}$ be an arbitrary* JB *algebra and let $\hat{\mathscr{A}} = \mathrm{Prim}\, \mathscr{A}$ denote its primitive spectrum with the Jacobson topology. Then the sets ${}_n\hat{\mathscr{A}}$ corresponding to factor representations of type $\mathrm{I}_{\leqq n}$ are closed.*

PROOF. Since in JB algebra one has the functional calculus of continuous functions the usual C*-proof shows that for each $x \in \mathscr{A}$ the function $f_x \colon \hat{\mathscr{A}} \ni P \to |x(P)|$ is lower semicontinuous. Now define for each $x \in \mathscr{A}_+ = \{y^2 \mid y \in \mathscr{A}\}$ the function

$$g_x \colon \mathscr{A} \ni P \to \sup \sum |x_i(P)| \,,$$

where the supremum is taken over all finite sets $\{x_i\} \subset \mathscr{A}_+$ with $0 \leqq \sum x_i \leqq x$. Then $\sum |x_i(P)| \leqq \sum \mathrm{Tr}\, x_i(P) \leqq \mathrm{Tr}\, x(P)$ and thus $g_x(P) \leqq \mathrm{Tr}\, x(P)$. Conversely we can always choose $\{x_i\} \subset \mathscr{A}_+$ such that $\sum x_i \leqq x$ and such that $\sum |x_i(P)| = \mathrm{Tr}\, x(P)$. Hence the function $g_x(\cdot) = \mathrm{Tr}\, x(\cdot)$ is lower semicontinuous on $\check{\mathscr{A}}$. Clearly

$${}_n\hat{\mathscr{A}} = \{P \in \mathscr{A} \mid \mathrm{Tr}\, 1(P) \leqq n\}$$

is closed.

This result can probably be shown also be using polynomial identities. However we remark that the usual polynomial identities for associative rings do not work. As an example consider

$$\mathscr{N} = \{x \circ (y \circ z) - (x \circ y) \circ z \mid x, y, z \in \mathscr{A}\}$$

the associator set of $\mathscr{A}$ then every factor representation of type $\mathrm{I}_{\leqq 2}$ maps $\mathscr{N}^2 = \{a \circ b \mid a, b \in \mathscr{N}\}$ onto a central element. In other words $a \circ (b \circ c) - (a \circ b) \circ c$, $a \in \mathscr{N}^2$, $b, c \in \mathscr{A}$ is annihilated by all elements of ${}_2\mathscr{A}$. Every factor representation π of type I_2 maps $\mathscr{A}$ onto a spin factor and we say π is of type $\mathrm{I}_{2,n}$ if $\dim \pi(\mathscr{A}) = n+1$.

Using $\mathscr{N}$ and a refined argument, one can even show that the set of all factorrepresentations of type $\mathrm{I}_{\leqq 2, \leqq n}$ are closed.

Theorem 2 implies in particular that any essentially exceptional JB algebra is of type $\mathrm{I}_{\leqq 3}$. There is a different way to view theorem 2. Let $\mathscr{A}$ be an essentially exceptional JB algebra. Let $a \in \mathscr{A}$ and $P \in \hat{J}$. Then $a(P)$ is defined. By using the center of J and a suitable coordinate bundle for $\mathscr{B}(J)$, it is easy to see that $\hat{J} \ni P \to a(P)$ defines a bounded continuous cross section of $\mathscr{B}(J)$. Clearly the map $\mathscr{A} \ni a \to (a(P))$ is a J-isomorphism of $\mathscr{A}$ into the JB algebra $M(J)$ of all bounded continuous cross sections of $\mathscr{B}(J)$. In fact $M(J)$ can be considered as the multiplier algebra of J. Let $a \in M(J)$ be positive and let (u_λ) be an bounded increasing central approximate unit of J. Then au_λ is a bounded increasing family of elements of J. Hence $(a \circ u_\lambda)$ converges strongly in $\tilde{J}$, the enveloping algebra of J. Here $\tilde{J}$ denotes the strong closure of J in $\tilde{\mathscr{A}}$. Thus $M(J)$ can also be viewed as the idealizer of J in $\tilde{J}$. In particular we see, $\mathscr{A} \subset M(J) \subset \tilde{J} = \mathscr{C}(\Omega) \otimes M_3^8$ [6]. This can be used to improve theorem 2 slightly. (The authors thank the referee for providing a proof of this result).

COROLLARY. *Let $\mathscr{A}$ be an essentially exceptional* JB *algebra. Then $\mathscr{A}$ is of type* $\mathrm{I}_{\leqq 3}$ *and every factor representation maps $\mathscr{A}$ onto a subfactor of M_3^8.*

PROOF. We may consider $\mathscr{A}$ a subalgebra of $\mathscr{B} = \tilde{J} = \mathscr{C}(\Omega) \otimes M_3^8$. Let ϱ be any pure state of $\mathscr{A}$ and let $\bar{\varrho}$ be an arbitrary pure state extension of ϱ to $\mathscr{B}$. By $c(\varrho)$ respective $c(\bar{\varrho})$ denote the central support ϱ, $\bar{\varrho}$ in $\mathscr{A}^{**}$ respectively $\mathscr{B}^{**}$. Considering $\mathscr{A}^{**}$ as a subalgebra of $\mathscr{B}^{**}$ the map $x \to xc(\varrho)$ defined an embedding of $\mathscr{A}^{**}c(\varrho)$ into $\mathscr{B}^{**}c(\bar{\varrho}) \cong M_3^8$.

Conversely any subfactor $\mathscr{D}$ of M_3^8 can be obtained from a factor representation of a suitable essentially exceptional JB algebra. To see this choose a canonical basis of M_3^8 $b_1, \ldots, b_{27}$ such that $b_1, \ldots, b_r$ is a basis of $\mathscr{D}$. Now let $\mathscr{A}$ be the algebra of all sequence (a_n) with $a_n \in M_3^8$ and $a_n = \sum \alpha_{n,k} b_k$ where we assume $\alpha_{n,k} \to 0$ for $27 \geqq k \geqq r+1$ and $\alpha_{n,k} \to \alpha_k \in \mathbf{R}$ for $1 \leqq k \leqq r$. Then $J = c_0 \otimes M_3^8$, $J^0 = (0)$ and $\mathscr{A}/J \cong \mathscr{D}$.

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