MATH. SCAND. 43 (1978), 44–48

FIBRES OF HUREWICZ AND APPROXIMATE FIBRATIONS

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A mapping $p\colon E \to B$ is a *Hurewicz fibration* if given mappings $H\colon X \times [0,1] \to B$ and $h\colon X \to E$ such that $ph(x) = H(x,0)$ for all $x \in X$, then there exists $G\colon X \times [0,1] \to E$ such that $pG = H$ and $G(x,0) = h(x)$ for all $x \in X$. If $x \in B$, then $p^{-1}(x)$ is called a *fibre* of p. Suppose that $p\colon E \to B$ is a Hurewicz fibration between closed connected manifolds of dimensions m and n, respectively. If E, B and p are smooth, then p is locally trivial and the fibres are smooth manifolds of dimension $m-n$. If p is not smooth, then p need not be locally trivial nor need the fibre be a manifold [9]. In [12], F. Raymond showed that the fibre must be a generalized manifold. One of the main results of this paper is the following.

THEOREM 1. *Let $p\colon E \to B$ be a Hurewicz fibration from a closed connected m-dimensional* CAT *manifold onto an n-dimensional* TOP *manifold with fibre F. If $m-n \geqq 5$ and if* Wh $(\pi_1(F) \oplus \mathbf{Z}^r) = 0$ *for all $r \leqq n-1$, then the fibre of p has the homotopy type of a closed* CAT *manifold of dimension $m-n$.*

Wh $(\pi_1(F) \oplus \mathbf{Z}^r)$ denotes the Whitehead group [11] of the direct sum of the fundamental group of F and the free Abelian group of rank r. Recall that if F is simply connected or has fundamental group isomorphic to a free Abelian group, then Wh $(\pi_1(F) \oplus \mathbf{Z}^r) = 0$ for all r [11]. CAT denotes one of the three categories: DIFF = differentiable, PL = piecewise linear or TOP = topological.

A mapping $p\colon E \to B$ is an *approximate fibration* if given an open cover $\mathscr{U}$ of B and mappings $H\colon X \times [0,1] \to B$ and $h\colon X \to E$ such that $ph(x) = H(x,0)$ for all $x \in X$, then there exists $G\colon X \times [0,1] \to E$ such that pG and H are $\mathscr{U}$-close (i.e., given $(x,t) \in X \times [0,1]$, then there exists $U \in \mathscr{U}$ such that $pG(x,t)$ and $H(x,t)$ are elements of U) and $G(x,0) = h(x)$ for all $x \in X$. Coram and Duvall [3] introduced the concept of approximate fibrations and showed that they have many similar properties as Hurewicz fibrations if one uses shape-theoretic

[1] Research was conducted while the author was on leave at the University of Zagreb and was supported under an exchange program between the National Academy of Sciences (U.S.A.) and the Yugoslav Council of Academies.

Received June 1, 1977.

concepts in place of their homotopy-theoretic counterparts. For example, if $p: E \to B$ is an approximate fibration between compact ANR's and if B is connected, then each fibre is an FANR and any two fibres have the same shape. Coram and Duvall construct an approximate fibration from the torus to the 1-sphere such that all the fibres except one are 1-spheres and the exceptional fibre is a Warsaw circle.

THEOREM 2. *Let $p: E \to B$ be an approximate fibration from a closed connected m-dimensional* CAT *manifold onto an n-dimensional* TOP *manifold with fibre F. If $m-n \geqq 5$,* Wh $(\check{\pi}_1(F) \oplus \mathbf{Z}^r)=0$ *for $r \leqq n$, and if F has the shape of a finite complex, then F has the shape of a closed* CAT *manifold of dimension $m-n$.*

$\check{\pi}_1(F)$ denotes the shape fundamental group [1] of F.

The author does not know whether the hypotheses on the fibre are necessary. However, it should be noted that S. Ferry [6] has constructed an approximate fibration from a compact ANR onto the 1-sphere such that the fibre does not have the shape of a finite complex.

First, we recall some results from the thesis of L. C. Siebenmann [13]. Although these results are stated in the DIFF category, they are also valid in any of the three categories CAT = DIFF, PL or TOP by the work of Kirby and Siebenmann [10]. Let M be an open connected r-dimensional CAT manifold with a finite number of ends $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$. The end ε_i is *stable* if there exists a sequence of connected neighborhoods $\{U_i\}$ of the end ε_i such that $U_i \supseteq U_{i+1}$ for all i, $\bigcap_i \mathrm{cl}(U_i) = \emptyset$, and if $\alpha_i: \pi_1(U_{i+1}, x_{i+1}) \to \pi_1(U_i, x_i)$ denotes the homomorphism induced by inclusion and a path between x_{i+1} and x_i, then $\alpha_i \mid \mathrm{image}\, \alpha_{i+1}: \mathrm{image}\, \alpha_{i+1} \to \mathrm{image}\, \alpha_i$ is an isomorphism of finitely presented groups. The inverse limit of this sequence of groups is called the *fundamental group of the end* ε_i and is denoted by $\pi_1(\varepsilon_i)$. A submanifold $V \subseteq M$ is called a 1-*neighborhood* of the end ε_i if V is a closed connected neighborhood of ε_i with compact connected boundary ∂V such that the inclusions $\partial V \subseteq V$ and of the end ε_i into V induce isomorphisms of fundamental groups. Siebenmann shows that if $r \geqq 5$ and if the end ε_i is stable, then ε_i has arbitrarily small 1-neighborhoods.

