MATH. SCAND. 43 (1978), 35–43

ON THE GAUSSIAN MARKOFF SPECTRUM

HENRY W. HANSEN

Abstract.

A method for determining small Markoff constants over imaginary quadratic fields is described. In particular, the first two gaps of the field $\mathbf{Q}(i)$ are determined.

1. Introduction.

Let $f(x,y)=ax^2+bxy+cy^2$ be a quadratic form with complex coefficients and with discriminant $D(f)=b^2-4ac\neq 0$. For such a form, define the *minimum* $M(f)=\inf|f(x,y)|$ over the set of pairs of Gaussian integers $(x,y)\neq(0,0)$, and the *Markoff constant* $\mu(f)=M(f)^{-1}|D(f)|^{\frac{1}{2}}$ if $M(f)\neq 0$ and $\mu(f)=\infty$ if $M(f)=0$. The *Markoff spectrum* of $Q(i)$ is the set of all Markoff constants $\mu(f)$ as f runs through the set of all non-singular complex forms.

Ford [3] in 1925 and Perron [5] in 1930 independently proved that the smallest value of this Gaussian Markoff spectrum is $3^{\frac{1}{2}}$. Cassels [2] in 1952 showed that this value is isolated in the spectrum in the sense that there exists a constant $M_1>3^{\frac{1}{2}}$ such that if $0<\mu(f)<M_1$, then f is equivalent to a multiple of $f_1(x,y)=x^2+xy+y^2$. A. L. Schmidt [6] in 1967 developed the theory of Farey triangles and quadrangles in the complex plane to show that $M_1>1.80$. Finally, in 1975, Schmidt [7] introduced the theory of regular and dually regular chains, the Gaussian analog of real continued fractions, to completely determine all complex binary quadratic forms with $\mu(f)<2$.

In this paper, another method for finding some small Markoff constants in imaginary quadratic fields will be described. In particular, we will give a proof of the following

THEOREM. *The Markoff constants for the field* $\mathbf{Q}(i)$ *in the interval* $(0,1.961)$ *are* $3^{\frac{1}{2}}$ *and* $(3/5)^{\frac{1}{2}}41^{\frac{1}{4}}=1.9600700\ldots$.

The forms associated with the second constant can be shown to be isolated in the same manner as those for the first.

Received April 29, 1977.

2. Basic properties.

Two complex quadratic forms are *equivalent* if there exist Gaussian integers r, s, t, u such that $|ru-st|=1$ and $g(x,y)=f(rx+sy, tx+uy)$. It is easily verified that if g is equivalent to a multiple of f, then $\mu(g)=\mu(f)$.

Further, in a manner completely similar to that of Cassels in the real case, [1, pp. 20–21], we can show that if r, s are relatively prime Gaussian integers and $f(r,s)=a'\neq 0$, then $f(x,y)$ is equivalent to a form $g(x,y)=a'x^2+b'xy+c'y^2$ where $|b'|\leqq 2^{\frac{1}{2}}|a'|$. So if f attains its minimum, i.e., if there is a pair of Gaussian integers r, s such that $f(r,s)=M(f)>0$, it then suffices to assume that $M(f)=f(1,0)$ since we are now only interested in Markoff constants. And by definition of $\mu(f)$, we may certainly assume that $M(f)>0$.

If f does not attain its minimum, then we can continue to paraphrase Cassels [1] and find a sequence of forms $f_n(x,y)=a_nx^2+b_nxy+c_ny^2$ satisfying the conditions

$$a_n = f(r_n, s_n)\,,$$
$$0 < M(f) < |a_n| < M(f)+n^{-1}, \qquad |b_n| \leqq 2^{\frac{1}{2}}a_n\,,$$
$$|D(f_n)| = |D(f)|, \qquad M(f_n) = M(f)\,,$$

for $n=1,2,3,\ldots$. By compactness, there is a subsequence with coefficients converging to $F(x,y)=Ax^2+Bxy+Cy^2$, and it follows immediately that $\mu(F)=\mu(f)$ and $M(F)=A=f(1,0)$.

Furthermore, since $\mu(\lambda f)=\mu(f)$, we can divide by the coefficient of x^2 and consider only those forms $f(x,y)=(x-ay)(x-by)$ with roots a, b satisfying the conditions

$$|D(f)| = |a-b|^2 > 0\,, \tag{1}$$

$$|(x-ay)(x-by)| \geqq 1 \quad \text{for every } (x,y)\neq(0,0),\ x,y\in \mathbf{Z}[i]\,. \tag{2}$$

Note that $\mu(f)=|a-b|$.

