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ON THE DUAL WEIGHTS FOR CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS I
Removing separability conditions

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Introduction.

Let $M\otimes_\alpha G$ be the crossed product of a von Neumann algebra M and a locally compact group G (cf. [12]). By a modification of Digernes' and Sauvageot's methods in [3], [4] and [10] we construct the dual weights on the crossed product, without separability conditions on M or G. Also the commutation theorem for crossed products [4, Theorem 3.14] is valid in the general case. In the last section we prove that when G is abelian, the "dualisation" map: $\varphi \to \tilde{\varphi}$ is a bijection of the set of normal, faithful, semifinite (n.f.s.) weights on M onto the set of n.f.s. weights on $M\otimes_\alpha G$, which are invariant under the dual action.

1. Preliminaries.

1.1 *Definition of the crossed product.* Let M be a von Neumann algebra and let $\alpha\colon G \to \operatorname{aut}(M)$ be a σ-weakly continuous action of a locally compact group on M. A *covariant representation* of (M,G,α) is a pair (π,λ) of a normal, nondegenerate representation π of M and a strongly continuous unitary representation λ of G, on a Hilbert space H, such that

$$\pi(\alpha_g(x)) = \lambda(g)\pi(x)\lambda(g)^* \qquad x \in M,\ g \in G .$$

If M acts on a Hilbert space H one can define a covariant representation (π,λ) of (M,G,α) on the Hilbert space $H\otimes L^2(G)=L^2(G,H)$ by [12, definition 3.1]:

$$(\pi(x)\xi)(g) = \alpha_g^{-1}(x)\xi(g) \qquad x \in M,\ \xi \in L^2(G,H)$$

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h) \qquad g \in G,\ \xi \in L^2(G,H) .$$

The von Neumann algebra generated by $\pi(M)$ and $\lambda(G)$ is called the crossed product of M and G, and will be denoted $M\otimes_\alpha G$.

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1.2 *The canonical implementation of an automorphism group.* Following [6] we say that a von Neumann algebra M is on *standard form* if it acts on a Hilbert space H, equipped with a conjugate linear isometric involution J and a selfdual cone P, such that

(1) $JMJ=M'$
(2) $JcJ=c^*, \quad c \in M \cap M'$
(3) $J\xi=\xi, \quad \xi \in P$
(4) $xJx(P) \subseteq P, \quad x \in M$.

When M is on standard form the group aut (M) of all *automorphisms of M has a unique unitary implementation (the *canonical* implementation) $g \to u_g$ on H, such that for any $g \in$ aut (M)

(a) $g(x)=u_g x u_g^*, \quad x \in M$
(b) $u_g J = J u_g$
(c) $u_g(P)=P$

(cf. [6, Theorem 3.2]). If $\alpha\colon G \to$ aut (M) is a σ-weakly continuous action of a locally compact group on M, then the canonical implementation $g \to u(g) \to u_{\alpha_g}$ of α is a strongly continuous unitary representation of G on H (cf. [6, corollary 3.6]).

1.3 *Relative modular theory.* (Cf. [2, § 1] and [4, § 2]). Let M be a von Neumann algebra. The set of normal, faithful, semifinite (n.f.s) weights on M is denoted $P(M)$. For $\varphi \in P(M)$ we let $(\pi_\varphi, H_\varphi, \Lambda_\varphi)$ be the representation of M induced by φ. For $\varphi, \psi \in P(M)$ the map

$$\Lambda_\varphi(x) \to \Lambda_\psi(x^*) \qquad x \in n_\varphi \cap n_\psi^*$$

is preclosed from H_φ to H_ψ and its closure $S_{\psi,\varphi}$ has the polar decomposition $S_{\psi,\varphi} = J_{\psi,\varphi} \Delta_{\psi,\varphi}^{\frac{1}{2}}$. Moreover

$$\Delta_{\psi,\varphi}^{it} = (D\psi : D\varphi)_t \Delta_\varphi^{it} .$$

Following [3] we put

$$\sigma_t^{\psi,\varphi}(x) = (D\psi : D\varphi)_t \sigma_t^\varphi(x) = \sigma_t^\psi(x)(D\psi : D\varphi)_t, \qquad x \in M, \ \varphi, \psi \in P(M) .$$

A simple computation shows that $t \to \sigma_t^{\psi,\varphi}$ is a one parameter group of isometries on M, and that

$$\sigma_t^{\psi,\varphi}(xy) = \sigma_t^{\psi,\omega}(x) \sigma_t^{\omega,\varphi}(y), \qquad x, y \in M, \ \varphi, \psi, \omega \in P(M) .$$

The closure P_φ of $\{\pi_\varphi(x) J_\varphi \Lambda_\varphi(x) \mid x \in n_\varphi\}$ is a selfdual cone in H_φ, such that $\pi_\varphi(M)$ is on standard form with respect to $(H_\varphi, J_\varphi, P_\varphi)$, (cf. [6] and [4, lemma 2.5]). Assume now that M itself is on standard form with respect to (H, J, P).

By the uniqueness of the standard form [6, Theorem 2.3] we can identify all the Hilbert spaces H_φ with H in such a way that π_φ is the identity $J_\varphi = J$ and $P_\varphi = P$ for any $\varphi \in P(M)$. Since the unitary operator

$$v_{\psi,\varphi} = J_\psi J_{\psi,\varphi} = J_{\psi,\varphi} J_\varphi$$

is the unique coupling operator of π_φ and π_ψ for which $V_{\psi,\varphi}(P_\varphi) = P_\psi$ (cf. [4, proposition 2.6]) we have $v_{\psi,\varphi} = 1$ under the above identification. Hence $J_{\psi,\varphi} = J$ for all $\varphi, \psi \in P(M)$.

2. The commutation theorem for crossed products.

THEOREM 2.1 (cf. [4, Theorem 3.14]). *Let M be a von Neumann algebra on a Hilbert space H, and α: $G \to$ aut (M) a σ-weakly continuous action of a locally compact group on M. Let $g \to u(g)$ be a unitary representation of G on H such that*

$$\alpha_g(x) = u(g)xu(g)^* \qquad x \in M,\ g \in G$$

and let U be the unitary operator on $L^2(G,H)$ given by

$$(U\xi)(g) = u(g)\xi(g) \qquad \xi \in L^2(G,H)\,.$$

Then

(1) *$M \otimes_\alpha G$ is generated by $U^*(M \otimes 1)U$ and $1 \otimes \mathscr{L}(G)$*

(2) *$(M \otimes_\alpha G)'$ is generated by $M' \otimes 1$ and $U^*(1 \otimes \mathscr{R}(G))U$ where $\mathscr{L}(G)$ and $\mathscr{R}(G)$ are the von Neumann algebras associated with the left and right regular representations of G on $L^2(G)$.*

Throughout the paper we let α: $G \to$ aut (M) be a fixed σ-weakly continuous action of a locally compact group on a von Neumann algebra M.

LEMMA 2.2. *The map $G \times M \to M$ given by $(g,x) \to \alpha_g x$ is σ-strong*-continuous on bounded sets.*

PROOF. We may assume that M is on standard form. Let $g \to u(g)$ be the canonical implementation of G on H. We have

$$\alpha_g(x) = u(g)xu(g)^*, \qquad x \in M,\ g \in G\,,$$

and $g \to u(g)$ is strongly continuous. Since strong and σ-strong*-topology coincide on the unitary group, and since the product is σ-strong*-continuous on bounded sets of $B(H)$ we get the required result.

Let dg be a fixed Haarmeasure on G, and let $\Delta_G(g)$ be the module function on G.

