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C*-ACTIONS

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I. Introduction.

In this paper we show that the basic structure theorem of Białynicki-Birula [1] for G_m actions on a complete non-singular variety goes over without change to holomorphic $\mathbf{C}^*$-actions with fixed points on a connected compact Kaehler manifold X. Roughly speaking [cf. section III for a full statement], X can be decomposed in either of two functorial ways into the union of a finite number of $\mathbf{C}^*$-invariant sets X_j which have the following properties: each X_j is Zariski open in its closure which is an analytic set; there exists a $\mathbf{C}^*$-equivariant maximal rank surjection of X_j onto a connected component F_j of the fixed point set of X with vector space fibres; F_j is a section of X_j; and the normal bundle of F_j in X_j is a specific subbundle of the normal bundle of F_j in X. One corollary is that X is projective if and only if the fixed point set of $\mathbf{C}^*$ in X is projective. Another is that X is rational if the fixed point set of $\mathbf{C}^*$ in X is. This fact is due, in the algebraic case, to D. Lieberman [8].

The proof breaks into two halves. The first studies certain local fibrations around the fixed point set of compact groups acting by means of biholomorphisms on a general compact complex manifold X. Somewhat amazingly, though a complex exponential map doesn't exist, it almost does and this lets one talk about what would be the image on X of certain subbundles of the normal bundle of the fixed point set. These results can be looked at as globalizations of some of the classical results of H. Cartan [cf. 2, Chap. I] on Reinhardt and other circular domains.

The second half of the proof is a straightforward application of the key result in [15] on the closure of orbits of a holomorphic $\mathbf{C}^*$ action with non-empty fixed point set on a compact connected Kaehler manifold.

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II. Local fibrations around the fixed points.

Let X be a compact complex manifold and V a C^∞ vector field on X. The one parameter group $\Phi: \mathbf{R} \times X \to X$ gotten by integrating V is a one parameter group of biholomorphisms of X if and only if $V-iJV$ is a holomorphic vector field, where J is the complex structure tensor of X. This remark allows one to show that certain bundles associated to the fixed point set of a compact Lie group G acting smoothly on X via biholomorphisms are holomorphic. It also allows one to pass from an S^1-action on X (via biholomorphisms) to a *holomorphic* $\mathbf{C}^*$-action on X; that is an action of $\mathbf{C}^*$ on X for which the natural map $\mathbf{C}^* \times X \to X$ is homomorphic.

Now let $\Phi: G \times X \to X$ be a C^∞ action of a connected compact Lie group on a not necessarily compact complex manifold by means of biholomorphisms. Then the fixed point set of Φ is a complex submanifold of X. To see that it is a manifold is not hard; e.g. [16, p. 213]. To see that it is a complex submanifold, use the fact that F is the set of common zeros of the holomorphic vector fields $V-iJV$ where V denotes the vector field on X generated by a one-parameter subgroup of G. Note that Φ gives rise to a fibrewise linear action $d\Phi$ of G on $T(X)|F$, where $T(X)$ denotes the holomorphic tangent bundle of X. In fact, since G is compact, hence reductive, $d\Phi$ extends to a holomorphic action of the complexification $G_{\mathbf{C}}$ of G on $T(X)|F$.

A continuous linear representation of G on a finite dimensional complex vector space V determines a unique direct sum decomposition $V = \oplus V_\lambda \oplus V_0$, where λ runs over all irreducible representations of G on V and G acts trivially on V_0. Here V_λ denotes the direct sum of all representations of type λ in V. This *isotypic* decomposition is an immediate consequence of the existence of a G-invariant Hermitian metric on V and Schur's Lemma which says that an equivariant linear map between two inequivalent irreducible representations is trivial. If $\Phi: G \times X \to X$ is a C^∞ action of G on X via biholomorphisms, then $T(X)|F$ decomposes correspondingly into a direct sum of holomorphic subbundles $\oplus E_\lambda \oplus T(F)$ and G acts trivially on $T(F)$. This decomposition can be seen directly from the existence of a holomorphic embedding of a G-invariant neighborhood of each $x \in F$ in a $\mathbf{C}^N$ which is equivariant for some representation of G in $\mathrm{GL}(N, \mathbf{C})$ [12].