3. Method of proof.

By using an equivalent form if necessary, we may assume that the root a lies in the square $X=(0, -i, -1-i, -1)$. Being interested in only the smaller Markoff constants, we can further assume that

$$\mu(f) = |a-b| < 2\,, \tag{3}$$

and up to equivalence, this forces the root b into the square $(-1-i, -1+2i, 2+2i, 2-i)$. Finally, by using the diagonal symmetry $(a,b)\to(i\bar{a}, i\bar{b})$, $\bar{a}$ denoting the complex conjugate of a, we may even assume that b lies in the triangle $\Delta(-1-i, 2+2i, 2-i)$.

The proof will be split into six cases, with the root $a \in X$, and b in one of the six squares X, $Y=(0,1,1-i,-i)$, $Z=(1,2,2-i,1-i)$, $U=(i,1+i,1,0)$, $V=(1+i,2+i,2,1)$, $W=(1+2i,2+2i,2+i,1+i)$.

Inequality (2) can be rewritten as

$$(4) \qquad |xy^{-1}-a| \geqq |y|^{-2}|xy^{-1}-b|^{-1} .$$

For a fixed pair of Gaussian integers x_0, y_0, $x_0y_0^{-1} \in X$, $|x_0y_0^{-1}-b|$ is bounded above by some positive constant K since in each case b has been restricted to a bounded region. Hence the inequality

$$|x_0y_0^{-1}-a| \geqq K^{-1}|y_0|^{-2}$$

forces the root a outside the circle with center $x_0y_0^{-1}$ and radius $K^{-1}|y_0|^{-2}$. Similarly,

$$(5) \qquad |xy^{-1}-b| \geqq |y|^{-2}|xy^{-1}-a|^{-1}$$

puts similar restrictions on b. Enough such circles can be constructed in each case to either cover an entire square, in which case no form arises, or to restrict the roots a and b to small regions, thus enabling certain isolation techniques to be applied in order to obtain the desired gaps.

4. The first gap.

CASE I. $a \in X$, $b \in X$. Since a and b both lie in the same unit square, we get from (4) and (5) that

$$|x-a| \geqq 2^{-\frac{1}{2}}, \qquad |x-b| \geqq 2^{-\frac{1}{2}}$$

for $x=0, -1-i, -1$ and $y=1$. This forces $a=b=(-1-i)/2$, contradicting (1), so no form arises in this case.

CASE II. $a \in X$, $b \in Y$. For convenience translate each root by i, so that $a \in (-1+i,i,0,-1)$, $b \in U$. Applying (4), (5) with $(x,y)=(1,1)$ gives in sequence

$$|1-b| \geqq |1-(-1+i)|^{-1} = 5^{-\frac{1}{2}} ,$$

$$|(-1+i)-a| \geqq |(-1+i)-(1+i5^{-\frac{1}{2}})|^{-1} > 0.4819 .$$

So by symmetry,

$$(6) \qquad |x-a| > 0.48 \quad \text{for } x=-1,-1+i ,$$

$$(7) \qquad |x-b| > 0.48 \quad \text{for } x= 1,1+i .$$

Using these restrictions on a and b, we then get

$$(8) \qquad |i-a|, |i-b|, |a|, |b| > |1+0.52i|^{-1} > 0.887 .$$

These few constraints (6)–(8) on a and b are already sufficient to apply our isolation methods, but we can restrict the roots much further with little effort. This kind of finer analysis will be necessary for the later cases.

First, note that

$$|(2+i)/2-b| \geqq 4^{-1}|(2+i)/2-(-1+0.48i)|^{-1} > 0.1249 \tag{9}$$

and by symmetry

$$|(-2+i)/2-a| \geqq 0.1249\,. \tag{10}$$

Now, $0.89+0.46i$ lies inside the circles (9) and (6a), so this point can be used to give better approximations than $1+0.48i$. Namely,

$$|x-a| > |(-1+i)-(0.89+0.46i)|^{-1} > 0.508 \quad \text{for } x=-1+i,\,-1; \tag{11}$$

$$|x-b| > 0.508 \quad \text{for } x=1,\,1+i; \tag{12}$$

$$|-1/(1+i)-a| > 2^{-1}|-1/(1+i)-(0.89+0.46i)|^{-1} > 0.359; \tag{13}$$

$$|1/(1-i)-b| > 0.359\,. \tag{14}$$

Therefore the roots a and b are forced into the regions $\mathscr{A}$ and $\mathscr{B}$ bounded by circles (10), (11), (13) and (9), (12), (14), respectively.