LEMMA 2.3. *Let $K(G,M)$ be the space of σ-strong*-continuous functions from G to M with compact support.*

(a) *$K(G,M)$ is an involutive algebra with product*

$$(x*y)(g) = \int_G \alpha_h x(gh)y(h^{-1})\,dh, \qquad x,y \in K(G,M)$$

and involution

$$x^\#(g) = \Delta_G(g)^{-1}\alpha_g^{-1}x(g^{-1})^* \qquad x \in K(G,M)$$

(b) *For $x,y \in K(G,M)$: $(x^\# * y)(e) = \int_G x(g)^* y(g)\,dg$*

(c) *$K(G,M)$ is a two sided module over M with the following multiplications:*

$$(x\cdot a)(g) = x(g)a, \qquad x \in K(G,M),\ a \in M$$

$$(a\cdot x)(g) = \alpha_g^{-1}(a)x(g), \qquad x \in K(G,M),\ a \in M$$

(d) *For $x,y \in K(G,M)$ and $a \in M$*

$$a\cdot(x*y) = (a\cdot x)*y,\ (x*y)\cdot a = x*(y\cdot a) \text{ and } (x\cdot a)^\# = a^*\cdot x^\#$$

(e) *Let (π,λ) be a covariant representation of (M,G,α). Then*

$$\mu(x) = \int_G \lambda(g)\pi(x(g))\,dg$$

*defines a *representation of the involutive algebra $K(G,M)$.*

Moreover $\mu(x\cdot a) = \mu(x)\pi(a)$ and $\mu(a\cdot x) = \pi(a)\mu(x)$ for $x \in K(G,M)$ and $a \in M$.

(f) *The representation μ maps $K(G,M)$ onto a σ-weakly dense subalgebra of the von Neumann algebra generated by $\pi(M)$ and $\lambda(G)$.*

PROOF. Let $x \in K(G,M)$. Since x has compact support, it follows from the principle of uniform boundedness that $\sup_{g\in G}\|x(g)\| < \infty$. Let now $x,y \in K(G,M)$. Since the product in M is σ-strong* continuous on bounded sets, we get from lemma 2.2 that the function

$$z(g,h) = \alpha_h x(gh)y(h^{-1})$$

is σ-strong* continuous from $G\times G$ to M. Moreover, it has compact support, because $gh \in \operatorname{supp}(x)$ and $h^{-1} \in \operatorname{supp}(y)$ imply that $g \in \operatorname{supp}(x)\operatorname{supp}(y)$ and $h \in \operatorname{supp}(y)^{-1}$. Let $g_0 \in G$ be fixed, and let p be a strong*-seminorm on M. Then

$$p((x*y)(g)-(x*y)(g_0)) \leqq \int_G p(z(g,h)-z(g_0,h))\,dh\,.$$

Since $(g, h) \to p(z(g,h)-z(g_0,h))$ is a continuous real function on $G\times G$ with compact support, the integral on the right side is a continuous function of $g \in G$. Hence

$$p((x*y)(g)-(x*y)(g_0)) \to 0 \quad \text{for } g \to g_0 .$$

This proves that $g \to (x*y)(g)$ is strong* continuous. Since $\operatorname{supp}(x*y) \subseteqq \operatorname{supp}(x)\operatorname{supp}(y)$, it follows that $x*y \in K(G,M)$.

The verification of the rest of lemma 2.3 is straight forward, and will be left to the reader (compare with [4, § 3] and [10, p. 942]).

REMARK. The above algebra $K(G,M)$ is analogue of $L^1(G,A)$ used in the crossed product construction for C*-algebras (cf. [5]). However, it is more convenient to define the algebraic structure of $K(G,M)$ such that the map $x \to \int_G \lambda(g)\pi(x(g))\,dg$ is a *representation instead of $x \to \int_G \pi(x(g))\lambda(g)\,dg$ as in [5].

In the following we let (π,λ) be the covariant representation of (M,G,α) used in the crossed product construction (cf. § 1.1), and we let μ be the associated representation of $K(G,M)$. We let $K(G,H)$ denote the subset of $L^2(G,H)$ consisting of continuous functions from G to H with compact support. Note that $K(G,H)$ is dense in $L^2(G,H)$.

LEMMA 2.4. *Let $x \in K(G,M)$ and $\xi \in K(G,H)$, then $\mu(x)\xi \in K(G,H)$ and*

$$(\mu(x)\xi)(g) = \int_G \alpha_h x(gh)\xi(h^{-1})\,dh$$

PROOF. It is easily seen that the integral on the right side defines an element of $K(G,H)$. Let $\eta \in K(G,H)$, then

$$\begin{aligned}(\mu(x)\xi \mid \eta) &= \int_G \left(\int_G ((\lambda(k)\pi(x(k))\xi)(g) \mid \eta(g))\,dg\right)dk \\ &= \int_G \left(\int_G ((\alpha_{g^{-1}k}x(k))\xi(k^{-1}g) \mid \eta(g))\,dk\right)dg \\ &= \int_G \left(\int_G \alpha_h x(gh)\xi(h^{-1})\,dh \mid \eta(g)\right)dg .\end{aligned}$$

Hence $(\mu(x)\xi)(g) = \int_G \alpha_h x(gh)\xi(h^{-1})\,dh$.

We will now assume that the von Neumann algebra M is on standard form with respect to (H,J,P). We may identify all the Hilbert spaces H_φ, $\varphi \in P(M)$

with H as in § 1.3. Hence $\pi_\varphi = \text{identity}$, $J_{\psi,\varphi} = J$ for $\varphi, \psi \in P(M)$ and $P_\varphi = P$ for $\varphi \in P(M)$.

Moreover we let $g \to u(g)$ be the canonical implementation of G on H. Let φ be a fixed n.f.s. weight on M, and put as usual

$$n_\varphi = \{a \in M \mid \varphi(a^*a) < \infty\} \quad \text{and} \quad m_\varphi = n_\varphi^* n_\varphi .$$

We put

$$B_\varphi = K(G,M)\cdot n_\varphi = \text{span}\{x\cdot a \mid x \in K(G,M),\ a \in n_\varphi\} .$$

Note that B_φ is a left ideal in $K(G,M)$.

Since for $y \in K(G,M)$ and $a \in n_\varphi$ we have $y(g)a \in n_\varphi$ and $\Lambda_\varphi(y(g)\cdot a) = y(g)\Lambda_\varphi(a)$ it follows that $\Lambda_\varphi(x(g))$ is defined for any $x \in B_\varphi$ and $g \in G$, and that the function

$$g \to \Lambda_\varphi(x(g))$$

belongs to $K(G,M) \subseteqq L^2(G,H)$. Hence one can define a map $\tilde{\Lambda}_\varphi : B_\varphi \to L^2(G,H)$ by

$$(\tilde{\Lambda}_\varphi(x))(g) = \Lambda_\varphi(x(g)) .$$

LEMMA 2.5. (1) *For* $x \in K(G,M)$ *and* $y \in B_\varphi$, $\tilde{\Lambda}_\varphi(x*y) = \mu(x)\tilde{\Lambda}_\varphi(y)$

(2) *If* $x,y \in B_\varphi$ *then* $(y^\# * x)(e) \in m_\varphi$ *and* $(\tilde{\Lambda}_\varphi(x) \mid \tilde{\Lambda}_\varphi(y)) = \varphi((y^\# * x)(e))$,

(3) $\mu(B_\varphi \cap B_\varphi^\#)$ *is* σ*-weakly dense in* $M \otimes_\alpha G$

(4) $\tilde{\Lambda}_\varphi(B_\varphi \cap B_\varphi^\#)$ *is dense in* $L^2(G,H)$.

PROOF. (1) We may assume that $y = z\cdot a$, $z \in K(G,M)$, $a \in n_\varphi$. Applying lemma 2.4 we get:

$$\begin{aligned}(\tilde{\Lambda}_\varphi(x*y))(g) &= \Lambda_\varphi((x*z)(g)a)\\ &= (x*z)(g)\Lambda_\varphi(a)\\ &= \int_G \alpha_h x(gh) z(h^{-1})\Lambda_\varphi(a)\,dh\\ &= (\mu(x)\xi)(g)\end{aligned}$$

where

$$\xi(g) = z(g)\Lambda_\varphi(a) = \Lambda_\varphi((z\cdot a)(g)) = (\tilde{\Lambda}_\varphi(y))(g) .$$

Hence $\tilde{\Lambda}_\varphi(x*y) = \mu(x)\tilde{\Lambda}_\varphi(y)$.