THEOREM 3 (Siebenmann). *Let M be an open connected* CAT *manifold of dimension $\geqq 6$ with a finite number of ends $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$. M is homeomorphic to the interior of a compact* CAT *manifold if and only if*

3.1 *each end is stable;*

3.2 *each end has arbitrarily small* 1-*neighborhoods which are dominated by a finite complex;*

3.3 *for each $i = 1, 2, \ldots, t$, a certain invariant $\sigma_i \in \tilde{K}_0(\pi_1(\varepsilon_i))$ must be zero.*

The invariant σ_i vanishes if and only if ε_i has arbitrarily small 1-neighborhoods which have the homotopy type of a finite complex.

Let T^r denote the product of r 1-spheres.

THEOREM 4. *Let F be a compact connected* ANR *such that for some $r \geqq 0$, $F \times T^r$ has the homotopy type of a closed connected* CAT *n-manifold, $n \geqq r+5$. If* Wh $(\pi_1(F) \oplus \mathbf{Z}^i)=0$ *for $i \leqq r$, then F has the homotopy type of a closed connected $(n-r)$-dimensional* CAT *manifold.*

PROOF. The proof will be by induction on r. The theorem is trivially true for $r=0$. Let $f\colon V \to F \times T^r$ be a homotopy equivalence of a closed connected CAT n-dimensional manifold onto $F \times T^r$. Since Wh $(\pi_1(F) \oplus \mathbf{Z}^r)=0$, the torsion [2] of f is zero. Let Q denote the Hilbert cube. By [14], $f \times \mathrm{id}\colon V \times Q \to F \times T^r \times Q$ is homotopic to a homeomorphism h.

Let $p\colon F \times \mathbf{R} \times T^{r-1} \to F \times T^r$ be a covering map (R = real numbers) and let $q\colon \tilde{V} \to V$ be the pull-back of p by f. By covering space theory, there exists a homeomorphism $\tilde{h}\colon \tilde{V} \times Q \to F \times \mathbf{R} \times T^{r-1} \times Q$ such that $(p \times \mathrm{id})\tilde{h} = h(q \times \mathrm{id})$. Note that $\tilde{V}$ is a CAT manifold which is a proper deformation retract of $\tilde{V} \times Q$. Hence, $\tilde{V}$ has two ends ε_1 and ε_2, $\tilde{V}$ is stable at each end and $\pi_1(\varepsilon_i)$ is isomorphic to $\pi_1(F \times T^{r-1})$. Let $V_1 \subseteq \tilde{V}$ be a 1-neighborhood of one of the ends of $\tilde{V}$. Without loss of generality, we may assume that there exists $a \in \mathbf{R}$ such that $F \times [a, +\infty) \times T^{r-1} \times Q \subseteq \operatorname{int} \tilde{h}(V_1 \times Q)$. Let C be the compact subset of $F \times \mathbf{R} \times T^{r-1} \times Q$ whose frontier is $(F \times \{a\} \times T^{r-1} \times Q) \cup \tilde{h}(\partial V_1 \times Q)$. Note that C is a deformation retract of $\tilde{h}(V_1 \times Q)$ and since C is a compact ANR, V_1 has the homotopy type of a finite complex [15]. By Theorem 3. $\tilde{V}$ is homeomorphic to the interior of a compact CAT manifold W. If W_1 is a component of ∂W, then it is straightforward to check that the inclusion of W_1 into W is a homotopy equivalence. Hence $F \times T^{r-1}$ is homotopy equivalent to the closed connected CAT manifold W_1. The induction hypothesis implies that F has the homotopy type of a closed, connected CAT manifold of dimension $n-r$.

PROOF OF THEOREM 1. Suppose F is not connected. Since F is a compact ANR, F has a finite number of components. Hence, from the long exact homotopy sequence of a fibration, $p_*(\pi_1(E))$ has finite index in B. Let $\tau\colon \tilde{B} \to B$ be the covering space corresponding to $p_*(\pi_1(E))$. Let $\tilde{p}\colon \tilde{E} \to \tilde{B}$ be the pull-back of p by τ. It is easily checked that $\tilde{E}$ and $\tilde{B}$ are compact manifolds, $\tilde{p}$ is a Hurewicz fibration the fibre of $\tilde{p}$ is homeomorphic to a component of F and each component of F is some fibre of $\tilde{p}$. Thus it suffices to consider the case when F is connected.