LEMMA I. *Let $\Phi: G \times X \to X$ be a C^∞ action of a compact connected Lie group G on a complex manifold X by means of biholomorphisms. Let Y be a not necessarily reduced complex analytic subspace of X that is invariant under G. Let $A: Y \to \mathbf{C}^N$ be a holomorphic map equivariant with respect to some representation $\varrho: G \to \mathrm{GL}(N, \mathbf{C})$. Then if A extends to a holomorphic map $\tilde{A}: X \to \mathbf{C}^N$, one can find an equivariant extension, $\tilde{\tilde{A}}$. In particular if X is Stein, then an equivariant extension of A exists.*

PROOF. Let dg denote the normalized Haar measure on G. Let

$$\tilde{\tilde{A}}(p) = \int_G \varrho(g^{-1})\circ\tilde{A}\circ\Phi(g,p)\,dg \quad \text{where } g\in G \text{ and } p\in X\,.$$

Note that

$$\begin{aligned}\tilde{\tilde{A}}\circ\Phi(g_0,p) &= \int_G \varrho(g^{-1})\circ\tilde{A}\circ\Phi(gg_0,p)\,dg\\ &= \int_G \varrho(g_0g^{-1})\circ\tilde{A}\circ\Phi(g,p)\,dg = \varrho(g_0)\circ\tilde{\tilde{A}}(p)\,.\end{aligned}$$

Also it is clear that $\tilde{\tilde{A}}\,|\,Y=\tilde{A}\,|\,Y=A$. Thus $\tilde{\tilde{A}}$ is the desired extension.

LEMMA II. *Let Φ: $G\times X \to X$ and Ψ: $G\times Y\to Y$ be C^∞ actions of a compact connected Lie group G on connected complex manifolds X and Y by means of biholomorphisms. Let A: $X \to Y$ be an equivariant holomorphic map and let $A(F)\subseteq H$ where F is a component of the fixed point set of X and H is a submanifold of the fixed point set of Y. If for each $f\in F$, no irreducible subrepresentation of G in $N(H)_{A(f)}$ occurs among the subrepresentations of $N(F)_f^{(t)}$ for any $t>0$, where $N(F)^{(t)}$ is the t-th symmetric power of the normal bundle of F, then $A(X)\subseteq H$.*

PROOF. Using Lemma I, $T(X)\,|\,F$ contains $N(F)$ as a direct summand. dA_f thus equivariantly maps $N(F)_f$ to $N(H)_{A(f)}$. By the above hypothesis for $t=1$ and Schur's lemma, dA_f: $N(F)_f \to N(H)_{A(f)}$ is the 0 map. In this situation one gets an equivariant map from $N(F)_f^{(2)}$ to $N(H)_{A(f)}$ which by the above is also zero. Iterating this one has shown that if w is a holomorphic function on a neighborhood U of $A(f)$ such that $w(H)=0$ then $w\circ A$ has zero Taylor series on $A^{-1}(U)\cap V$ where V is a neighborhood of f. Thus $A(f)\subseteq H$.

It is convenient when Φ: $G\times X \to X$ is a C^∞ action of a compact Lie group G on X by means of biholomorphisms to introduce some notation. Let F be a connected component of the fixed point set of G on X. We have

$$T(X)\,|\,F = T(F)\oplus E_{\lambda_1}\oplus\ldots\oplus E_{\lambda_n}$$

where λ_i are the nontrivial irreducible representations that occur in the fibre of $T(X)_f$ over any point. For $I\subseteq\{1,\ldots,n\}$, let

$$\lambda_I = \bigoplus_{j\in I}\lambda_j \quad \text{and} \quad E_I = \bigoplus_{j\in I} E_{\lambda_j}\,.$$

PROPOSITION I. *Let Φ: $G\times X \to X$ be a C^∞ action of a compact connected Lie* *group G on a complex manifold X by means of biholomorphisms. Let F be a connected component of the fixed point set of G in X and let*

$$T(X)|F = E_{\lambda_1}\oplus\ldots\oplus E_{\lambda_n}\oplus T(F).$$

If $I\subseteqq\{1,\ldots,n\}$ is such that no subrepresentation of $\lambda_{I'}$ (I' is the complement of I in $\{1,\ldots,n\}$) occurs in any symmetric tensor power of λ_I, then there exists a connected complex manifold $\mathscr{F}_I$ with a C^∞ action of G by biholomorphisms and a unique equivariant embedding Φ_I: $\mathscr{F}_I \to X$ such that:

A) *there is a maximal rank equivariant holomorphic surjection p_I: $\mathscr{F}_I \to F$ with a section s: $F \to \mathscr{F}_I$.*

B) *$\varphi_I s$ is the identity on F and $d\varphi_I\,|\,s(F)$ gives an equivariant isomorphism of $T(\mathscr{F}_I)\,|\,s(F)$* and *$E_I\oplus T(F)$.*