(2) Let $x = x_1\cdot a$ and $y = y_1\cdot b$, $x_1, y_1 \in K(G,M)$, $a,b \in n_\varphi$. Then

$$(x^\# * y)(g) = \alpha_g^{-1}(b^*)(x_1^\# * y_1)(g)a, \qquad g \in G .$$

Hence

$$(x^{\#} * y)(e) = b^*(x_1^{\#} * y_1)(e)a \in n_\varphi^* n_\varphi = m_\varphi .$$

Moreover

$$(\tilde{\Lambda}_\varphi(x_1 \cdot a))(g) = \Lambda_\varphi(x_1(g)a) = x_1(g)\Lambda_\varphi(a) .$$

Thus

$$(\tilde{\Lambda}_\varphi(x) \mid \tilde{\Lambda}_\varphi(y)) = \int_G (x_1(g)\Lambda_\varphi(a) \mid y_1(g)\Lambda_\varphi(b))\, dg$$

$$= \int_G \varphi(b^* y_1(g)^* x_1(g) a)\, dg .$$

Since $x \to \psi(b^*xa)$ is a σ-weakly continuous functional on M for $a, b \in n_\varphi$ we get using lemma 2.3(c):

$$(\tilde{\Lambda}_\varphi(x) \mid \tilde{\Lambda}_\varphi(y)) = \varphi\left(b^* \left(\int_G y_1(g)^* x_1(g)\, dg \right) a \right)$$

$$= \varphi(b^*(y_1^{\#} * x_1)(e)a) = \varphi(y^{\#} * x)(e) .$$

(3) Note that $B_\varphi \cap B_\varphi^{\#} \supseteq n_\varphi^* \cdot K(G, M) \cdot n_\varphi$.
For $a, b \in n_\varphi$ and $x \in K(G, M)$ we have

$$\mu(b^* \cdot x \cdot a) = \pi(b)^* \mu(x) \pi(a) .$$

Since $\mu(K(G, M))$ is σ-weakly dense in $M \otimes_\alpha G$ (lemma 2.3(f)), and since n_φ is σ-weakly dense in M, it follows that $\mu(B_\varphi^{\#} \cap B_\varphi)$ is σ-weakly dense in $M \otimes_\alpha G$.

(4) *Let* $a, b \in n_\varphi$ and let f be a continuous function on G, with compact support. Since we may consider f as a function in $K(G, M)$ we have $b^* \cdot f \cdot a \in B_\varphi \cap B_\varphi^{\#}$. Moreover,

$$(\tilde{\Lambda}_\varphi(b^* \cdot f \cdot a))(g) = \Lambda_\varphi(\alpha_g^{-1}(b)^* f(g) a) = \alpha_g^{-1}(b^*) f(g) \Lambda_\varphi(a)$$

$$= (\pi(b)^*(\Lambda_\varphi(a) \otimes f))(g) ,$$

where we have identified $L^2(G, H)$ and $H \otimes L^2(G)$.

By taking a net $(b_i)_{i \in I}$ in n_φ, that converges strongly to 1, it is seen that $\Lambda_\varphi(a) \otimes f$ is in the closure of $\tilde{\Lambda}_\varphi(B_\varphi \cap B_\varphi^{\#})$ for $f \in K(G)$ and $a \in n_\varphi$. Since $K(G)$ is dense in $L^2(G)$ and $\Lambda_\varphi(n_\varphi)$ is dense in $H_\varphi = H$, the lemma is proved.

LEMMA 2.6. *The map* $(g, t) \to (D\varphi \circ \alpha_g : D\varphi)_t$ *is σ-strong continuous from* $G \times \mathbf{R}$ *into the unitary group in* M.

PROOF. By [4, proposition 2.7] the map is separately continuous. (The separability conditions in [4] are not essential for the proof). The joint continuity follows easily from the proof of [4, proposition 2.7] by using lemma 2.2.

Since the map $g \to (D\varphi\circ\alpha_g : D\varphi)_t$ is continuous for any $t \in \mathbb{R}$, and since

$$\sigma_t^{\varphi\circ\alpha_g,\varphi}(x) = (D\varphi\circ\alpha_g : D\varphi)_t\sigma_t^\varphi(x), \quad x \in M,\ g \in G$$

we can for each $t \in \mathbb{R}$ define a map ϱ_t^φ on $K(G,M)$ by

$$(\varrho_t^\varphi(x))(g) = \Delta_G(g)^{it}\sigma_t^{\varphi\circ\alpha_g,\varphi}(x(g)), \quad x \in K(G,M)\,.$$

LEMMA 2.7. (cf. [4, lemma 3.7]). (1) *The family* $(\varrho_t^\varphi)_{t\in\mathbb{R}}$ *is a one parameter group af* $\#$ *automorphisms of* $K(G,M)$.

(2) *For* $x \in K(G,M)$ *and* $a \in M$

$$\varrho_t^\varphi(x\cdot a) = \varrho_t^\varphi(x)\cdot\sigma_t^\varphi(a) \quad \text{and} \quad \varrho_t^\varphi(a\cdot x) = \sigma_t^\varphi(a)\cdot\varrho_t^\varphi(x)$$

(3) B_φ *and* $B_\varphi^\#$ *are invariant under* ϱ_t^φ.

PROOF. (1) can be proved by a direct calculation as in the proof of [4, lemma 3.7]. It is easy to verify (2), and (3) follows from (2) because n_φ and n_φ^* are σ_t^φ-invariant.

LEMMA 2.8. *There exists an injective selfadjoint operator* $\tilde{\Delta}_\varphi$ *and a conjugate linear isometric involution* $\tilde{J}$ *on* $L^2(G,H)$ *such that*

$$(\tilde{\Delta}_\varphi^{it}\xi)(g) = \Delta_G(g)^{it}\Delta_{\varphi\circ\alpha_g,\varphi}^{it}\xi(g), \qquad \xi \in L^2(G,H)\,,$$

$$(\tilde{J}\xi)(g) = \Delta_G(g)^{-\frac{1}{2}}u(g)^{-1}J\xi(g^{-1}), \quad \xi \in L^2(G,H)\,,$$

where $g \to u(g)$ *is the canonical implementation of* G.

PROOF. Since $\Delta_{\varphi\circ\alpha_g,\varphi}^{it} = (D\varphi\circ\alpha_g : D\varphi)_t\Delta_\varphi^{it}$ it follows that for each $t \in \mathbb{R}$, the formula

$$(u_t^\varphi\xi)(g) = \Delta_G(g)^{it}\Delta_{\varphi\circ\alpha_g,\varphi}^{it}\xi(g)$$

defines an operator on $K(G,H)$. It is easily seen that u_t^φ can be extended to a unitary operator on $L^2(G,H)$ given by the same formula. Moreover $(u_t)_{t\in\mathbb{R}}$ is a one parameter group. For $\xi,\eta \in K(G,H)$ we get:

$$\lim_{t\to 0}(u_t^\varphi\xi \mid \eta) = \lim_{t\to 0}\int_G (\Delta_G(g)^{it}\Delta_{\varphi\circ\alpha_g,\varphi}^{it}\xi(g) \mid \eta(g))\,dg = \int_G (\xi(g)\mid\eta(g))\,dg\,.$$

Hence $t \to u_t^\varphi$ is weakly, and thus strongly continuous. This proves the existence of $\tilde{\Delta}_\varphi$. It is easily seen that $\tilde{J}$ is a conjugate linear isometry on $L^2(G,H)$. Moreover for $\xi \in L^2(G,H)$

$$(\tilde{J}\tilde{J}\xi)(g) = \Delta_G g)^{-\frac{1}{2}}u(g)^{-1}J(\Delta_G(g)^{\frac{1}{2}}u(g)J\xi(g)) = \xi(g)$$

because J and $u(g)$ commutes (cf. section 1.2).

