MATH. SCAND. 43 (1978), 74–80

ENHANCING THE CONVERGENCE REGION OF A SEQUENCE OF BILINEAR TRANSFORMATIONS

JOHN GILL

Consider a sequence of bilinear transformations, $\{t_n(z)\}$, where

$$t_n(z) = (a_n z + b_n)/(c_n z + d_n), \quad a_n d_n - b_n c_n \neq 0, \quad n = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} t_n(z) = t(z) = (az+b)/(cz+d), \quad ad - bc \neq 0,$$

with $a_n \to a$, $b_n \to b$, $c_n \to c$, and $d_n \to d$ as $n \to \infty$.

Define

$$(1) \qquad T_1(z) = t_1(z), \quad T_n(z) = T_{n-1}(t_n(z)), \quad n = 2, 3, \ldots$$

Each non-parabolic $t_n(z)$ with finite fixed points can be written

$$\frac{t_n(z)-\alpha_n}{t_n(z)-\beta_n} = K_n \cdot \frac{z-\alpha_n}{z-\beta_n}, \quad \text{where } K_n = \frac{a_n - c_n\alpha_n}{a_n - c_n\beta_n}.$$

If $|K_n| \leqq 1$, then α_n is the *attractive* fixed point and β_n the *repulsive* fixed point of $t_n(z)$. Let $K = \lim_{n\to\infty} K_n$, $\alpha = \lim_{n\to\alpha} \alpha_n$, and $\beta = \lim_{n\to\infty} \beta_n$, where α and β are finite and distinct.

The following theorem is a modification of a theorem due to Mandell and Magnus, [1], and may be found in [2].

THEOREM 1. *If* $|K_n| < 1$, $n = 1, 2, \ldots$, *and* $|K| < 1$, *then* $\lim_{n\to\infty} T_n(\alpha)$ *exists and* $\underline{\lim}\, |\mu_n - \beta_n| > 0$ *implies* $\lim_{n\to\infty} T_n(\mu_n)$ *exists and* $\lim_{n\to\infty} T_n(\mu_n) = \lim_{n\to\infty} T_n(\alpha)$.

Examples given in [2] illustrate the efficacy of modifying sequences of bilinear transformations, $\{T_n(z)\}$, in accordance with theorem 1, in order to accelerate convergence. The following simple example demonstrates another aspect of this modification.

Received April 12, 1977.

EXAMPLE 1. Let $F(x)=1/1-x$. Then $F(x)=1+x+x^2+\ldots$ in $|x|<1$. The T-fraction expansion of this power series is

$$1+\frac{x}{1-2x}+\frac{x}{1-x}+\frac{x}{1-x}+\ldots, \tag{2}$$

converging to $F(x)$ in $|x|<1$, and converging to $\frac{1}{2}$ in $|x|>1$. The nth approximant of (2) can be written $1+T_n(0)$, where $t_1(z)=x/(1-2x+z)$ and $t_n(z)=x/(1-x+z)$ for $n=2,3,\ldots$. Observe that $\alpha=\alpha(x)=x$ and $\beta=-1$ for $|x|<1$, whereas $\alpha=-1$ and $\beta=\beta(x)=x$ for $|x|>1$.

Modify (2) by considering $1+T_n(x)$ instead of $1+T_n(0)$. Now,

$$\lim_{n\to\infty}(1+T_n(x)) = F(x) \quad \text{in } |x|<1\,,$$

agreeing with theorem 1. However, it is also true that

$$\lim_{n\to\infty}(1+T_n(x)) = F(x) \quad \text{in } |x|>1\,.$$

Consequently we have extended the set of points upon which (1) converges "properly" by employing the sequence $\{T_n(\beta)\}$.

One purpose of this paper is to show that this idea may be fruitful in the context of more general limit-periodic continued fractions of the form

$$\frac{b_1(x)}{d_1(x)}+\frac{b_2(x)}{d_2(x)}+\ldots,$$

where $b_n(x)\to b(x)$, $d_n(x)\to d(x)$ in some region $\Delta\subset\mathbb{C}$.

We begin with the following basic theorem concerning the sequence (1).

THEOREM 2. *Suppose* $|K|<1$, *or* $|K|=1$, $K\neq 1$ *and* $\prod_{n=1}^{\infty}|K_n|=0$. *If there exists an* $h_0>0$ *such that*

(i) $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$,

(ii) $\sum_{n=1}^{\infty}[|\beta_n-\beta_{n-1}|\cdot\prod_{h_0+1}^{n-1}|K_{j-1}|^{-1}]<\infty$,

(iii) $\lim_{n\to\infty}[|\mu_n-\beta_n|\cdot\prod_{h_0+1}^{n}|K_{j-1}|^{-1}]=0$,

are all satisfied, then $\lim_{n\to\infty}T_n(\mu_n)$ *exists.*

REMARK. (ii) and (iii) will hold for all values of $h>h_0$.

