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ON A CONJECTURE OF ALPERIN AND MCKAY

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For any finite group G and rational prime p, denote by $m_p(G)$ the number of irreducible complex characters (henceforth referred to as "characters") whose degree is prime to p. Such characters will be referred to as p'-characters. Alperin has conjectured (see [1] for details and background) that for any G:

(1) *$m_p(G)=m_p(N_G(S))$, where $N_G(S)$ is the normalizer of a p-Sylow subgroup S of G.*

Alperin (loc. cit.) proves (1) when $G=\mathrm{GL}(n,q)$ and $q=p^e$. Since then, (1) has been proved for $G=\mathrm{GL}(n,q)$ or $G=\Sigma_n$ (the symmetric group on n symbols) and p any prime by Olsson [7]. The purpose of this note is twofold: firstly, we show that the results of [2] and [3] imply (1) for almost all finite Lie groups of "conformal type" over F_q (e.g. the general symplectic group rather than $\mathrm{Sp}(2n,q)$) and $q=p^e$. Secondly, we prove (1) for $G=\mathrm{SL}(n,q)$ and $q=p^e$, with the aid of the results of [5] and [6]. The former result follows very easily from those of [2] and [3] by Alperin's method. In contrast, to prove the result for $\mathrm{SL}(n,q)$ one needs much more detailed information about the p'-characters of $\mathrm{GL}(n,q)$ and their restriction to $\mathrm{SL}(n,q)$. The way in which distinct arithmetical paths lead to the same result here lends support to the conjecture.

For an abelian group A, $(A)\hat{}$ will denote its complex character group.

1. The case of an algebraic group over a finite field.

Our notation is as follows: $\boldsymbol{G}$ is a connected, reductive group defined over F_q; it is assumed that $\boldsymbol{G}$ has connected centre $\boldsymbol{Z}$, and that the characteristic p is good for $\boldsymbol{G}$ (see [3]; only a few small characteristics are excluded in some cases). G will denote the group F_q-rational points of $\boldsymbol{G}$, and l the F_q-rank of $\boldsymbol{G}/\boldsymbol{Z}$.

Let B, T and U be as in § 5 of [3]. Then U is a Sylow p-subgroup of G, $B=N_G(U)$ and B is the semidirect product $B=T.U$.

We shall prove:

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THEOREM 1. *With notation as above,*

$$m_p(G) = m_p(B) = |Z|q^l$$

where Z is the group of rational points of the centre $\mathbf{Z}$ *of* $\mathbf{G}$.

PROOF. Theorem 3 of [2] asserts, inter alia, that $m_p(G)=|Z|q^l$. Hence one needs only to show that $m_p(B)=|Z|q^l$. For this, following Alperin, one notes the following facts:

(3) *$m_p(B)$ is the number of irreducible complex characters of B/U'.*

The proof is the same as Alperin's for the case GL (n, q).

Now there is a canonical isomorphism η: $U/U' \to X_1 \times \ldots \times X_S$ where $X_i = \mathrm{GF}(q^{n_i})^+$ and $n_1 + \ldots + n_s = l$ (see [3, p. 258]). Thus for any linear (i.e. 1-dimensional) character λ of U, one defines its support by

$$\operatorname{supp}(\lambda) = \{i \mid \lambda|_{X_i} \neq 1\} .$$

As in [1], one then has

(4) *Two linear characters of U are conjugate under the action of B (or, equivalently, T) if and only if they have the same support.*

This is clear from the arguments used to establish Theorem B′ in [3]. The proof of the theorem is now completed by showing

(5) *The number of irreducible complex characters of B/U' is $|Z| \cdot q^l$.*

We have a canonical isomorphism: $B/U' \to T.U/U'$. Since T is abelian and has p'-order, all the irreducible characters of B/U' are constructed as follows: one takes an irreducible (linear) character μ of U/U', extends to its centralizer $T(\mu).U/U'$ and induces to B/U'. The set of $|T(\mu)|$ characters $\{(\varphi\mu)^{B/U'} \mid \varphi \in (T(\mu))\hat{}\}$ depends only on the T-orbit $\Omega(\mu)$. Hence the number of irreducible characters of B/U' is

$$m(B/U') = \sum_{\Omega} |T(\Omega)| \tag{6}$$

the sum being over the T-orbits Ω of characters of U/U', and $T(\Omega)$ denoting the stabilizer of any element of Ω. But by (4), the orbits corresponds to subsets $I \subset \{1, \ldots, l\}$. Moreover by the arguments in § 5 of [2], it is easy to see that

$$|T(I)| = |T| \Big/ \prod_{i \in I} (q^{n_i} - 1)$$

$$= |Z| \cdot \prod_{i \notin I} (q^{n_i} - 1) .$$

Thus

$$\begin{aligned} m(B/U') &= m_p(B) \\ &= \sum_{I \subset \{1,\dots,l\}} |Z| \prod_{i \notin I} (q^{n_i}-1) \\ &= |Z| \prod (q^{n_i}-1+1) \\ &= |Z| q^{\sum n_i} \\ &= |Z| q^l . \end{aligned}$$

This completes the proof of Theorem 1.