Let $C_c^\infty(\mathbf{R})$ denote the space of C^∞-functions on $\mathbf{R}$ with compact support. For $\varphi \in C_c^\infty(\mathbf{R})$ we put

$$\hat{\varphi}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)e^{-izx}\,dx, \qquad z \in \mathbf{C}\,.$$

Note that $\hat{\varphi}$ is the analytic extension of the Fourier transformed of φ. Since $\varphi \in C_c^\infty$ it follows that for any $n \in \mathbf{N}$ and $t \in \mathbf{R}$

$$\hat{\varphi}(s+it)|s|^n \to 0 \qquad \text{for } s \to \infty\,.$$

In particular $\int_{-\infty}^{\infty} |\hat{\varphi}(s+it)|\,ds < \infty$ for any $t \in \mathbf{R}$.

LEMMA 2.9. *Let K be a injective, positive selfadjoint (non necessarily bounded) operator on a Hilbert space $\mathscr{H}$. Let $\alpha \in \mathbf{R}$. For $\xi, \eta \in \mathscr{H}$ the following conditions are equivalent*:

(1) *$\xi \in D(K^\alpha)$ and $\eta = K^\alpha\xi$*
(2) *For any $\varphi \in C_c^\infty(\mathbf{R})$*

$$\int_{-\infty}^{\infty} \hat{\varphi}(t)K^{it}\eta\,dt = \int_{-\infty}^{\infty} \hat{\varphi}(t+i\alpha)K^{it}\xi\,dt\,.$$

PROOF. By the inversion formula for Fourier transformation we have

$$\varphi(x) = \int_{-\infty}^{\infty} \hat{\varphi}(t)e^{ixt}\,dt$$

and

$$e^{\alpha x}\varphi(x) = \int_{-\infty}^{\infty} \hat{\varphi}(t+i\alpha)e^{ixt}\,dt\,.$$

Hence

$$\varphi(\log K) = \int_{-\infty}^{\infty} \hat{\varphi}(t)K^{it}\,dt \qquad \text{(strongly)}$$

$$K^\alpha\varphi(\log K) = \int_{-\infty}^{\infty} \hat{\varphi}(t+i\alpha)K^{it}\,dt \qquad \text{(strongly)}\,.$$

Thus (2) is equivalent with

(3) For any $\varphi \in C_c^\infty(\mathbf{R})$: $K^\alpha\varphi(\log K)\xi = \varphi(\log K)\eta$.

(1) $\Rightarrow$ (3) is trivial because $\varphi(\log K)\, K^\alpha \subseteq K^\alpha \varphi(\log K)$.

(3) $\Rightarrow$ (1). Let $(\varphi_n)_{n\in\mathbf{N}}$ be a sequence of C^∞-functions with compact support, such that $\varphi_n(x) \to 1$ uniformly on compact sets. Put

$$\xi_n = \varphi_n(\log K)\xi \quad \text{and} \quad \eta_n = \varphi_n(\log K)\eta .$$

By (3) $\xi_n \in D(K^\alpha)$ and $\eta_n = K^\alpha \xi_n$. Since K^α is closed, we get $\xi \in D(K^\alpha)$ and $\eta = K^\alpha \xi$.

LEMMA 2.10. *Let $\xi, \eta \in L^2(G,H)$ and $\alpha \in \mathbf{R}$, such that $\xi(g) \in D(\Delta^\alpha_{\varphi\circ\alpha_s,\varphi})$ and $\eta(g) = \Delta^\alpha_{\varphi\circ\alpha_s,\varphi}\xi(g)$ for almost any $g \in G$. Then*

$$\xi \in D(\tilde{\Delta}^\alpha) \quad \textit{and} \quad \eta = \tilde{\Delta}^\alpha \xi .$$

PROOF. Let $\varphi \in C_c^\infty(\mathbf{R})$ and let $\zeta \in L^2(G,H)$. By lemma 2.9 we have

$$\int_{-\infty}^{\infty} \hat{\varphi}(t)(\Delta_G(g)^{it}\Delta^{it}_{\varphi\circ\alpha_s,\varphi}\eta(g) \mid \zeta(g))\, dt =$$

$$\int_{-\infty}^{\infty} \hat{\Phi}(t+i\alpha)(\Delta_G(g)^{it}\Delta^{it}_{\varphi\circ\alpha_s,\varphi}\xi(g) \mid \zeta(g))\, dt$$

for almost any $g \in G$.

Integrating over G we get

$$\int_{-\infty}^{\infty} \hat{\varphi}(t)(\tilde{\Delta}^{it}_\varphi \eta \mid \zeta)\, dt = \int_{-\infty}^{\infty} \hat{\varphi}(t+i\alpha)(\tilde{\Delta}^{it}_\varphi \xi \mid \zeta)\, dt .$$

Hence by lemma 2.9, $\xi \in D(\tilde{\Delta}^\alpha)$ and $\eta = \tilde{\Delta}^\alpha \xi$.

LEMMA 2.11. *The canonical implementation $g \to u(g)$ of G on H satisfies*

$$u(g)(\Lambda_\varphi(x)) = \Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g x), \qquad x \in n_\varphi,\ g \in G .$$

PROOF. Clearly the map $\Lambda_\varphi(x) \to \Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g x)$, $x \in n_\varphi$, can be extended to a unitary operator v_g on H. To show that $u(g) = v_g$, we need only to prove that (cf. § 1.2)

(a) $\alpha_g x = v_g x v_g^*, \quad x \in M$

(b) $J v_g = v_g J$

(c) $v_g(P) = P$.

For $x \in M$ and $y \in n_\varphi$

$$(\alpha_g x)v_g\Lambda_\varphi(y) = (\alpha_g x)\Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g y)$$
$$= \Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g(xy)) = v_g\Lambda_\varphi(xy) = v_g x\Lambda_\varphi(y) .$$

Hence $(\alpha_g x)v_g = v_g x$, $x \in M$, $g \in G$. This proves (a). For $x \in n_\varphi \cap n_\varphi^*$

$$v_g S_\varphi \Lambda_\varphi(x) = v_g\Lambda_\varphi(x^*) = \Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g x^*)$$
$$= S_{\varphi\circ\alpha_g^{-1}}\Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g x) = S_{\varphi\circ\alpha_g^{-1}}v_g\Lambda_\varphi(x) .$$

Since $\Lambda_\psi(n_\psi \cap n_\psi^*)$ is a core of S_ψ for any $\psi \in P(M)$ we get $v_g S_\varphi v_g^* = S_{\varphi\circ\alpha_g^{-1}}$. By polar decompostion it follows that

$$v_g\Delta_\varphi v_g^* = \Delta_{\varphi\circ\alpha_g^{-1}} \quad \text{and} \quad v_g J v_g^* = J$$

Hence (b). By section 1.3 we have

$$P = P_\varphi = \{xJ\Lambda_\varphi(x) \mid x \in n_\varphi\}^- .$$

Thus

$$v_g(P) = \{v_g xJ\Lambda_\varphi(x) \mid x \in n_\varphi\}^-$$
$$= \{(\alpha_g x)Jv_g\Lambda_\varphi(x) \mid x \in n_\varphi\}^-$$
$$= \{(\alpha_g x)J\Lambda_{\varphi\circ\alpha_g^{-1}}(\alpha_g x) \mid x \in n_\varphi\}^-$$
$$= \{yJ\Lambda_{\varphi\circ\alpha_g^{-1}}(y) \mid y \in n_{\varphi\circ\alpha_g^{-1}}\}^- = P_{\varphi\circ\alpha_g^{-1}} = P .$$

This proves (c). Hence $u(g) = v_g$ for any $g \in G$.