Region $\mathscr{B}$ is enclosed in a square with vertices $0.85+0.52i$, $0.89+0.52i$, $0.89+0.48i$, $0.85+0.48i$. Noting that $3^{\frac{1}{2}}/2+i/2 \in \mathscr{B}$ and using symmetry for the root a, we can write

$$a = (-3^{\frac{1}{2}}/2+\varepsilon_1)+i(\tfrac{1}{2}+\varepsilon_2) \tag{15}$$

$$b = (3^{\frac{1}{2}}/2+\varepsilon_3)+i(\tfrac{1}{2}+\varepsilon_4) \tag{16}$$

where the ε_i are real and

$$|\varepsilon_i| < 0.03 \qquad (i=1,2,3,4)\,. \tag{17}$$

It will next be shown that $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=0$.

The law of arithmetic and geometric means applied to (2) gives

$$\gamma|x-ay|^2+\gamma^{-1}|x-by|^2 \geqq 2 \tag{18}$$

where x, y are fixed Gaussian integers not both zero and $\gamma>0$ is a constant. Upon the substitution of the eight pairs

$$(x,y) = (0,1), (i,1), (1,1), (1+i,1), (2+i,2), (-2+i,2), (-1+i,1), (-1,1)$$

into (18), along with the choice

$$\gamma = |x-(3^{\frac{1}{2}}/2+i/2)y|^2\,,$$

and a, b given by (15), (16), eight inequalities of the type

$$A\varepsilon_1+B\varepsilon_2+C\varepsilon_3+D\varepsilon_4+E(\varepsilon_1^2+\varepsilon_2^2)+F(\varepsilon_3^2+\varepsilon_4^2) \geqq 0$$

are obtained. Upon taking the proper linear combination of these inequalities, (see [4, pp. 23–24] for the explicit combination), we obtain

$$-\sum_{i=1}^{4} |\varepsilon_i|+(3+3^{\frac{1}{2}}) \sum_{i=1}^{4} \varepsilon_i^2 \geqq 0 .$$

Rewriting this last inequality, we see that if some term of

$$\sum_{i=1}^{4} [(3+3^{\frac{1}{2}})|\varepsilon_i|^2-|\varepsilon_i|] \geqq 0$$

were positive, then for some i we would have $|\varepsilon_i|>(3+3^{\frac{1}{2}})^{-1}>0.21$, contradicting (17). Hence each term is zero, so that $\varepsilon_i=0$ or $|\varepsilon_i|=(3+3^{\frac{1}{2}})^{-1}>0.21$. Therefore $\varepsilon_i=0(i=1,2,3,4)$ and we have proved the following

THEOREM 1. *If* $f(x,y)=(x-ay)(x-by)$ *has roots* $a \in (-1+i,i,0,-1)$, $b \in (i,1+i,1,0)$ *satisfying the conditions* $D(f)=|a-b|^2>0$ *and* $|f(x,y)|\geqq 1$ *for all pairs of Gaussian integers* $(x,y)\neq(0,0)$, *then*

$$f(x,y) = (x-(-3^{\frac{1}{2}}/2+i/2)y)(x-(3^{\frac{1}{2}}/2+i/2)y) = x^2-ixy+y^2 .$$

After the investigation of the remaining cases, it will be seen that $\mu(x^2-ixy+y^3)=3^{\frac{1}{2}}$ is the only value of the Markoff spectrum in the interval $0<\mu(f)<(3/5)^{\frac{1}{2}}41^{\frac{1}{4}}$.

5. The second gap.

We next assume that $a \in X$ and $b \in U$; by making use of the symmetry $(a,b) \to (i\bar{a},i\bar{b})$ we can even suppose that $b \in \Delta(0,i,1+i)$. Since there is an infinite chain of forms occuring in this case (see [7]), inequalities (4) and (5) are not sufficient to locate the next form, as they were in section 4. If, however, we also insist that

$$(19) \qquad 0 < |a-b| < 1.961 ,$$

we can then show that any form $f(x,y)=(x-ay)(x-by)$ satisfying conditions (2) and (19) must have roots a and b lying either in the triangles $\Delta_1=((-1-i)/2,(-4-3i)/5,(-4-2i)/5)$ and $\Delta_2=((3+4i)/5,(1+2i)/2,1+i)$, respectively, or in a pair of triangles symmetric to Δ_1 and Δ_2 about the origin or about the diagonal $\mathrm{Re}(z)=\mathrm{Im}(z)$.