2. The case $G = \mathrm{SL}\,(n, q)$.

For this section, notation will be as follows: $G = \mathrm{GL}\,(n,q)$, B is the group of upper triangular matrices in G, T is the group of diagonal matrices in G, U ($< B$) is the (p-group) of upper unitriangular matrices. For any subgroup $H \leq G$. H_1 will denote $H \cap \mathrm{SL}\,(n,q)$. Thus $G_1 = \mathrm{SL}\,(n,q)$, $U_1 = U$ and $B_1 = N_{G_1}(U_1)$. We shall prove, (for p = characteristic of F_q):

THEOREM 2. $m_p(G_1) = m_p(B_1)$.

To prove this, it will be necessary to go into more detail concerning the p'-characters of G and B. In fact we shall in effect set up an *explicit* bijection between the two sets, something which was not necessary for the proof of theorem 1. We first note the following elementary facts:

LEMMA 3. *Let H, K be finite groups, $K \trianglelefteq H$ with H/K cyclic, of p'-order. Then*

(i) *For any irreducible character χ of H, $\chi|_K = \mu_1 + \dots + \mu_e$ where the sum is precisely over one H-orbit $\{\mu_1, \dots, \mu_e\}$ of characters of K.*

(ii) *If χ_1 and χ_2 are characters of H then their restrictions to K either coincide or are disjoint, and $\chi_1|_K = \chi_2|_K \Leftrightarrow \chi_2 = \theta\chi_1$ with $\theta \in (H/K)\hat{}$.*

(iii) *The p'-characters of K are precisely the irreducible constituents of the restrictions of the p'-characters of H.*

(iv) *For any character χ of H, let $f(\chi)$ be the number of characters χ' of H such that $\chi' = \theta\chi$ for some $\theta \in (H/K)\hat{}$, and let $e(\chi)$ be the number of irreducible components of $\chi|_K$. Then $e(\chi) \cdot f(\chi) = |H/K|$.*

Putting these facts together, we obtain

$$m_p(K) = \sum \frac{e(\chi)}{f(\chi)} = \sum \frac{e(\chi)^2}{|H/K|} = \frac{|K|}{|H|} \sum e(\chi)^2 \tag{7}$$

where the sum is over the p'-characters χ of H. Notice also that in the action of $(H/K)\hat{}$ on the characters of H, the stabilizer $S(\chi)$ of χ has order $|H/K|/f(\chi) = e(\chi)$. Hence (7) can be rewritten

$$m_p(K) = \frac{|K|}{|H|} \sum |S(\chi)|^2 . \tag{8}$$

Here $S(\chi)$ is the stabilizer in $(H/K)\hat{}$ of χ, and the sum is over the p'-characters of H.

We now set up a bijection between the p'-characters of G and B, such that $|S(\chi)|$ is the same for corresponding characters. This will prove theorem.

p' characters of G. The p'-characters of GL (n,q) are precisely the characters $= J^{\langle\psi_1\rangle}(\{r_1\})\circ\ldots\circ J^{\langle\psi_1\rangle}(\{r_k\})$, in the notation of [5]. Here $\langle\psi_i\rangle$ is an n_i-simplex, $\{r_i\}$ is the partition of r_i consisting of one part, and $\sum_{i=1}^k n_i r_i = n$. Now $\langle\psi_i\rangle$ corresponds ([6]) to an irreducible monic polynomial f_i of degree n_i over F_q ($\{\psi_i, \psi_i^q, \ldots\}$ is regarded as the set of roots of f_i). Thus the character χ above may be written

$$\begin{aligned} \chi &= f_1^{r_1} f_2^{r_2} \cdots f_k^{r_k} \\ &= f = t^n + a_1 t^{n-1} + \ldots + a_n \\ &= [a_1, \ldots, a_n] \qquad (a_n \neq 0) . \end{aligned} \tag{9}$$

This identification of the p'-characters with polynomials of degree n immediately gives $m_p(G) = q^{n-1}(q-1)$, as this is the number of monic polynomials of degree n over F_q, with non-zero constant term.