LEMMA 2.12. (1) *$\mathfrak{A}_\varphi = \tilde{\Lambda}_\varphi(B_\varphi \cap B_\varphi^\sharp)$ is a left Hilbert algebra with product*

$$\tilde{\Lambda}_\varphi(x)\cdot\tilde{\Lambda}_\varphi(y) = \tilde{\Lambda}_\varphi(x * y) \qquad x, y \in B_\varphi \cap B_\varphi^\sharp$$

and involution

$$\tilde{\Lambda}_\varphi(x)^\sharp = \tilde{\Lambda}_\varphi(x^\sharp) \qquad x \in B_\varphi \cap B_\varphi^\sharp$$

(2) *The closure of the involution $\sharp$ in $L^2(G,H)$ has the polar decomposition*

$$\tilde{S}_\varphi = \tilde{J}\tilde{\Delta}_\varphi^{\frac{1}{2}}$$

(3) *For $x \in B_\varphi \cap B_\varphi^\sharp$ and $t \in \mathbf{R}$*

$$\tilde{\Lambda}_\varphi(\varrho_t^\varphi(x)) = \tilde{\Delta}_\varphi^{it}\tilde{\Lambda}_\varphi(x)$$

(4) *$\mathscr{L}(\mathfrak{A}_\varphi) = M \otimes_\alpha G$.*

PROOF. (3) For $x \in B_\varphi \cap B_\varphi^\sharp$ and $t \in \mathbf{R}$

$$\begin{aligned}\tilde{\Lambda}_\varphi(\varrho_t^\varphi(x))(g) &= \Lambda_\varphi(\Delta_G(g)^{it}\sigma_t^{\varphi\circ\alpha_g,\varphi}(x(g)))\\ &= \Delta_G(g)^{it}(D\varphi\circ\alpha_g\colon D\varphi)_t\Lambda_\varphi(\sigma_t^\varphi x(g))\\ &= \Delta_G(g)^{it}(D\varphi\circ\alpha_g\colon D\varphi)_t\Delta_{\varphi\circ\alpha_g,\varphi}^{it}\Lambda_\varphi(x(g))\\ &= (\tilde{\Delta}_\varphi^{it}\tilde{\Lambda}_\varphi(x))(g)\,.\end{aligned}$$

(2) For $x \in B_\varphi \cap B_\varphi^\#$ we get using lemma 2.11:

$$\begin{aligned}(\tilde{J}\tilde{\Lambda}_\varphi(x^\#))(g) &= \Delta_G(g)^{-\frac{1}{2}}u_g^{-1}J\Lambda_\varphi(x^\#(g^{-1}))\\ &= \Delta_G(g)^{\frac{1}{2}}Ju_g^{-1}\Lambda_\varphi(\alpha_g x(g)^*)\\ &= \Delta_G(g)^{\frac{1}{2}}J\Lambda_{\varphi\circ\alpha_g}(x(g)^*)\\ &= \Delta_G(g)^{\frac{1}{2}}JS_{\varphi\circ\alpha_g,\varphi}\Lambda_\varphi(x(g))\\ &= \Delta_G(g)^{\frac{1}{2}}\Delta_{\varphi\circ\alpha_g,\varphi}^{\frac{1}{2}}(\tilde{\Lambda}_\varphi(x))(g)\,.\end{aligned}$$

Hence by lemma 2.10 $\tilde{\Lambda}_\varphi(x) \in D(\tilde{\Delta}_\varphi^{\frac{1}{2}})$ and

$$\tilde{J}\tilde{\Lambda}_\varphi(x^\#) = \tilde{\Delta}_\varphi^{\frac{1}{2}}\tilde{\Lambda}_\varphi(x)$$

or equivalently

$$\tilde{\Lambda}_\varphi(x^\#) = \tilde{J}\tilde{\Delta}_\varphi^{\frac{1}{2}}\tilde{\Lambda}_\varphi(x)\,.$$

Thus # is preclosed and its closure $\tilde{S}_\varphi$ satisfies $\tilde{S}_\varphi \subseteqq \tilde{J}\tilde{\Delta}_\varphi^{\frac{1}{2}}$.

By (3) and lemma 2.5 (4) it follows that $\mathfrak{A}_\varphi = \tilde{\Lambda}_\varphi(B_\varphi \cap B_\varphi^\#)$ is a $\tilde{\Delta}_\varphi^{it}$-invariant, dense subset of $L^2(G,H)$. Let q be the projection on $((1+\tilde{\Delta}_\varphi)^{\frac{1}{2}}\mathfrak{A}_\varphi)^\perp$.

Since q commutes with $\tilde{\Delta}_\varphi^{it}$ for any $t \in \mathbf{R}$, we have

$$q(1+\tilde{\Delta}_\varphi)^{\frac{1}{2}} \subseteqq (1+\tilde{\Delta}_\varphi)^{\frac{1}{2}}q\,.$$

Hence for any $\xi \in \mathfrak{A}_\varphi$

$$(1+\Delta_\varphi)^{\frac{1}{2}}(q\xi) = q(1+\tilde{\Delta}_\varphi)^{\frac{1}{2}}\xi = 0\,,$$

which proves that $q\xi = 0$ for any $\xi \in \mathfrak{A}_\varphi$. Hence $q = 0$ and thus $(1+\tilde{\Delta}_\varphi)^{\frac{1}{2}}\mathfrak{A}_\varphi$ is dense in $L^2(G,H)$. Therefore $\mathfrak{A}_\varphi$ is a core of $\tilde{\Delta}_\varphi^{\frac{1}{2}}$. Hence $\tilde{S}_\varphi = \tilde{J}\tilde{\Delta}_\varphi^{\frac{1}{2}}$ and by the uniqueness of the polar decomposition we get (2).

(1). We check the four conditions in the definition of a left Hilbert algebra (cf. [11, definition 2.1]). By lemma 2.5 (1) we have

$$\tilde{\Lambda}_\varphi(x * y) = \mu(x)\tilde{\Lambda}_\varphi(y), \qquad x,y \in B_\varphi \cap B_\varphi^\#\,.$$

Hence the map $\eta \to \xi\eta$ is continuous for any $\xi \in \mathfrak{A}_\varphi$. Moreover $\pi_l(\tilde{\Lambda}_\varphi(x)) = \mu(x)$, $x \in B_\varphi \cap B_\varphi^\#$. It follows from the formula $\mu(x^\#) = \mu(x)^*$, $x \in K(G,M)$ that

$$(\xi\eta \mid \zeta) = (\eta \mid \xi^\#\zeta) \qquad \forall\, \xi,\eta,\zeta \in \mathfrak{A}_\varphi\,.$$

Since $\tilde{\Lambda}_\varphi(x*y)=\mu(x)\tilde{\Lambda}_\varphi(y)$, $x,y \in B_\varphi \cap B_\varphi^\sharp$ and since $\mu(B_\varphi \cap B_\varphi^\sharp)$ is σ-weakly dense in $M\otimes_\alpha G$ by lemma 2.5 (3) it follows that $(\mathfrak{A}_\varphi)^2$ is dense in $L^2(G,H)$. From (2) we get that $\sharp$ is preclosed. Hence $\mathfrak{A}_\varphi$ is a left Hilbert algebra.

(4) We have $\pi_l(\mathfrak{A}_\varphi)=\mu(B_\varphi \cap B_\varphi^\sharp)$ is σ-weakly dense in $M\otimes_\alpha G$. Hence $\mathscr{L}(\mathfrak{A}_\varphi) = M\otimes_\alpha G$.

REMARK. The method in the proof of lemma 2.12 (2) can be used to prove that condition VIII in the definition of modular Hilbert algebras (cf. [11, Definition 2.1]):

$$(1+\Delta(t))\mathfrak{A} \text{ is dense in } \mathfrak{A} \qquad \forall t \in \mathbf{R}$$

can be deduced from the other seven conditions.