Proof. The main idea is to produce an equivariant embedding of a neighborhood of an arbitrary $x\in F$ into $\mathbf{C}^m$ where $m=\dim X$. As G is compact, each point $x\in F$ has arbitrarily small G-invariant neighborhoods U_x. By a result of R. Richardson [12], there exists, for some U_x, an holomorphic embedding φ_x: $U_x \to \mathbf{C}^N$, equivariant with respect to some representation $G\to \mathrm{GL}(N,\mathbf{C})$, such that $d\varphi_y$ is injective for all $y\in F\cap U_x$. By Lemma I, there exists an equivariant linear map π: $\mathbf{C}^N\to T(X)_x\cong\mathbf{C}^m$ for which $\pi d\varphi_x$ is the identity. By the inverse function theorem, $\pi\varphi$ is an embedding of a (possibly smaller) invariant neighborhood onto an open set in $\mathbf{C}^m$ which has an equivariant local inverse g. Therefore for some G-invariant neighborhood W_x of the zero section of $N(F)\,|\,U_x\cap F$, one can define a holomorphic map Ψ_x: $W_x\to X$ by first denoting the points of W_x by (y,v) where $y\in F$ and $v\in N(F)_y$ and setting

$$\Psi_x(y,v) = g\big(\pi\varphi(y)+\pi d\varphi_y(v)\big).$$

Since any translation by a fixed point of a map equivariant with respect to a linear action is again equivariant, Ψ_x is equivariant. Furthermore, $\Psi_x\,|\,F\cap U_x$ is the inclusion into X where we have identified F as usual with the zero section of $N(F)$. Finally, note that the maps Ψ_x define a system of local coordinates near F, since if $W_x\cap W_y\neq\emptyset$, then $\Psi_y^{-1}\Psi_x$ is biholomorphic.

Given E_I as above, let $\mathscr{F}_I$ denote the union of the images of the maps $\Psi_x\,|\,E_I\cap W_x$ as x ranges over F. By Lemma II and the assumption on I, these images fit together in the sense that if $W_x\cap W_y\neq\emptyset$, then $\Psi_x(W_x\cap W_y\cap E_I)$ $=\Psi_y(W_x\cap W_y\cap E_I)$ is a well defined submanifold of X satisfying A and B when φ_I is the inclusion and p_I the map defined by composing a Ψ^{-1} with the bundle projection of $N(F)$ on F.

A question that immediately poses itself is when is $\mathscr{F}_I$ a neighborhod Y of the zero section F of a vector bundle (resp. when does an equivariant exponential map on $Y \subset N(F)$ exist)? The answer is obvious; each $\Psi_y \Psi_x^{-1}$ must be linear (resp. the identity). An effective criterion (cf. [1, p. 490]) for this to happen in terms of the representation on a fibre of E_I now follows.

LEMMA III. *Let $G \to \mathrm{GL}\,(V_{\mathbf{C}})$ be a representation of a compact Lie group on a complex vector space $V_{\mathbf{C}}$. Let $f\colon U \to V_{\mathbf{C}}$ be a G equivariant holomorphic function that takes the origin to the origin, where U is an invariant open set around the origin. f is the restriction of a linear mapping $V_{\mathbf{C}} \to V_{\mathbf{C}}$ if no subrepresentation of G in $V_{\mathbf{C}}$ occurs in the representation of G in any symmetric tensor power $S^k V_{\mathbf{C}}$ of $V_{\mathbf{C}}$ for $k>1$.*

PROOF. This is really just Lemma II. Let df_0 denote the linear mapping from $V_{\mathbf{C}}$ to $V_{\mathbf{C}}$ gotten by identifying $T(V_{\mathbf{C}})_0$ with $V_{\mathbf{C}}$. Now $f - df_0 = \varphi$ is still equivariant. $d\varphi_0$ is so one gets a map from $S^2 V_{\mathbf{C}}$ to $V_{\mathbf{C}}$; by Schur's lemma and the hypothesis this map is 0. Repeating one gets the Maclaurin–Taylor expansion of φ is 0.

REMARK. One should note if no subrepresentation of $V_{\mathbf{C}}$ occurred in any symmetric power $S^k V_{\mathbf{C}}$ for $k>n$ then one could conclude φ was a polynomial mapping of degree at most n.

Let us give a few examples that illustrate the above.