Denote the circle $|z-a|=r$ by $(a;r)$. Then, assuming $a \in \Delta_1$ and $b \in \Delta_2$, and using only (4) and (5), we find that the root a is restricted to a region bounded by circles $((-4-2i)/5; 0.1049)$, $((-4-3i)/5; 0.0976)$, $((-3-2i)/4; 0.03218)$, $((-23-15i)/29; 0.01738)$, and b to a region bounded by $((1+2i)/2; 0.1256)$, $((3+4i)/5; 0.1043)$, $((8+12i)/13; 0.0348)$, $((15+23i)/26; 0.01977)$. Note that all centers are vertices of the Farey partition of the complex plane defined by A. L. Schmidt [6].

Denoting by a_2 and b_2 the roots of $f_2(x,y)=x^2+(\frac{1}{5}-2i/5)xy-iy^2$, we can then write

$$a = a_2+(\delta_1+i\delta_2), \qquad b = b_2+(\delta_3+i\delta_4)$$

where

$$|\delta_1| < 0.0015, \qquad |\delta_j| < 0.0013 \qquad \text{for } j=2,3,4 .$$

By using these bounds to obtain similar ones for the error term in $a+b$ and ab, we obtain the

LEMMA. *Let $f(x,y)=(x-ay)(x-by)$ have roots $a \in \Delta_1$, $b \in \Delta_2$ satisfying inequalities* (1) *and* (2). *Then*

$$f(x,y) = x^2+[(\tfrac{1}{5}+\varepsilon_1)-(\tfrac{2}{5}+\varepsilon_2)i]xy-[\varepsilon_3+(1+\varepsilon_4)i]y^2 \tag{20}$$

where

$$|\varepsilon_i| < 0.0004 \qquad \textit{for } i=1,2,3,4 . \tag{21}$$

For each pair of Gaussian integers (x,y) substituted into $|f(x,y)|^2 \geqq 1$, f defined by (20), an inequality

$$A\varepsilon_1+B\varepsilon_2+C\varepsilon_3+D\varepsilon_4+E(\varepsilon_1^2+\varepsilon_2^2)+F(\varepsilon_3^2+\varepsilon_4^2)$$
$$+G(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)+H(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) \geqq 0$$

with rational integral coefficients is obtained. We use the inequalities

$$\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4, \varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4 \leqq 0.5 \sum_{i=1}^{4} \varepsilon_i^2$$

to eliminate the terms with coefficients G and H.

By choosing $(x,y)=(0,1)$, $(i,1)$, $(-1,1)$, $(1+i,1)$, $(-i,1+i)$, $(1+2i,2)$, $(2,-2+i)$, $(-1+2i,1+2i)$, $(2-i,-1+2i)$, $(4i,3+2i)$, $(5+i,-5+2i)$, and taking the proper positive linear combinations of these inequalities (see [4]), we obtain

$$\sum_{i=1}^{4} (-|\varepsilon_i|+115\varepsilon_i^2) \geqq 0 .$$

If some term were positive, then $|\varepsilon_i|>115^{-1}>0.008$, contradicting (21).

Therefore all terms are zero and $\varepsilon_i = 0$ for $i = 1, 2, 3, 4$.

We have therefore proved

THEOREM 2. *Let $f(x,y) = (x - ay)(x - by)$ have roots $a \in X$, $b \in U$, satisfying (2), (19). Let a_2, b_2 be the roots of*

$$f_2(x,y) = x^2 + (\tfrac{1}{5} - 2i/5)xy - iy^2 .$$

Then (a,b) must be one of the four pairs

$$(a_2, b_2),\ (-b_2, -a_2),\ (i\bar{a}_2, i\bar{b}_2),\ (-i\bar{b}_2, -i\bar{a}_2) .$$

Furthermore, $f_2(x,y)$ is the only such form satisfying (1), (2) *and having roots*

$$a \in \Delta_1 = ((-1-i)/2, (-4-3i)/5, (-4-2i)/5)$$

and

$$b \in \Delta_2 = ((3+4i)/5, (1+2i)/2, 1+i) .$$

COROLLARY. $\mu(f_2) = (\frac{3}{5})^{\frac{1}{2}} 41^{\frac{1}{4}}$.

PROOF. Note that $(1+2i)f_2(x,y)$ can not attain any of the values i^k or $\pm 1 \pm i$. Therefore $M(f_2) = 1$.

6. Other cases.

If $a \in X$ and $b \in V$ or $b \in W$, then we can show that no forms arise satisfying (2) and (19). Enough of each region can be covered to force $|a-b| > 1.961$.