PROOF OF THEOREM 2.1. Let l and r be the left and right regular representations of G on $L^2(G)$:

$$(l(g)f)(h) = f(g^{-1}h), \qquad f \in L^2(G)$$

$$(r(g)f)(h) = \Delta_G^{\frac{1}{2}}(h)f(hg), \qquad f \in L^2(G)\,.$$

Put $(U\xi)(g)=u(g)\xi(g)$, $\xi \in L^2(G,H)$, then U is a unitary operator on $L^2(G,H)$. We get

$$(\pi(x)\xi)(g) = (\alpha_g^{-1}x)\xi(g) = u(g)^*xu(g)\xi(g)$$

$$= (U^*(x\otimes 1)U\xi)(g), \qquad \xi \in L^2(G,H)\,.$$

Hence $\pi(x)=U^*(x\otimes 1)U$, $x \in M$. Moreover

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h) = ((1\otimes l(g))\xi)(h)\,.$$

Hence $M\otimes_\alpha G=\{\pi(M),\lambda(G)\}''$ is generated by $U^*(M\otimes 1)U$ and $1\otimes\mathscr{L}(G)$.

(2) As in the proof of [4, Theorem 3.14] one can reduce the general case, to the case where M is on standard form, and $u(g)$ is the canonical implementation of G. Let $\mathfrak{A}_\varphi$ be the Hilbert algebra on $L^2(G,H)$ constructed in lemma 2.12. We have

$$M\otimes_\alpha G = \mathscr{L}(\mathfrak{A}_\varphi)\,.$$

Hence by the fundamental theorem of the Tomita–Takesaki theory we have

$$(M\otimes_\alpha G)' = \tilde{J}(M\otimes_\alpha G)\tilde{J}$$

where

$$(\tilde{J}\xi)(g) = \Delta(g)^{-\frac{1}{2}}u(g)^{-1}J\xi(g^{-1}), \qquad \xi \in L^2(G,H)\,.$$

(cf. lemma 2.11 (2)). An elementary calculation shows that (cf. proof of [4, corollary 3.13]):

$$(\tilde{J}\pi(x)\tilde{J}\xi)(g) = JxJ\xi(g), \quad x \in M,\ g \in G.$$

and

$$(\tilde{J}\lambda(g)\tilde{J}\xi)(h) = \Delta_G(g)^{+\frac{1}{2}}u(g)\xi(hg), \quad g,h \in G.$$

Hence

$$\tilde{J}\pi(M)\tilde{J} = (JMJ)\otimes 1 = M'\otimes 1$$

and

$$\tilde{J}\lambda(g)\tilde{J} = u(g)\otimes r(g) = U^*(1\otimes r(g))U, \quad g \in G,$$

which proves that $(M\otimes_\alpha G)'$ is generated by $M'\otimes 1$ and $U^*(1\otimes\mathscr{R}(G))U$.

3. The dual weights on $M\otimes_\alpha G$.

DEFINITION 3.1. Let $\varphi \in P(M)$. *The weight* $\tilde{\varphi}$ *on* $M\otimes_\alpha G$ associated with the left Hilbert algebra $\mathfrak{A}_\varphi$ in lemma 2.12 is called the dual weight of φ.

THEOREM 3.2. (1) *For any* $x \in B_\varphi$

$$\tilde{\varphi}(\mu(x^\# * x)) = \varphi((x^\# * x)(e)).$$

(2) *The automorphism group* $\sigma_t^{\tilde{\varphi}}$ *is given by*

$$\sigma_t^{\tilde{\varphi}}(\pi(x)) = \pi(\sigma_t^\varphi(x)), \quad x \in M,\ t \in \mathbf{R}$$

$$\sigma_t^{\tilde{\varphi}}(\lambda(g)) = \Delta_G(g)^{it}\lambda(g)\pi((D\varphi\circ\alpha_g : D\varphi)_t), \quad g \in G,\ t \in \mathbf{R}.$$

(3) *For* $\varphi,\psi \in P(M)$

$$(D\tilde{\psi} : D\tilde{\varphi})_t = \pi((D\psi : D\varphi)_t), \quad t \in \mathbf{R}.$$

PROOF. (1) By [1, Definition 2.12] we have for $x \in \mathscr{L}(G)_+$:

$$\tilde{\varphi}(x) = \begin{cases} \|\xi\|^2 & \text{if } x=\pi_l(\xi)^*\pi_l(\xi),\ \xi \in L^2(G,H), \text{ is left bounded} \\ \infty & \text{otherwise}. \end{cases}$$

Let $x \in B_\varphi$. We can choose a net $(a_i)_{i\in I}$ of operators in n_φ^*, that converges strongly to 1. For any $i \in I$

$$a_ix \in n_\varphi^*\cdot K(G,M)\cdot n_\varphi \subseteqq B_\varphi \cap B_\varphi^\#.$$

Moreover $\mu(a_ix)=\pi(a_i)\mu(x) \to \mu(x)$ strongly and

$$\tilde{\Lambda}_\varphi(a_ix) = \pi(a_i)\tilde{\Lambda}_\varphi(x) \to \tilde{\Lambda}_\varphi(x) \quad \text{in } L^2(G,H).$$

Since $\tilde{\Lambda}_\varphi(a_i x) \in \mathfrak{A}_\varphi$ it follows that $\tilde{\Lambda}_\varphi(a_i x)$ is left bounded, and $\pi_l(\tilde{\Lambda}_\varphi(a_i x)) = \mu(a_i x)$ for any $i \in I$. Let $\eta \in \mathfrak{A}'_\varphi$ (the associated right Hilbert algebra). Then

$$\pi_r(\eta)\tilde{\Lambda}_\varphi(x) = \lim \pi_r(\eta)\tilde{\Lambda}_\varphi(a_i x) = \lim \mu(a_i x)\eta = \mu(x)\cdot\eta .$$

Hence $\tilde{\Lambda}_\varphi(x)$ is left bounded and $\pi_l(\tilde{\Lambda}_\varphi(x)) = \mu(x)$. Hence by lemma 2.5 (2)

$$\tilde{\varphi}(\mu(x^\# * x)) = \tilde{\varphi}(\mu(x)^*\mu(x)) = (\tilde{\Lambda}_\varphi(x) \mid \tilde{\Lambda}_\varphi(x)) = \varphi((x^\# * x)(e)) .$$

(2) follows from the equation $\sigma_t^{\tilde{\varphi}}(x) = \tilde{\Delta}^{it} x \tilde{\Delta}^{it}$, $x \in M \otimes_\alpha G$. (cf. proof of [4, corollary 3.10] and [10, proposition 1]).

For the proof of (3) we need some lemmas:

LEMMA 3.3. *Let ω be a* n.f.s. *weight on $M \otimes_\alpha G$ that satisfies* (1) *and* (2) *in theorem* 3.2 *(with $\tilde{\varphi}$ replaced by ω) then $\omega = \tilde{\varphi}$.*

PROOF. Using (2) we get for $x \in K(G, M)$:

$$\sigma_t^{\tilde{\varphi}}(\mu(x)) = \int_G (\sigma_t^{\tilde{\varphi}}\lambda(g))(\sigma_t^{\tilde{\varphi}}\pi(x(g)))\,dg$$

$$= \int_G \Delta_G(g)^{it}\lambda(g)\pi((D\varphi\circ\alpha_g : D\varphi)_t)\sigma_t^\varphi(x)\,dg = \mu(\varrho_t^\varphi(x)) .$$

Since B_φ and $B_\varphi^\#$ are ϱ_t^φ-invariant by lemma 2.7 it follows that $\mu(B_\varphi^\# B_\varphi)$ is a $\sigma_t^{\tilde{\varphi}}$-invariant subalgebra of $n_{\tilde{\varphi}}^* n_{\tilde{\varphi}} = m_{\tilde{\varphi}}$. Moreover $\mu(B_\varphi^\# B_\varphi)$ is σ-weakly dense in $M \otimes_\alpha G$. By the assumptions ω and $\tilde{\varphi}$ coincide on

$$\mu(B_\varphi^\# B_\varphi) = \operatorname{span}\{\mu(x^\# * x) \mid x \in B_\varphi\} .$$

Moreover ω and $\tilde{\varphi}$ has the same modular automorphism group. Hence by [9, proposition 5.9] it follows that $\omega = \tilde{\varphi}$.

Let F_2 be the algebra of 2×2-matrices, and let $(e_{ij})_{i,j} = 1,2$ be the natural basis for F_2.