EXAMPLE I. Let S^1 act on $\mathbf{C}^2$ by $(a,b) \to (za, z^t b)$. The map $(a,b) \to (a, b+a^t)$ is not linear though equivariant. Note that the representation on the second factor is contained in the tth symmetric power of $\mathbf{C}^2$. The representation space V_1 of eigenvectors $(a,0)$ of weight one admits two equivariant imbeddings satisfying A and B: namely, the identity and $(a,0) \to (a,a^2)$. Since their images are clearly distinct, the hypothesis on I in Proposition I cannot be removed.

EXAMPLE II. Let SU (2), the compact real form of SL $(2,\mathbf{C})$, act on $\mathbf{C}^2$ via the usual representation of SL$(2,\mathbf{C})$. Since any symmetric power of this representation is irreducible the lemma applies; but consider the direct sum of this representation on itself. One has SU (2) acting on the 2×2 matrices and the determinant is an invariant quadratic function on these matrices! This is because in taking symmetric tensor powers of $\mathbf{C}^2 \oplus \mathbf{C}^2$ one gets actual tensor powers of $\mathbf{C}^2$ with itself from the cross terms. One can compute when the lemma applies by using formulas such as the Clebsch–Gordan series [11, p. 33].

Now let us specialize to S^1 actions and give a more complete description of the $\mathscr{F}_I$ of Proposition I. If S^1 acts complex linearly on a vector space $V_{\mathbf{C}}$ it splits $V_{\mathbf{C}}$ into a direct sum $\oplus_{n \in \mathbf{Z}} V_n$, where $n \in \mathbf{Z}$ is an integer and $v \in V_n$ goes to $z^n V$ under the action of $z \in S^1$; the isomorphism type of an irreducible representation is given by an integer. The $n \in \mathbf{Z}$ for which V_n is nontrivial are called weights.

Now note $S^k V_n$, the kth symmetric power of V_n, is a direct sum of representations of type kn. Thus:

COROLLARY I. *Let $\Phi: S^1 \times X \to X$ be a C^∞ action of S^1 on a complex manifold X by means of biholomorphisms. Let F be a fixed point component and let*

$$T(X)|F = T(F) \oplus \left(\bigoplus_{j \in I} E_j \right) \oplus \left(\bigoplus_{j \in J} E_j \right)$$

where $I = \{i_1, \ldots, i_r\}$ with $i_1 < i_2 < \ldots < i_r < 0$ and $J = \{j_1, \ldots, j_s\}$ with $0 < j_1 < \ldots < j_s$. Then:

(A) *$\mathscr{F}_I$ and $\mathscr{F}_J$ are both defined. In fact for $k > 0$, $\mathscr{F}_{k^+}$ is defined where $k^+ = \{i \in I : i > k > 0\}$. Similarly, for $k < 0$, $\mathscr{F}_{k^-}$ is defined where $k^- = \{i \in J : i < k < 0\}$.*

(B) *Similarly, $\mathscr{F}_{\mathscr{P}^+}$ is defined where $\mathscr{P}^+ = \{r \in I : r$ is divisible by the prime $p\}$ and likewise $\mathscr{F}_{\mathscr{P}^-}$.*

(C) *More generally, $\mathscr{F}_{\mathscr{M}}$ is defined if $\mathscr{M} = (I \cup J) \cap S$, where S is a sub-semigroup of $\mathbf{Z}$ not containing* 0.

(D) *$\mathscr{F}_{\mathscr{M}}$ is a tubular neighborhood of the zero section of a vector bundle if $\mathscr{M} = \{i_n\}$ or $\{j_s\}$ or more generally $(I \cap J) \cup S$, where S is a sub-semigroup of $\mathbf{Z}$ such that $(S+S) \cap I \cup J$ is null.*

(E) *If Φ is the restriction of a holomorphic $\mathbf{C}^*$-action on X, then each $\mathscr{F}_{k^+}$* (respectively $\mathscr{F}_{k^-}$) *of* (i) *may be assumed to have vector space fibres with respect to the morphism p_{k^+}* (respectively p_{k^-}) *of Proposition* I, *part* A.

PROOF. To prove parts $A-D$, simply use the fact that $S^k V_n$, the kth symmetric power of V_n, is of type kn with respect to the induced S^1-action and apply Lemma III. If X is compact, then the identity component of the group of all biholomorphisms of X is a complex Lie group acting holomorphically on X so Φ has an extension to $\mathbf{C}^* \times X$. This $\mathbf{C}^*$ takes any tubular neighborhood of the zero section of E_I (respectively E_J) onto all of E_I (respectively E_J). Now for any $x \in F$, the map Ψ_x defined in the proof of Proposition I extends to E_{Ix} (respectively E_{Jx}) and every nearby fibre by the duplication formula $\Psi_x(y, v) = \lambda^{-1} \Psi_x(y, \lambda v)$, where (y, v) denotes a point of E_{Iy} (respectively E_{Jy}) and $\lambda \in \mathbf{C}^*$ sends (y, v) into the domain of Ψ_x. The independence of λ in this extension follows from the S^1-equivariance of ψ_x and the identity principal. This completes the proof of E.

