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AN EXAMPLE CONCERNING THE TOPOLOGICAL CHARACTER OF THE ZERO-SET OF A HARMONIC FUNCTION

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Let $f(x_1,x_2)=(f_1(x_1,x_2), f_2(x_1,x_2))$ be a harmonic mapping (i.e. f_1 and f_2 are harmonic functions) of a neighbourhood of the origin in $\mathbf{R}^2$ into $\mathbf{R}^2$ and suppose that $f(0,0)=(0,0)$. The following result is due to H. Lewy ([2], Theorem 1):

The mapping f is one-to-one in some neighbourhood of the origin if and only if the Jacobian $\partial(f_1,f_2)/\partial(x_1,x_2)$ does not vanish at the origin.

A simple proof of this theorem has been given by L. Bers ([1, Lemma 3.1], see also [4, § 136]):

Let $x=(x_1,x_2)$ and let $\|x\|$ denote the Cartesian norm of x. By the Taylor formula,

$$f_i(x_1,x_2) = p_i(x_1,x_2)+r_i(x_1,x_2) \qquad (i=1,2)\,,$$

where p_i is a homogeneous polynomial of degree k_i and $r_i=o(\|x\|^{k_i})$ as $x \to 0$. Assume that the Jacobian of f vanishes at the origin. Then either $k_i>1$ for some index i or $k_1=k_2=1$ and $p_1=\alpha p_2$ for some real number α. In the first case the zero-set of p_i consists of k_i lines crossing the origin (since $p_i(x_1,x_2) = \operatorname{Re}(c_i z^{k_i})$ for some complex number c_i and $z=x_1+ix_2$). It follows that there is no neighbourhood V of the origin such that the zero-set of $f_i \mid V$ can be topologically embedded into the real line. Hence f is not one-to-one in any neighbourhood of the origin. In the second case, set $g_1=f_1-\alpha f_2$, $g_2=f_1+\alpha f_2$. Since $\partial g_1/\partial x_1=\partial g_1/\partial x_2=0$ at the origin, (g_1,g_2) satisfies the assumptions of the previous case and hence is not one-to-one. So (f_1,f_2) cannot be one-to-one either.

The problem of generalizing this theorem to higher dimensions seems to be open. A partial solution, for harmonic gradient mappings of $\mathbf{R}^3$, has been obtained by H. Lewy in [3]. In connection with Bers' proof it is natural to ask the following question (which has been posed to me by professor H. S. Shapiro):

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Let $f(x_1,\ldots,x_n)$ $(n>2)$ be a real harmonic function, defined in a neighbourhood of the origin in $\mathbf{R}^n$ and suppose that f and grad f vanish at the origin. Is it true that there is no neighbourhood of the origin such that the zero-set of f in this neighbourhood can be embedded into $\mathbf{R}^{n-1}$?

An affirmative answer would immediately lead to a generalization of H. Lewy's theorem to higher dimensions. However, we demonstrate by an example that this question actually has a negative answer.

EXAMPLE. Let $f(x,y,z)=x^3-3xy^2+z^3-\frac{3}{2}(x^2+y^2)z$. This is a homogeneous harmonic polynomial and $f(0,0,0)=0$, grad $f(0,0,0)=(0,0,0)$. We shall show that the zero-set of f is homeomorphic to $\mathbf{R}^2$.

Introduce polar coordinates:

$$x = r\cos\theta\cos\varphi \qquad y = r\sin\theta\cos\varphi \qquad z = r\sin\varphi\,.$$

The set Z of zeros of f on the unit sphere is given by the equations:

$$\cos^3\theta\cos^3\varphi-3\sin^2\theta\cos\theta\cos^3\varphi+\sin^3\varphi-\tfrac{3}{2}\sin\varphi\cos^2\varphi = 0\,,$$

$$r = 1\,,$$

which after a simple modification become

$$\cos 3\theta = \tfrac{3}{2}\tan\varphi-\tan^3\varphi\,,$$

$$r = 1\,.$$

Note that the function $g(t)=\frac{3}{2}t-t^3$ has a maximum less than 1 for $t\geqq 0$ and that $\lim g(t)=-\infty$ as $t\to\infty$. It follows that there is a number $\alpha\in(0,\pi/2)$ such that $\frac{3}{2}\tan\alpha-\tan^3\alpha=-1$ and $\frac{3}{2}\tan\varphi-\tan^3\varphi\in[-1,1]$ for any $\varphi\in[-\alpha,\alpha]$. Therefore, for $3\theta\in[0,\pi]+k\pi$ (k is a fixed integer), θ is a continuous function of φ in $[-\alpha,\alpha]$ and the set Z is the union of six arcs l_k, given by the equations

$$\theta = \frac{1}{3}(-1)^{k+1}\arccos\left(\frac{3}{2}\tan\varphi-\tan^3\varphi\right)+\frac{2}{3}\pi\left[\frac{k}{2}\right],$$

where $k=1,\ldots,6$, $\varphi\in[-\alpha,\alpha]$, $\arccos t\in[0,\pi]$ and $[k/2]$ denotes the integer part of $k/2$. It is easy to verify that the arcs l_k and l_{k+1} (for $k=1,\ldots,5$) have precisely one endpoint in common, so do the arcs l_1 and l_6, and all other pairs of arcs are disjoint. So we see that the set Z is homeomorphic to the circle. The zero-set of f consists of all rays emanating from the origin and passing through the points of Z. Since no ray passes through more than one point of Z, the zero-set of f is homeomorphic to $\mathbf{R}^2$.

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