In the remaining case, $a \in X$ and $b \in Z$, the analysis proceeds exactly as in section 5. In particular, the transformation $(x,y) \to (ix+y, x)$ maps $(1+2i)f_2(x,y)$ into the form

$$(1-2i)f'_2(x,y) = (1-2i)x^2 + (-3+2i)xy + (1+2i)y^2 \tag{22}$$

with roots

$$a'_2 = (a_2 - i)^{-1} \qquad X' = (-1+i, i, 0, -1) ,$$

$$b'_2 = (b_2 - i)^{-1} \qquad V = (1+i, 2+i, 2, 1) .$$

Note that X', V are just translations by i of X, Z.

Proceeding as before, we can show that any root a satisfying (2) and (19) lies in a region bounded by the circles $((-1+3i)/5; 0.1049)$, $((-4+7i)/13; 0.03840)$, $((-1+2i)/4; 0.03219)$, and the root b in a region bounded by $((5+i)/3; 0.05700)$, $((43+7i)/26; 0.01978)$, $((49+8i)/29; 0.01743)$. This allows us to write a form with roots $a \in \Delta_3 = (-\frac{1}{2}+i/2, i, i/2)$ and $b \in V$ and satisfying (2) and (19) as

$$f'(x,y) = x^2 - [(\tfrac{7}{5}+\varepsilon_1)+(\tfrac{4}{5}+\varepsilon_2)i]xy + [(-\tfrac{3}{5}+\varepsilon_3)+(\tfrac{4}{5}+\varepsilon_4)i]y^2$$

where

$$(23) \qquad |\varepsilon_i| < 0.0035, \quad \text{for } i=1,2,3,4 .$$

But as before, we can obtain an inequality

$$\sum_{i=1}^{4} (-|\varepsilon_i| + 130\varepsilon_i^2) \geqq 0 ,$$

and since $|\varepsilon_i| \geqq 130^{-1} > 0.007$ contradicts (23), it follows that $\varepsilon_i = 0$ for $i = 1,2,3,4$.

Therefore we have proved

THEOREM 3. *Let $f(x,y) = (x-ay)(x-by)$ have roots $a \in X'$, $b \in V$ satisfying (2) and (19). Let a', b' be roots of $f'_2(x,y)$ defined by (22). Then (a,b) is one of the four pairs*

$$(a'_2, b'_2),\ (1+i-a'_2, 1+i-b'_2),\ (1-\bar{a}'_2, 1-\bar{b}'_2),\ (i+\bar{a}'_2, i+\bar{b}'_2) .$$

7. Summary.

The above results are summarized below.

THEOREM. *Let $f(x,y)$ be a quadratic form with complex coefficients satisfying the conditions*

$$M(f) = \inf|f(x,y)| = f(1,0) = 1, \qquad (x,y) \neq (0,0),\ x,y \in \mathbf{Z}[i] ,$$

$$0 < \mu(f) = |D(f)|^{\frac{1}{2}} M(f)^{-1} < 1.961 .$$

Then $f(x,y)$ is equivalent to a multiple of either

$$f_1(x,y) = x^2 - ixy + y^2 ,$$

$$(1+2i)f_2(x,y) = (1+2i)x^2 + xy + (2-i)y^2 ,$$

or its conjugate form

$$(1-2i)\bar{f}_2(x,y) = (1-2i)x^2 + xy + (2+i)y^2 .$$

It follows that the only Markoff constants for the field $\mathbf{Q}(i)$ in the interval $(0, 1.961)$ are $3^{\frac{1}{2}}$ and $(\frac{3}{5})^{\frac{1}{2}} 41^{\frac{1}{4}}$.

We will show in another paper that the forms f_2 and $\bar{f}_2$ are isolated, that is, *any* form satisfying $3^{\frac{1}{2}} < \mu(f) < 1.961$ is equivalent to a multiple of f_2 or $\bar{f}_2$.

REFERENCES

1. J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge University Press, New York, 1957.
2. J. W. S. Cassels, *Über einen Perronschen Satz*, Arch. Math. (Basel) 3 (1952), 10–14.
3. L. R. Ford, *On the closeness of approach of complex rational fractions to a complex irrational number*, Trans. Amer. Math. Soc. 27 (1925), 146–154.
4. H. W. Hansen, *The second gap of the Markoff spectrum of* Q(*i*), Dissertation, The Ohio State University, 1973.
5. O. Perron, *Über die Approximation einen komplexen Zahl durch Zahlen des Körpers K(i)* I-II, Math. Ann. 103 (1930), 533–544; 105 (1931), 160–164.
6. A. L. Schmidt, *Farey triangles and Farey quadrangles in the complex plane*, Math. Scand. 21 (1967), 241–295.
7. A. L. Schmidt, *Diophantine approximation of complex numbers*, Acta Math. 134 (1975), 1–85.

THE OHIO STATE UNIVERSITY
NEWARK, OHIO
U.S.A.