We note that one has a filtration $\mathscr{F}_{i_r} \subset \mathscr{F}_{\{i_r, i_{r-1}\}} \subset \ldots$. It is interesting that though they are not in general tubular neighborhoods of the zero sections of vector bundles, quotients are well defined and the graded object gotten is simply a tubular neighborhood of F in $\oplus_{j \in I} E_j$ (and similarly for J). This follows immediately from:

LEMMA IV. *Let S^1 act complex linearly on a complex vector space $V_{\mathbf{C}}$. Let $V_{\mathbf{C}} = V_I \oplus V_J$ where*

$$I = \{\lambda_j \in \mathbf{Z} \mid 0 < \lambda_1 < \ldots < \lambda_r \leqq a\}$$

and

$$J = \{\mu_j \in \mathbf{Z} \mid a < \mu_1 < \ldots < \mu_s\}$$

for some a, where V_I and V_J are the subrepresentation spaces containing all subrepresentations of type $j \in I$ and $j \in J$ respectively. Let f: $U \to V_{\mathbf{C}}$ be an equivariant holomorphic map where U is a neighborhood of the origin in $V_{\mathbf{C}}$. Let p_I: $V_{\mathbf{C}} \to V_I$ denote the usual projection. Then f extends to $V_{\mathbf{C}}$ and there is a unique equivariant map f_I: $V_I \to V_I$ such that $f_I \circ p_I = p_I \circ f$.

PROOF. f extends to $V_{\mathbf{C}}$ by the remark. If such an f_I exists it is clearly unique since f_I: $V_I \to V_I$ must be gotten by composing the inclusion $V_I \to (V_I, 0)$ with $p_I \circ f$. If we write f as $p_I \circ f \oplus p_J \circ f$ what we must show is that $p_I \circ f$ does not depend on the V_J variables. Let V_I have coordinates $\{z_{11}, z_{12}, \ldots, z_{21}, z_{22}, \ldots\}$ and V_J have coordinates $\{w_{11}, w_{12}, \ldots, w_{21}, w_{22}, \ldots\}$ where $\eta \in S^1$ takes $\{z_1, z_{12}, \ldots, z_{21}, z_{22}, \ldots, w_{11}, w_{12}, \ldots, w_{21}, w_{22}, \ldots\}$ to $\{\eta^{\lambda_1} z_{11}, \eta^{\lambda_1} z_{12}, \ldots, \eta^{\lambda_2} z_{22}, \ldots, \eta^{\mu_1} w_{11}, \ldots, \eta^{\mu_2} w_{21}, \ldots\}$. Now by projecting V_I onto a one dimensional irreducible vector subspace one gets a holomorphic function $\tilde{f}_I$: $V \to \mathbf{C}$ such that $\tilde{f}_I(\Phi(\eta, (z, w))) = \eta^\delta \tilde{f}_I(z, w)$ where $\delta \leqq a$. One concludes from power series expansions that $\tilde{f}_I$ is a sum of monomials $a_{ijst} z_{ij}^{r_{ij}} w_{st}^{q_{st}}$ with $r_{ij}\lambda_i + q_{st}\mu_s = \delta$. But since $\lambda_i > 0$ and $\mu_s > a$ and $\delta \leqq a$ we see that $q_{st} = 0$.

III. The invariant decomposition.

In this section we will be solely concerned with holomorphic $\mathbf{C}^*$- (and $(\mathbf{C}^*)^n$-) actions on a compact Kaehler manifold X that have at least one fixed point on each component of X. By the main result of [15], this happens if the Lie algebra of vector fields $(C^*)^n$ induces on X is annihilated by every holomorphic one form. The following is due to Białynicki-Birula in the algebraic case [1].

PROPOSITION II. *Let $\mathbf{C}^*$ act holomorphically with at least one fixed point on a connected compact Kaehler manifold X. Let $\{F_j \mid 1 \leqq j \leqq r\}$ be the connected* *components of the fixed point set of* $\mathbf{C}^*$ *on* X. *There exist two functorial decompositions (the plus and the minus decompositions) into* $\mathbf{C}^*$*-invariant sets* $\{V_j^+ \mid 1 \leqq j \leqq r\}$ *(respectively* $\{V_j^- \mid 1 \leqq j \leqq r\}$*) such that*

A) V_j^+ *(respectively* V_j^-*) is Zariski open in its closure which is an analytic subset of* X.