In view of (8), we describe how $(G/G_1)\hat{}$ acts on χ above: we have $(G/G_1)\hat{} \cong F_q^*$, and by Corollary 5.23 in [5], the latter acts on χ by multiplicatively translating the roots of f. If we denote χ above (cf. (9)) by

$$\chi = [a_1, a_2, \ldots, a_n]$$

(recall $a_i \in \mathsf{F}_q$, $i = 1, \ldots, n-1$, $a_n \in \mathsf{F}_q^*$), then for $a \in (G/G_1)\hat{} \cong \mathsf{F}_q^*$ it is easily seen that

$$\chi^a = [aa_1, a^2 a_2, a^3 a_3, \ldots, a^n a_n] . \tag{10}$$

For the same χ, define

$$\operatorname{supp}(\chi) = I = \{i \mid a_i = 0\} \subset \{1, \ldots, n-1\} .$$

Then for the stabilizer $S(\chi)$ of χ in F_q^*, we have (from (10)).

$$|S(\chi)| = |\{a \in \mathsf{F}_q^* \mid a^i = 1 \text{ for } i \notin I\}| . \tag{11}$$

p'-characters of B. From the discussion preceding formula (6) it is apparent that for each p'-character χ of B there is a subset $I \subset \{1,\dots,n-1\}$ such that χ corresponds to a character φ of $T(I)$. Moreover I and φ are uniquely determined by χ. We write

$$\chi = (I, \varphi)\,. \tag{12}$$

To identify $T(I)$, consider the isomorphism $\alpha\colon \mathrm{T} \to \mathbf{F}_q^* \times \dots \times \mathbf{F}_q^*$ (n times) given by

$$\alpha(\mathrm{diag}\,(a_1,\dots,a_n)) = (a_1 a_2^{-1},\dots,a_{n-1}a_n^{-1},a_n)\,. \tag{13}$$

Identifying T with $\mathbf{F}_q^* \times \dots \times \mathbf{F}_q^*$ using α, we have

$$T(I) = \{(t_1,\dots,t_n) \mid t_i \in \mathbf{F}_q^*,\ t_i = 1 \text{ for } i \in I\}\,. \tag{14}$$

In this way we identify χ with a symbol

$$\chi = [\varphi_j]_{j \notin I}, \quad \varphi_j \in (\mathbf{F}_q^*)^\wedge,\ j \in \{1,2,\dots,n\}\,. \tag{15}$$

The character φ of $\chi = (I, \varphi)$ is defined by

$$\varphi(t_1,\dots,t_n) = \prod_{i \notin I} \varphi_i(t_i)\,. \tag{16}$$

We now describe the action of $(B/B_1)^\wedge$ on χ. Using the notation of the discussion preceding (6) above, we have, for $\theta \in (B/B_1)^\wedge$

$$\theta\,.\,(\varphi\mu)^{B/U'} = ((\theta\varphi)\cdot\mu)^{B/U'} \tag{17}$$

where $\theta\varphi$ denotes the product of φ and the restriction of θ to $T(I)$, and μ is the linear character of U in (6). Since B/B_1 may be canonically identified with T/T_1 (and hence similarly for their character groups), we therefore have in the notation of (12), for $\theta \in (T/T_1)^\wedge = (B/B_1)^\wedge$

$$\chi^\theta = \theta\,.\,(I, \varphi) = (I,\, \theta\varphi)\,. \tag{18}$$

Any character θ of T which is trivial on T_1 is of the form

$$\begin{aligned}\theta(\mathrm{diag}\,(a_1,\dots,a_n)) &= \psi(a_1 a_2 \dots a_n), \quad \text{for some } \psi \in (\mathbf{F}_q^*)^\wedge \\ &= \psi(a_1a_2^{-1})\psi(a_2a_3^{-1})^2 \dots \psi(a_{n-1}a_n^{-1})^{n-1}\psi(a_n)^n\,,\end{aligned}$$

that is,

$$\theta(t_1,\dots,t_n) = \psi(t_1)\psi^2(t_2)\dots\psi^n(t_n) \tag{19}$$

(here $\psi \in (\mathbf{F}_q^*)^\wedge$).

Thus from (15) and (18) we have, for $\theta \in (T/T_1)^\wedge$

$$\chi^\theta = [\psi^j\varphi_j]_{j \notin I} \quad (\psi \in (\mathbf{F}_q^*)^\wedge)\,. \tag{20}$$

It follows that the order of the stabilizer $S(\chi)$ of χ in $(B/B')\hat{}\ (=(T/T_1)\hat{})$ has order given by

$$(21)\qquad |S(\chi)| = |\{\psi \in (\mathsf{F}_q^*)\hat{} \mid \psi^j = 1 \text{ for } j \notin I\}| .$$

Thus in both (11) and (21), $|S(\chi)|$ depends only on $I \subset \{1,\ldots,n-1\}$, and has the same value in each case. Since the number of χ corresponding to a given I is the same in both cases $(=(q-1)^{n-|I|})$, Theorem 2 now follows from (8) applied to the pairs G, G_1 and B, B_1.

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