PROOF OF THEOREM 2. The decomposition of $T_n(z)$ given in [3] is employed. Set

$$K_n(z) = K_n\cdot z, \qquad Y_n(z) = (z-\alpha_n)/(z-\beta_n), \qquad n=1,2,\ldots\,.$$

Then

$$t_n(z) = Y_n^{-1}\circ K_n\circ Y_n(z), \qquad n=1,2,\ldots\,.$$

Let $w_j(z)=K_j\circ Y_j\circ Y_{j+1}^{-1}(z)$, $S_j(z)=K_j\circ Y_j(z)$, $j=1,2,\ldots$, and $W_n^h(z)=w_h\circ w_{h+1}\circ\ldots\circ w_{n-1}(z)$, $n>h$. Write

$$w_{n-1}(z) = (p_n z+q_n)/(r_n z+1)\ ,$$

where $p_n=K_{n-1}(\beta_n-\alpha_{n-1})/(\beta_{n-1}-\alpha_n)$, $q_n=K_{n-1}(\alpha_{n-1}-\alpha_n)/(\beta_{n-1}-\alpha_n)$, and $r_n=(\beta_n-\beta_{n-1})/(\beta_{n-1}-\alpha_n)$, $n=1,2,\ldots$. For sufficiently large n all denominators are $\neq 0$, since $\alpha\neq\beta$.

Finally, write

$$W_n^h(z) = \frac{A_n^h z+B_n^h}{C_n^h z+D_n^h}, \quad \text{for } n>h\ .$$

As in [3], we observe that $\lim_{n\to\infty}B_n^h=l(B,h)$ exists and $\lim_{h\to\infty}l(B,h)=0$. Further $\lim_{n\to\infty}D_n^h=l(D,h)$ exists and $\lim_{h\to\infty}l(D,h)=1$. Also, $\lim_{h\to\infty}B_{h+m}^h=0$ and $\lim_{h\to\infty}D_{h+m}^h=1$, for fixed m. Set

$$\mathring{A}_n^h = A_n^h\Big/\left(\prod_{h+1}^{n} p_i\right), \quad \mathring{B}_n^h = B_n^h\Big/\left[\left(\prod_{h+1}^{n} p_i\right)S_n(\mu_n)\right],$$

$$\mathring{C}_n^h = C_n^h\Big/\left(\prod_{h+1}^{n} p_i\right), \quad \text{and} \quad \mathring{D}_n^h = D_n^h\Big/\left[\left(\prod_{h+1}^{n} p_i\right)S_n(\mu_n)\right].$$

For $n>j\geqq 1$. Then

$$W_n^h(S_n(\mu_n)) = \frac{\mathring{A}_n^h+\mathring{B}_n^h}{\mathring{C}_n^h+\mathring{D}_n^h}, \quad n>h\geqq 1\ . \tag{3}$$

The analysis of the convergence behavior of (3) begins with an examination of $\mathring{B}_n^h$.

$$\mathring{B}_n^h = \frac{B_n^h}{\left[\left(\prod_{h}^{n-1} K_j\right)\dfrac{\mu_n-\alpha_n}{\mu_n-\beta_n}\right]\cdot\left[K_n\prod_{h+1}^{n}\left(1+\dfrac{\beta_j-\beta_{j-1}+\alpha_j-\alpha_{j-1}}{\beta_{j-1}-\alpha_j}\right)\right]}\ . \tag{4}$$

For h sufficiently large, condition (iii) of theorem 2 guarantees the divergence to infinity of the first factor of the denominator of (4), whereas the second factor of the denominator converges to a finite non-zero value under hypotheses (i) and (ii). Hence $\lim_{n\to\infty}\mathring{B}_n^h=0$, for h sufficiently large.

In an entirely analogous fashion $\lim_{n\to\infty}\mathring{D}_n^h=0$, for h sufficiently large.