B) *There exists a* $\mathbf{C}^*$*-invariant maximal rank holomorphic surjection* $\pi_j^+ : V_j^+ \to F_j$ *(respectively* $\pi_j^- \to F_j$*) with vector space fibres.*

C) $T(V_j^+) \mid F_j = N(F_j)^+ \oplus T(F_j)$ *(respectively* $T(V_j^-) \mid F_j = N(F_j)^- \oplus T(F_j)$*), where* $T(X) \mid F_j = T(F_j) \oplus N(F_j)^+ \oplus N(F_j)^-$ *is the usual decomposition in terms of the fixed part and the positive and negative weights of the associated* S^1*-action.*

D) *There is precisely one component* F_1 *(respectively* F_r*) called the source (respectively the sink) such that* $T(X) \mid F_1 = T(F_1) \oplus N(F_j)^+$ *(respectively* $T(X) \mid F_r = T(F_r) \oplus N(F_r)^-$.

E) X *fibres meromorphically over the source (respectively the sink) with generic fibre bimeromorphic to* $\mathbf{C}P^N$.

PROOF. We will prove the above for the plus decomposition; the minus decomposition is precisely the same. The V_j^+ are defined in accordance with Proposition I as the $\mathbf{C}^*$-invariant submanifolds which are associated to the $N(F_j)^+$. We first show that $X = \bigcup V_j^+$. By the basic Lemma II—A of [15], each point x of X has a well defined limit x' under $\lim_{\lambda \to 0} \lambda \cdot x$, where $\lambda \in \mathbf{C}^*$ and $\lambda \cdot x$ denotes the image of x under λ. It is clear that $x' \in F$. By this and the invariance of the V_j^+, it is sufficient to show that each point $x' \in F_j$ has a neighborhood U such that if $x \in U$ and $\lim_{\lambda \to 0} \lambda \cdot x$ is contained in F_j, then $x \in V_j^+$. Let U be the domain of an equivariant biholomorphism φ onto an open set in $\mathbf{C}^n$. By replacing φ by $\varphi - \varphi(x')$ one can assume $\varphi(x') = 0$. Choose local coordinates $(w_1, \ldots, w_n)$ on $\varphi(U)$ so that if $\lambda \in \mathbf{C}^*$, then

$$\lambda \cdot (w_1, \ldots, w_n) = (\lambda^{k_1} w_1, \ldots, \lambda^{k_s} w_s, w_{s+1}, \ldots, w_t, \lambda^{j_{t+1}} w_{t+1}, \ldots, \lambda^{j_n} w_n),$$

where each $k_i > 0$ and each $j_i < 0$. Here $s = \operatorname{rank} N(F_j)^+$ and $t = \dim V_j^+$.

Now $\{w \in \varphi(U) : \lim_{\lambda \to 0} \lambda \cdot w \in \varphi(U \cap F_j)\}$ is the intersection of $\varphi(U)$ and the hyperplane $H = \{w_{t+1} = \ldots = w_n = 0\}$, and by definition $\varphi(V_j^+ \cap U) \subset H \cap \varphi(U)$. It is clear that $V_j^+ \cap U$ is closed in U so in fact $\varphi(V_j^+ \cap U) = H \cap \varphi(U)$. Thus $x \in V_j^+$, and hence $X = \bigcup V_j^+$.

Each V_j^+ is Zariski open in compact analytic space $\overline{V_j^+}$ such that the $\mathbf{C}^*$-action on V_j^+ extends to $\overline{V_j^+}$ and π_j^+ extends to a holomorphic equivariant map $\pi_j^+ : \overline{V_j^+} \to F_j$. Now by a simple application of the main proposition of [13] (or by means of Lemma I-A, and Lemma II-A of [15] and Siu's extension theorem [14]), one sees that the inclusion of V_j^+ in X extends meromorphically to $\overline{V_j^+}$. This shows A. It also shows that one and only one V_j^+ say V_1^+ can be Zariski open; this proves E.