Let U_0 be an open n-cell in B and let $U \subseteq U_0$ be an open subset which is homeomorphic to $T^{n-1} \times \mathbf{R}$. Let $W = p^{-1}(U)$; note that W is an open connected CAT manifold with two ends. By [5], there exists a fibre homotopy equivalence $p^{-1}(U_0) \to F \times U_0$; consider the restriction α: $W \to F \times U$ of this fibre homotopy equivalence to W. It is easily checked that W is stable at each end and that the fundamental group of each end is isomorphic to $\pi_1(F \times T^{n-1})$. Let W_1 be a 1-neighborhood of an end of W. Identify U with $T^{n-1} \times \mathrm{R}$ such that there exists $a \in \mathrm{R}$ with $p^{-1}(T^{n-1} \times [a, \infty)) \subseteq W_1$. Analogous to the proof of the corresponding fact in the proof of Theorem 4, one can show that the compactum $C \subseteq W_1$ whose frontier (in W_1) is $p^{-1}(T^{n-1} \times \{a\})$ is a deformation retract of W_1. Thus C is a compact ANR and W_1 has the homotopy type of a finite complex [15]. By Theorem 3, W is homeomorphic to the interior of a compact CAT manifold S; let V be a component of ∂S. Again, it is easily checked that the inclusion $V \subseteq S$ is a homotopy equivalence. Therefore V is homotopy equivalent to $F \times T^{n-1}$ and the theorem follows from Theorem 4.

PROOF OF THEOREM 2. As in the proof of Theorem 1, it suffices to consider the case when the fibre is connected. Let K be a finite complex which has the same shape as F. We now attempt to follow the proof of Theorem 1. Choose U, U_0 and W as before; in general, there does not exists a fibre homotopy equivalence α: $W \to F \times U$. However, by [8], there exists a proper homotopy equivalence β: $W \to K \times U$. Hence, again it is easily checked that W is an open CAT manifold with two ends, W is stable at each end and the fundamental group of each end is isomorphic to $\pi_1(K \times T^{n-1}) \cong \check{\pi}_1(F \times T^{n-1})$. Let W_1 be a 1-neighborhood of an end of W and choose $a \in \mathrm{R}$ with $p^{-1}(T^{n-1} \times [a, +\infty)) \subseteq W_1$. Let C be the compactum in W_1 whose frontier is $p^{-1}(T^{n-1} \times \{a\})$; now C need not be locally connected and, hence, we cannot proceed as before to show that W satisfies the hypotheses of Theorem 3.

However, we claim that C is an FANR [1]. For, suppose that a was chosen such that

$$\partial W_1 \cap p^{-1}(T^{n-1} \times (a-1, a)) = \emptyset .$$

Let $B_i = p^{-1}(T^{n-1} \times (a-1, a+1/i))$ for each positive integer i. It follows from the long exact homotopy sequence of an approximate fibration [3] and the Whitehead theorem that the inclusion $B_{i+1} \subseteq B_i$ is a homotopy equivalence for each i. Hence $C \cup B_{i+1} \subseteq C \cup B_i$ is a homotopy equivalence for each i. Since $C = \bigcap_i (C \cup B_i)$, this implies that C has the pointed shape of a simplicial complex. Hence by [4], C is an FANR.

By [1; p. 254], C is pointed shape dominated by a finite complex. Since the inclusion of $C \cup B_i$ into W_1 is a homotopy equivalence (by exactly the same argument showing that $B_{i+1} \subseteq B_i$ is a homotopy equivalence), the inclusion of C into W_1 is a pointed shape equivalence. Hence W_1 is pointed shape dominated by a finite complex. Since W_1 is an ANR, this implies that W_1 is dominated by a finite complex [1; p. 102].

The author is unable to show that W_1 has the homotopy type of a finite complex. However, by the product formula for Wall's obstruction [7], $W_1 \times S^1$ ($S^1 = 1$-sphere) has the homotopy type of a finite complex. By Theorem 3, $W \times S^1$ is homeomorphic to the interior of a CAT compact manifold S; let V be a component of ∂S. Again, $V \subseteqq S$ is a homotopy equivalence and, hence, V is homotopy equivalent to $K \times U \times S^1$. Thus V is homotopy equivalent to $K \times T^n$; by Theorem 4, K is homotopy equivalent to a closed connected $(m-n)$-dimensional CAT manifold. Since K has the same shape as F, Theorem 2 follows.

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