The recurrence relation $A_n^h=P_nA_{n-1}^h+r_nB_{n-1}^h$, $n>h\geqq 1$, provides the key for an inductive proof of

(5)

$$\mathring{A}_n^h = 1+\sum_{j=1}^{m-1}\left[r_{h+j+1}\left(\prod_{h+1}^{h+j+1} p_i\right)^{-1}B_{h+j}^h\right]+r_n\left(\prod_{h+1}^{n} p_i\right)^{-1}B_{n-1}^h, \quad n=h+m\ .$$

Observe that

$$\frac{r_{h+j+1}}{\prod_{h+1}^{h+j} p_i} = \frac{\beta_{h+j+1}-\beta_{h+j}}{\prod_{h+1}^{h+j} K_{j-1}} \cdot \frac{1}{(\beta_{h+j}-\alpha_{h+j+1})\cdot \prod_{h+1}^{h+j}\left(1+\dfrac{\beta_i-\beta_{i-1}+\alpha_i-\alpha_{i-1}}{\beta_{i-1}-\alpha_i}\right)} .$$

For large h the hypotheses imply that the series

$$\sum_{j=1}^{\infty}\left|\frac{\beta_{h+j+1}-\beta_{h+j}}{\prod_{h+1}^{h+j} K_{i-1}}\right|$$

converges to a value $S(h)$ approaching 0 as $h \to \infty$, and, furthermore, that

$$\frac{1}{|\beta_{h+j}-\alpha_{h+j+1}|\cdot \prod_{h+1}^{h+j}\left|1+\dfrac{\beta_i+\beta_{i-1}+\alpha_i-\alpha_{i-1}}{\beta_{i-1}-\alpha_i}\right|} < M_1 .$$

Also $|B_{h+j}^h| < M$, so that

$$|\overset{*}{A}{}_n^h - 1| \leqq MM_1 S(h) .$$

Hence, the expansion (5) converges and

$$\lim_{n\to\infty} \overset{*}{A}{}_n^h = 1+\varepsilon(h), \qquad \text{where } \lim_{h\to\infty} \varepsilon(h) = 0 .$$

In an entirely analogous fashion $\lim_{n\to\infty} \overset{*}{C}{}_n^h = \nu(h)$, where $\lim_{h\to\infty} \nu(h) = 0$. Therefore, for large h,

$$W_n^h(S_n(\mu_n)) = \frac{\overset{*}{A}{}_n^h + \overset{*}{B}{}_n^h}{\overset{*}{C}{}_n^h + \overset{*}{D}{}_n^h} \to \frac{1+\varepsilon(h)}{\nu(h)} \approx \infty, \qquad \text{as } n \to \infty .$$

Consequently,

$$T_n(\mu_n) = T_{h-1}\circ Y_h^{-1}[W_n^h(S_n(\mu_n))] \to T_{h-1}\circ Y_h^{-1}\left(\frac{1+\varepsilon(h)}{\nu(h)}\right), \qquad \text{as } n \to \infty .$$

This completes the proof of theorem 2.

Let us now assume that a_n, b_n, c_n, and d_n are continuous functions of a complex variable x, and $a_n(x) \to a(x)$, etc. uniformly in some domain, D, of the complex plane. Let the fixed points of $t_n(x,z)$ be $u_n(x)$ and $v_n(x)$, and those of $t(x,z)$ be $u(x)$ and $v(x)$.

Define

$$K_n(x) = \frac{a_n(x)-c_n(x)u_n(x)}{a_n(x)-c_n(x)v_n(x)} \to K(x) = \frac{a(x)-c(x)u(x)}{a(x)-c(x)v(x)} .$$

Let $\Omega=\{x : |K(x)|<1\}$ and $\Gamma=\{x : |K(x)|>1\}$, and let Δ be a compact subset of $\Gamma\subset D$. Then $u(x)$ is attractive in Ω and $v(x)$ is attractive in Γ.

Theorem 2 applies, giving the following result for certain sequences $\{\mu_n(x)\}$ where $\mu_n(x)\to u(x)$ in Γ.

$$\lim_{n\to\infty} T_n[x;\mu_n(x)] = F(x), \quad \text{continuous in } \Delta .$$

Furthermore,

$$F(x) \neq \lim_{n\to\infty} T_n[x;0] = \lim_{n\to\infty} T_n[x;v(x)] .$$

In view of example 1, one suspects that, in the case of certain continued fractions, the use of the *repulsive* fixed point as a modifying factor might analytically extend a function from Ω into Γ.

More general limit-periodic continued fractions do indeed satisfy the hypotheses of theorem 2, as the next example illustrates.