There is an exactly analogous decomposition when $(\mathbf{C}^*)^n$ acts with at least one fixed point on a compact connected Kaehler manifold X except that $T(V_j^+)\,|\,F_j = N(F_j)^+ \oplus T(F_j)$, where

$$T(X)\,|\,F_j = T(F_j) \oplus N(F_j)^+ \oplus N(F_j)^- \oplus \Gamma\ ,$$

where $N(F_j)^+$ is the largest summand with no trivial direct summands on which $(\mathbf{C}^*)^n$ acts with semi-positive weights (and similarly for $T(V_j^-)\,|\,F_j$). This is proven by induction. One does the decomposition for $(C^*)^{n-1} \times \{1\}$ and notes that since $(C^*)^n$ is commutative, $\{1\} \times \mathbf{C}^*$ leaves the fixed point set and the decomposition invariant. One now simply applies the above proposition to the action of $\{1\} \times \mathbf{C}^*$ on the fixed point set of $(\mathbf{C}^*)^{n-1} \times \{1\}$ and notes everything is compatible. The following corollary is well known.

Corollary II. *Let $(\mathbf{C}^*)^n$ act holomorphically on a not necessarily connected compact Kaehler manifold X. The Euler characteristic of X is equal to the Euler characteristic of the fixed point set of X.*

Proof. One can assume X is connected. If $(\mathbf{C}^*)^n$ has no fixed point set on X then the Euler characteristic $\mathscr{E}(X)$ of X is 0. Thus one can assume $(\mathbf{C}^*)^n$ has a fixed point set with connected components $\{F_j \mid 1 \leqq j \leqq r\}$. By Proposition II $X = \bigcup_j V_j^+$. Thus $\mathscr{E}(X) = \sum_j \mathscr{E}(V_j^+)$. Now $\pi_j^+ \colon V_j^+ \to F_j$ is a maximal rank surjection with vector spaces or fibres, and thus $\mathscr{E}(V_j^+) = \mathscr{E}(F_j)$.

Corollary III. *If $(\mathbf{C}^*)^n$ acts on a compact Kaehler manifold X, then X is projective if and only if the source (sink) is projective. In particular, X is projective if and only if the fixed point set is projective.*

Proof. Only one assertion is non-trivial. Assume the source is projective and that X is connected. Since X is bimeromorphic to $\overline{V_1^+}$ and since a Moisezon Kaehler space is projective by Moisezon's theorem [9, p. 280, Cor. II], it suffices to show that $\overline{V_1^+}$ is Moisezon. One can do this directly but it is easier to blow up F_1 in $\overline{V_1^+}$ to get V_1^+ with a meromorphic surjection onto the blown up F_1 and generic fibre $\mathbf{C}P^1$. The blown up F_1 is projective and we can use the theorem of Kodaira–Kawai–Hironaka [10, Cor. 5.2].

Remark. If X is projective it can be shown [8] that X is birational to $F_1 \times \mathbf{C}P^a$, and similarly for F_r. This yields the pleasant consequence that a Kaehler X is birational to $\mathbf{C}P^N$ if F_1 (or F_r) is. If X is not projective, X is not necessarily bimeromorphic to $\mathbf{C}P^a \times F_1$. To see this choose a Kaehler torus T with no analytic subspaces and $L \to T$ a non-trivial line bundle. Add a copy T_∞ of T to infinity on L to get the compact Kaehler manifold $\bar{L}$ that $\mathbf{C}^*$ acts on with source the zero-section and T_∞ as the sink. If $\bar{L}=CP^1 \times T$ bimeromorphically, then L would have a meromorphic section. But then, since L is non-trivial, T would have a non-trivial analytic subset, i.e., the zero and pole loci, contradicting our choice of T.

The next example shows the situation can, as expected, be quite different for compact complex manifolds.

EXAMPLE II. Let H be the Hopf manifold $C^2-\{0\}/Z$ where the Z action is generated by $(z,w) \to (2z,2w)$. The C^*-action $\lambda\cdot(z,w)=(\lambda z,w)$ on $C^2-\{0\}$ gives rise to a C^*-action on H and CP^1 with respect to which the universal surjection $p\colon H \to CP^1$ is equivariant. On CP^1 the fixed point set is $[0,1]$ and $[1,0]$. On H it is only the torus $T=p^{-1}([0,1])$. Now $p^{-1}([1,0])$ is an invariant set, and $H-p^{-1}([1,0])\approx T\times C$ equivariantly with C^* acting on the second factor. The closure of any orbit in $H-p^{-1}([1,0])$ contains $p^{-1}([0,1])$. Thus there is a source but no sink. The sink is replaced by an invariant set.

EXAMPLE III. X can have isolated fixed points without being rational. For example, let X' be any compact Kaehler manifold, and let C^* act on $CP^1\times X'$ by acting only on the first factor. On the first factor, let the action be $\lambda[w,z]=[\lambda w,z]$. Blow up any point $([1,0],x)$: the action lifts, and the fixed point set consists of one copy of X', one copy of X' with $x\in X'$ blown up and one isolated point in the CP^n one replaced $([1,0],x)$ by.