We consider the crossed product of $M \otimes F_2$ and G with respect to the action $\beta = \alpha \otimes i$ (i = identity on F_2). With obvious identifications we have

$$(M \otimes F_2) \otimes_\beta G = (M \otimes_\alpha G) \otimes F_2 .$$

LEMMA 3.4. *Let $\varphi, \psi \in P(M)$ and define a* n.f.s. *weight θ on $M \otimes F_2$ by*

$$\theta(\sum x_{ij} \otimes e_{ij}) = \varphi(x_{11}) + \psi(x_{22}) \quad \textit{for } \sum x_{ij} \otimes e_{ij} \in (M \otimes F_2)_+$$

then

$$(D\theta\circ\beta_g : D\theta)_t = (D\varphi\circ\alpha_g : D\varphi)_t \otimes e_{11} + (D\psi\circ\alpha_g : D\psi) \otimes e_{22} .$$

PROOF. For $\sum x_{ij}\otimes e_{ij} \in (M\otimes F_2)_+$ we get:

$$(\theta\circ\beta_g)(\sum x_{ij}\otimes e_{ij}) = (\varphi\circ\alpha_g)(x_{11}) + (\psi\circ\alpha_g)(x_{22}) .$$

Hence by [2, lemma 1.2.2] it follows that $1\otimes e_{ii}$, $i=1,2$, are in the centralizer for $\theta\circ\beta_g$ for any $g \in G$. Using the formula (cf. § 1.3)

$$\sigma_t^{\psi,\varphi}(xy) = \sigma_t^{\psi,\omega}(x)\sigma_t^{\omega,\varphi}(y) \qquad x,y \in M,\ \varphi,\psi,\omega \in P(M)$$

twice we get:

$$\sigma_t^{\theta\circ\beta_g,\theta}(1\otimes e_{ii}) = \sigma_t^{\theta\circ\beta_g}(1\otimes e_{ii})\sigma_t^{\theta\circ\beta_g,\theta}(1\otimes e_{ii})\sigma_t^{\theta}(1\otimes e_{ii})$$
$$= (1\otimes e_{ii})\sigma_t^{\theta\circ\beta_g,\theta}(1\otimes e_{ii})(1\otimes e_{ii}) .$$

Hence

$$\sigma_t^{\theta\circ\beta_g,\theta}(1\otimes e_{ii}) = u_i\otimes e_{ii} \qquad \text{for some } u_i \in M,\ i=1,2 .$$

Using the K.M.S. conditions for $(D\psi : D\varphi)_t$ ([4 proposition 2.2]) one gets easily $u_1 = (D\varphi\circ\alpha_g : D\varphi)_t$. Similarly $u_2 = (D\psi\circ\alpha_g : D\psi)_t$. Hence:

$$(D\theta\circ\beta_g : D\theta)_t = \sigma_t^{\theta\circ\beta_g,\theta}(1) = \sigma_t^{\theta\circ\beta_g,\theta}(1\otimes e_{11}) + \sigma_t^{\theta\circ\beta_g,\theta}(1\otimes e_{22})$$
$$= (D\varphi\circ\alpha_g : D\varphi)_t\otimes e_{11} + (D\psi\circ\alpha_g : D\psi)_t\otimes e_{22} .$$

LEMMA 3.5. *Let φ, ψ and θ be as in lemma 3.4, and let $\tilde{\varphi}$, $\tilde{\psi}$ and $\tilde{\theta}$ be their dual weights. Then*

$$\tilde{\theta}(\sum y_{ij}\otimes e_{ij}) = \tilde{\varphi}(y_{11}) + \tilde{\psi}(y_{22}) \qquad \text{for } \sum y_{ij}\otimes e_{ij} \in ((M\otimes_\alpha G)\otimes F_2)_+ .$$

PROOF. Let $(\bar{\pi},\bar{\lambda})$ be the covariant representation of $(M\otimes F_2, G, \beta)$ that generates the crossed product $(M\otimes F_2)\otimes_\beta G = (M\otimes_\alpha G)\otimes F_2$. We have

$$\bar{\pi}(\sum x_{ij}\otimes e_{ij}) = \sum \pi(x_{ij})\otimes e_{ij}, \qquad x_{ij} \in M$$
$$\bar{\lambda}(g) = \lambda(g)\otimes 1, \qquad g \in G .$$

The associated representation $\bar{\mu}$ of $K(G, M\otimes F_2) = K(G,M)\otimes F_2$ is given by

$$\bar{\mu}(\sum y_{ij}\otimes e_{ij}) = \sum \mu(y_{ij})\otimes e_{ij}, \qquad y_{ij} \in K(G,M) .$$

By Theorem 3.2 (1) we get

$$\sigma_t^{\tilde{\theta}}(1\otimes e_{ii}) = \bar{\pi}(\sigma_t^{\theta}(1\otimes e_{ii})) = \bar{\pi}(1\otimes e_{ii}) = 1\otimes e_{ii}, \qquad i=1,2 .$$

Hence $1\otimes e_{11}$ and $1\otimes e_{22}$ are in the centrilizer of $\tilde{\theta}$. Thus by [9, proposition 4.1]

$$\tilde{\theta}(x) = \sum_{i=1}^{2} \tilde{\theta}((1\otimes e_{ii})x(1\otimes e_{ii})), \qquad x \in (M\otimes_\alpha G)\otimes F_2 ,$$

or equivalently

$$\tilde{\theta}(\sum x_{ij}\otimes e_{ij}) = \omega_1(x_{11})+\omega_2(x_{22}),\ x_{ij} \in M\otimes_\alpha G$$

where

$$\omega_i(x) = \tilde{\theta}(x\otimes e_{ii}), \quad i=1,2\,,$$

are n.f.s. weights on $M\otimes_\alpha G$. We will prove that $\omega_1=\varphi$ and $\omega_2=\psi$ by the use of lemma 3.3. Using Theorem 3.1(2) we get for $x \in M$:

$$\sigma_t^{\omega_1}(\pi(x))\otimes e_{11} = \sigma_t^{\tilde{\theta}}(\pi(x)\otimes e_{11}) = \sigma_t^{\tilde{\theta}}(\bar{\pi}(x\otimes e_{11}))$$
$$= \bar{\pi}(\sigma_t^{\theta}(x\otimes e_{11})) = \bar{\pi}(\sigma_t^{\varphi}(x)\otimes e_{11}) = \pi(\sigma_t^{\varphi}(x))\otimes e_{11}\,.$$

Hence $\sigma_t^{\omega_1}(\pi(x))=\sigma_t^{\tilde{\varphi}}(\pi(x)),\ \forall\, x \in M$.

Moreover by lemma 3.4

$$\sigma_t^{\omega_1}(\lambda(g))\otimes e_{11} = \sigma_t^{\tilde{\theta}}(\lambda(g)\otimes e_{11}) = \sigma_t^{\tilde{\theta}}(\lambda(g)\otimes 1)\sigma_t^{\tilde{\theta}}(1\otimes e_{11})$$
$$= \sigma_t^{\tilde{\theta}}(\bar{\lambda}(g))(1\otimes e_{11}) = \Delta_G(g)^{it}\bar{\lambda}(g)\bar{\pi}((D\theta\circ\beta_g : D\theta)_t)(1\otimes e_{11})$$
$$= \Delta_G(g)^{it}\lambda(g)\pi((D\varphi\circ\alpha_g : D\theta)_t)\otimes e_{11} = \sigma_t^{\tilde{\varphi}}(\lambda(g))\otimes e_{11}\,.$$

Hence $\sigma_t^{\omega_1}(\lambda(g))=\sigma_t^{\tilde{\varphi}}(\lambda(g))$.