EXAMPLE 2. Let

$$f_n(x) = x^2+(2^{-n}+n^{-1}+1)x+2^{-n}(1+n^{-1}), \qquad n=1,2,\dots,$$

and $f(x)=\lim_{n\to\infty} f_n(x)=x^2+x$. The continued fraction

$$\frac{f_1(x)}{f'_1(x)}\,\frac{}{-}\,\frac{f_2(x)}{f'_2(x)}\,\frac{}{-}\cdots\frac{}{-}\,\frac{f_n(x)}{f'_n(x)}\,\frac{}{-}\cdots$$

is derived from transformations having fixed points

$$u_n(x) = -(x+2^{-n}), \quad v_n(x) = -(x+1+n^{-1}), \quad n=1,2,\dots .$$

$\Omega = \{x : \operatorname{Re} x>-\tfrac{1}{2}\}$, $\Gamma = \{x : \operatorname{Re} x<\tfrac{1}{2}\}$, and

$$K_n(x) = u_n(x)/v_n(x) \to K(x) = x/(x+1) .$$

In the region

$$\Gamma' = \Gamma \cap \{x : |x+\tfrac{16}{7}|>\tfrac{12}{7}\}$$

we find that $|K^{-1}(x)|>\frac{3}{4}$. Hence $\frac{3}{4}<|K_n^{-1}(x)|<1$ for n sufficiently large. Now, (i) of theorem 2 is trivially satisfied and

$$|u_n(x)-u_{n-1}(x)| = |u_n(x)-u(x)| = \frac{1}{2^n} < \left(\frac{3}{4}\right)^n\left(\frac{3}{4}\right)^n < \left(\frac{3}{4}\right)^n\cdot\prod_{h+1}^{n-1}|K_{j-1}^{-1}(x)|$$

shows that (ii) and (iii) are satisfied when $\mu_n(x)\equiv u(x)$.

Convergence of $\{T_n[x;u(x)]\}_{n=1}^{\infty}$ occurs also on $B(\Omega)-\{1\}$. If we set $|K_n(x)| =1+y_n(x)$, it can be shown that $\sum y_n(x)=-\infty$, so that $\prod|K_n(x)|=0$, and theorem 2 applies.

Theorem 1 and subsequent remarks can be applied to the power series

(6) $$P(x) = c_0+c_1x+c_2x^2+\ldots, \quad |x| < R < \infty .$$

We first convert (6) to its equivalent continued fraction

(7) $$P(x) = c_0+\frac{c_1x}{1}\ \frac{c_2x/c_1}{-c_2x/c_1+1-}\cdots\frac{c_nx/c_{n-1}}{-c_nx/c_{n-1}+1-}\cdots,$$

which is in "fixed point form" (that is, $u_1v_1/(u_1+v_1)\_\ u_2v_2/(u_2+v_2)\_\ \ldots$).

Set $u_n(x)=c_nx/c_{n-1}$, $v_n(x)\equiv 1$. Then $K_n(x)=u_n(x)$. Let us assume $\lim_{n\to\infty} c_n/c_{n-1}=\eta$, $|\eta|=R^{-1}$, in which case (7) is limit-periodic. Consequently (6) can be modified by computing $\{T_n[x;\, u(x)]\}$ instead of $\{T_n[x;\, 0]\}$. These two sequences converge to a common limit in $|x|<R$.

The idea of using the modifying factor $u(x)$ in Γ, where it is *repulsive*, in order to extend the region of "proper" convergence of a continued fraction may be discerned in the foundations of a paper by Waadeland [4]. Waadeland considers a power series $P(x)=1+c_1x+c_2x^2+\ldots$, holomorphic in $|x|<R$, $R>2$, This series is converted into a *T-fraction*

$$1+d_0x+\frac{x}{1+d_1x+}\ \frac{x}{1+d_2x+}\cdots,$$

the nth approximant of which may be written $T_n[x;\, 0]$, employing the present notation. The hypotheses of the following theorem imply $d_n \to -1$, so that the fixed points of $t(x;\, z)$ are $u(x)=x$ and $v(x)=-1$. Waadeland's result can be paraphrased:

THEOREM 4. (Waadeland). *Suppose* $|P(x)-1|<\tilde{K}<\frac{1}{2}R-1$ *in* $|x|<R$, $R>2$. *Then* $\{T_n[x;\, x]\}_{n=1}^{\infty}$ *converges to* $P(x)$ *uniformly on any compact subset of* $|x|<\frac{1}{2}R$.

Hovstad, [5], later showed that "proper" convergence actually occurs in $|x|<R$.

It is easily seen that convergence of $\{T_n[x;\, x]\}_n$ to *a* continuous function on compact subsets of $\{x : 1<|x|<\frac{1}{2}R\}$ is implied by theorem 3.

The author's $K-D$ fractions, [6] can be modified in the same manner.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN COLORADO
PUEBLO, COLORADO 81001
U.S.A.