Let Ω_X denote any sheaf on X of the form $(\Omega_X^{p_1})^{\otimes a_1}\oplus\ldots\oplus(\Omega_X^{p_k})^{\otimes a_k}$ with some p_i and a_i both positive, where Ω_X^p denotes the sheaf of germs of holomorphic p-forms on X. Since X is bimeromorphic to $\overline{V_1^+}$ and since holomorphic sections of Ω_X are bimeromorphic invariants, it makes sense to talk about sections of Ω_X that are pullbacks of sections of Ω_{F_1} via $\pi_1^+ : \overline{V_1^+} \to F_1$ (and similarly for F_r).

COROLLARY IV. *Let $(C^*)^d$ act holomorphically with at least one fixed point on a compact Kaehler manifold X. Let F_1 and F_r denote the source and the sink. Then for any sheaf Ω_X defined above,*

$$H^0(X,\Omega_X)\approx H^0(F_1,\Omega_{F_1})\approx H^0(F_r,\Omega_{F_r}).$$

In particular, $H^0(X,\Omega_X^q)=0$ if $q>\inf\{\dim_C F_1,\dim_C F_r\}$.

PROOF. It suffices to do the part concerning F_1. Since $\pi_1^+\colon \overline{V_1^+}\to F_1$ is locally a product projection, the tangent sheaf of X restricted to a fibre is a direct sum of the normal sheaf to the fibre and the tangent sheaf to the fibre.

Now if $\eta \in H^0(\overline{V_1^+}, \Omega_{\overline{V_1^+}})$ was not the pullback of a section of Ω_{F_1}, then, by using the last line and the fact that the normal sheaf to a fibre is trivial, one could find some section χ of an Ω_{CP^n}, which is absurd.

The above sharpens and generalizes [3], [5], and [7, p. 161] for C*-actions.

REFERENCES

1. A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. 98 (1973), 480–497.
2. S. Bochner and W. T. Martin, *Several Complex Variables*, Princeton University Press, The Maple Press Co., Pennsylvania, 1948.
3. J. B. Carrell and D. Lieberman, *Holomorphic vector fields and Kaehler manifolds*, Invent. Math. 23 (1973), 303–309.
4. T. Frankel, *Fixed points and torsion on Kaehler manifolds*, Ann. of Math. (2) 70 (1959), 1–8.
5. A. Howard, *Holomorphic vector fields on algebraic manifolds*, Amer. J. Math. 94 (1972), 1282–1290.
6. B. Iversen, *Cohomology and torus actions*, preprint.
7. S. Koboyashi, *Transformation groups in differential geometry* (Ergebnisse Math. 70), Springer-Verlag, Berlin - New York, 1972.
8. D. Lieberman, *Vector fields on projective manifolds.*, Proceedings of Symposia in Pure Mathematics, Vol. XXX (1976).
9. B. G. Moisezon, *On n-dimensional compact varieties within algebraically independent meromorphic functions* I, II, III, Amer. Math. Soc. Transl. 63 (2) (1967), 51–177. (Izvestija Akad. Nauk. SSSR, Ser. Mat. 30 (1966), 133–174, 345–386, 621–656).
10. J. Morrow and H. Rossi, *Some theorems of algebraicity for complex spaces*, J. Math. Soc. Japan 27 (1975), 167–183.
11. O. Riemenschneider, *Characterizing Moisezon spaces by almost positive coherent analytic sheaves*, Math. Z. (1971), 263–284.
12. R. Richardson, *Principal orbit types for reductive groups acting on Stein manifolds*, Math. Ann. 208 (1974), 323–331.
13. H. Samelson, *Notes on Lie algebras*, Van Nostrand Reinhold Co., New York 1969.
14. Y.-T. Siu, *Extension of meromorphic maps into Kähler manifolds*, Ann. of Math. 102 (1975), 421–462.
15. A. J. Sommese, *Extension theorems for reductive group actions on compact Kaehler manifolds*, Math. Ann. 218 (1975), 107–116.
16. S. Sternberg, *Lectures on differential geometry*, Prentice–Hall, 1964.
17. P. Wagreich, *Algebraic varieties with group action* in *Algebraic Geometry*, Proceedings of Symposia in Pure Mathematics, Vol. 29 pp. 633–642, Amer. Math. Soc., Providence R.I., 1975.

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