Let $x \in B_\varphi=K(G,M)\cdot n_\varphi$. Then

$$x\otimes e_{11} \in (K(G,M)(1\otimes e_{11}))\cdot(n_\varphi\otimes e_{11}) \subseteq K(G,\ M\otimes F_2)\cdot n_\theta = B_\theta$$

because $n_\varphi\otimes e_{11}\subseteq n_\theta$. Put $y=x^{\#}*x$. Then

$$y\otimes e_{11} = (x\otimes e_{11})^{\#}*(x\otimes e_{11})\,.$$

Hence by Theorem 3.1(1)

$$\omega_1(\mu(x^{\#}*x)) = \tilde{\theta}(\mu(y)\otimes e_{11}) = \tilde{\theta}(\mu(y\otimes e_{11}))$$
$$= \tilde{\theta}((y\otimes e_{11})(e)) = \theta(y(e)\otimes e_{11}) = \varphi((x^{\#}*x)(e))\,.$$

Hence $\omega_1=\tilde{\varphi}$ by lemma 3.3. Similarly $\omega_2=\tilde{\psi}$. This completes the proof.

End of the proof of Theorem 3.2. (3). Let $\tilde{\varphi}$, $\tilde{\psi}$ and $\tilde{\theta}$ be as in lemma 3.5. Then by [2, lemma 1.2.2] and theorem 3.2(2) we get

$$(D\tilde{\psi} : D\tilde{\varphi})_t\otimes e_{21} = \sigma_t^{\tilde{\theta}}(1\otimes e_{21}) = \sigma_t^{\tilde{\theta}}(\bar{\pi}(1\otimes e_{21}))$$
$$= \bar{\pi}(\sigma_t^{\theta}(1\otimes e_{21})) = \bar{\pi}((D\psi : D\varphi)_t\otimes e_{21}) = \pi((D\psi : D\varphi)_t)\otimes e_{21}\,.$$

Hence (3) is proved.

When G is an abelian group, one can define a dual action $\hat{\alpha}$ of the dual group $\hat{G}$ on $M\otimes_\alpha G$ (cf. [12, definition 4.1]). The automorphisms $\hat{\alpha}_p$, $p \in \hat{G}$ can be characterized by their action on the generators

$$\hat{\alpha}_p(\pi(x)) = \pi(x), \qquad x \in M,\ p \in \hat{G}$$

$$\hat{\alpha}_p(\lambda(g)) = \overline{\langle p, g\rangle}\lambda(g), \qquad g \in G,\ p \in \hat{G}\ .$$

The following lemma is due to Landstad [8, § 2.5 Theorem 2]. For convenience we will give a short proof, using the same ideas.

LEMMA 3.6.

$$\pi(M) = \{x \in M\otimes_\alpha G \mid \hat{\alpha}_p(x) = x,\ \forall p \in \hat{G}\}\ .$$

PROOF. Put

$$N = \{x \in M\otimes_\alpha G \mid \hat{\alpha}_p(x) = x,\ \forall p \in \hat{G}\}\ .$$

Clearly $\pi(M) \subseteq N$. We may assume that M is represented on a Hilbert space H such that $\alpha: G \to \operatorname{aut}(M)$ has a strongly continuous unitary implementation $g \to u(g)$. By [8, equation 2.13] we get

$$\pi(M) = (M'\otimes 1)' \cap (U^*(1\otimes\mathscr{R}(G))U)' \cap (1\otimes L^\infty(G))'$$

where $(U\xi)(g) = u(g)\xi(g)$, $\xi \in L^2(G,H)$.

By the commutation Theorem (Theorem 2.1) we have

$$N \subseteq M\otimes_\alpha G \subseteq (M'\otimes 1)' \cap (U^*(1\otimes\mathscr{R}(G))U)'$$

(note that this inclusion can be proved by elementary means).

Since $\hat{\alpha}_p$ is implemented by the unitary $\mu(p)$ given by

$$(\mu(p)\xi)(g) = \overline{\langle p, g\rangle}\xi(g)$$

and since $\mu(p)$, $p \in \hat{G}$ generates $1\otimes L^\infty(G)$, we have $N \subseteq (1\otimes L^\infty(G))'$. Hence $N \subseteq \pi(M)$. This completes the proof.

THEOREM 3.7. *Let $M\otimes_\alpha G$ be the crossed product of a von Neumann algebra with an abelian locally compact group G. The map $\varphi \to \tilde{\varphi}$ is a bijection of $P(M)$ onto the set of* n.f.s. *weights on $M\otimes_\alpha G$, that are invariant under the dual action.*

PROOF. Let $\varphi \in P(M)$. Then $\tilde{\varphi}$ is $\hat{\alpha}$-invariant (same proof as in [4, proposition 4.1]). It follows from Theorem 3.2(3) that the map $\varphi \to \tilde{\varphi}$ is injective.

Let ω be an $\hat{\alpha}$-invariant weight on $M\otimes_\alpha G$, and choose $\varphi \in P(M)$. Since both $\tilde{\varphi}$ and ω are $\hat{\alpha}$-invariant we get by [4, corollary 2.3] that

$$\hat{\alpha}_p((D\omega : D\tilde{\varphi})_t) = (D\omega : D\tilde{\varphi})_t, \quad p \in \hat{G}.$$

Hence $(D\omega : D\tilde{\varphi})_t \in \pi(M)$ by lemma 3.6. Put $u_t = \pi^{-1}((D\omega : D\tilde{\varphi})_t)$. We have by Theorem 3.2(2) that

$$\pi(u_{s+t}) = (D\omega : D\tilde{\varphi})_{s+t} = (D\omega : D\tilde{\varphi})_s \sigma_s^{\tilde{\varphi}}(D\omega : D\tilde{\varphi})_t$$
$$= \pi(u_s \sigma_s^{\varphi}(u_t)).$$

Hence by [2, Theorem 1.2.4] there exists a n.f.s. weight ψ on M, such that

$$(D\psi : D\varphi)_t = u_t, \quad t \in \mathbf{R}.$$

By Theorem 3.2(3) it follows that

$$(D\tilde{\psi} : D\tilde{\varphi})_t = \pi(u_t) = (D\omega : D\tilde{\varphi})_t.$$

Hence $\tilde{\varphi} = \omega$. This proves that the map $\varphi \to \tilde{\varphi}$ is surjective.

Remark. In a subsequent paper we will give an alternative construction of the dual weights, by the use of operator valued weights [7]. It will follow that the map $\varphi \to \tilde{\varphi}$ has a natural extension to all normal weights on M, such that

$$(\varphi + \psi)^{\sim} = \tilde{\varphi} + \tilde{\psi}.$$

Moreover we obtain a slight extension of Theorem 3.2(1) namely

$$\tilde{\varphi}(\mu(x^{\#} * x)) = \varphi((x^{\#} * x)(e)), \quad x \in K(G, M).$$

REFERENCES

1. F. Combes, *Poids associé à une algèbre Hilbertienne à gauche*, Compositio Math. 23 (1971), 49–77.
2. A. Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup. 4 série, 6 (1973), 133–252.
3. T. Digernes, *Poids dual sur un produit croisé*, C.R. Acad. Sci. Paris. Sér. A 278 (1974), 937–940.
4. T. Digernes, *Duality for weights on covariant systems and its application*, Thesis, University of California, Los Angeles, 1975.
5. S. Doplicher, D. Kastler and D. W. Robinson, *Covariance algebras in field theory and statistical mechanics*, Comm. Math. Phys. 3 (1966), 1–28.
6. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. 37 (1975), 271–283.
7. U. Haagerup, *Operator valued weights in von Neumann algebras*, to appear in J. Functional Analysis.
8. M. Landstad, *Duality for covariant systems*, preprint, Trondheim, 1974.
9. G. K. Pedersen and M. Takesaki, *The Radon Nikodym Theorem for von Neumann algebras*, Acta Math. 130 (1973), 53–87.

10. L. Sauvageot, *Sur le type de produit croisé d'une algèbre de von Neumann par une groupe localement compacte automorphisms*, C.R. Acad. Sci. Paris, Sér. A 278 (1974), 941–944.

11. M. Takesaki, *Tomitas theory of modular Hilbert algebras and its applications* (Lecture Notes in Mathematics 128), Springer-Verlag, Berlin - Heidelberg - New York, 1970.

12. M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type* III, Acta Math. 131 (1973), 249–308.

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