MATH. SCAND. 43 (1978), 141–156

ON UNICITY OF THE RIESZ DECOMPOSITION OF AN EXCESSIVE MEASURE

JESPER LAUB

Let G be a locally compact abelian group and N a convolution kernel satisfying the domination principle. In a series of papers ([2]–[5]) M. Itô has studied positive measures ξ on G for which N satisfies the relative domination principle with respect to ξ. These measures, which here will be called N-excessive in analogy with the Hunt kernel case, are treated using the method of reduced measures. The use of reduced measures rely on the fact, that the set of N-excessive measures is stable under the formation of infimum.

The Riesz decomposition theorem for excessive measures which M. Itô proved in [3], [5] is proved in full generality including unicity for non-singular kernels. As an application it turns out that the invariant part in the Riesz decomposition is characterized by invariance under reduction on the complement of compact sets. Moreover we give a simple proof for the fact, that the regular part N_0 of N satisfies the relative balayage principle with respect to the singular part N' of N.

1. Excessive measures.

The concept of excessive measures will rely deeply on domination and balayage principles and therefore we first state a few facts concerning these principles.

We denote by C_K^+ the positive continuous functions on G with compact support. The integral of a function $\varphi \in C_K^+$ with respect to a positive measure μ on G will be written $\langle \mu, \varphi \rangle$.

DEFINITION. The convolution kernel N_1 is said to satisfy the relative (respectively transitive) domination principle with respect to N_2 if for all $f, g \in C_K^+$

$$N_1 * f \leqq N_2 * g \text{ (respectively } N_1 * f \leqq N_1 * g\text{) on supp } f$$

$$\text{implies } N_1 * f \leqq N_2 * g \text{ (respectively } N_2 * f \leqq N_2 * g\text{)}$$

where supp f denotes the support of f.

Received October 14, 1977.

If N_1 satisfies the relative (respectively transitive) domination principle with respect to N_2 we will write $N_1 \prec N_2$ (respectively $N_1 \sqsubset N_2$).

The convolution kernel N is said to satisfy the domination principle if $N \prec N$.

REMARK. M. Itô has recently proved ([4]), that the two principles are equivalent for non-zero kernels and moreover, that they are equivalent to the following principle.

DEFINITION. The convolution kernel N_1 is said to satisfy the relative balayage principle with respect to N_2 if the following statement holds:

For every positive measure μ with compact support and every open relatively compact set $\omega \subseteqq G$, there exists a positive measure μ_ω with the property

$$\operatorname{supp} \mu_\omega \subseteqq \bar{\omega}, \quad N_1 * \mu_\omega \leqq N_2 * \mu$$

$$N_1 * \mu_\omega = N_2 * \mu \quad \text{in } \omega .$$

The measure μ_ω is called a balayaged measure of μ on ω relative to (N_1, N_2).

We will not need the full equivalence of these principles but only the following more easily established proposition (cf. [3]).

PROPOSITION 1.1. *Let N_1 and N_2 be convolution kernels, for which $N_1 \neq 0$ and $N_1 \prec N_1 \prec N_2$. Then N_1 satisfies the relative balayage principle with respect to N_2.*

If N is a convolution kernel, then $D^+(N)$ will denote the set of positive measures μ for which $N * \mu$ exist.

The following domination principles for measures are easily proved by first considering measures with compact support and by regularization.

LEMMA 1.2. *Let N_1 and N_2 be non-zero convolution kernels satisfying $N_1 \prec N_2$ (respectively $N_1 \sqsubset N_2$). If $\mu \in D^+(N_1)$, $\nu \in D^+(N_2)$ (respectively $\mu, \nu \in D^+(N_1) \cap D^+(N_2)$) and ω is an open set with* $\operatorname{supp} \mu \subseteqq \omega$, *then*

$$N_1 * \mu \leqq N_2 * \nu \text{ in } \omega \text{ implies } N_1 * \mu \leqq N_2 * \nu$$

(respectively

$$N_1 * \mu \leqq N_1 * \nu \text{ in } \omega \text{ implies } N_2 * \mu \leqq N_2 * \nu)$$

In the rest of the paper N is a fixed non-zero convolution kernel satisfying the domination principle.

DEFINITION. A positive measure ξ is called N-excessive if $N \prec \xi$.

The set of N-excessive measures will be denoted $\varepsilon(N)$ and it is easily seen (cf. [2]), that $\varepsilon(N)$ is a vaguely closed convex cone. (The vague topology on the set of positive measures is defined by the requirement, that a net $(\mu_\alpha)_{\alpha \in A}$ of positive measures converges vaguely to μ ($\mu_\alpha \to \mu$) if $\forall \varphi \in C_K^+ \ \langle \mu_\alpha, \varphi \rangle \to \langle \mu, \varphi \rangle$.) Moreover every N-potential is N-excessive.

The next lemma will enable us to define reduced measures of N-excessive measures.

LEMMA 1.3. *The convex cone $\varepsilon(N)$ is infimum-stable, i.e. if $A \subseteqq \varepsilon(N)$ then* $\inf A \in \varepsilon(N)$.

PROOF. We will first prove that the infimum $\xi \wedge \eta$ of two N-excessive measures is N-excessive.

Let $f, g \in C_K^+$ and suppose that

$$N * f \leqq (\xi \wedge \eta) * g \quad \text{on } \operatorname{supp} f .$$

For every $x \in G$ we have

$$(\xi \wedge \eta) * g(x) = \inf \{ \xi * g_1(x) + \eta * g_2(x) \mid g_1, g_2 \in C_K^+, g_1 + g_2 = g \} .$$

For every $g_1, g_2 \in C_K^+$, such that $g_1 + g_2 = g$ we have

$$N * f \leqq \xi * g_1 + \eta * g_2 \quad \text{on } \operatorname{supp} f .$$

Let ω be an open relatively compact set, with

$$\omega \supseteqq \operatorname{supp} f - \operatorname{supp} g .$$

If ε_0 denotes the Dirac-measure at the neutral element of G, Proposition 1.1 shows the existence of balayaged measures μ_ω and ν_ω of ε_0 on ω relative to (N, ξ) and (N, η) respectively. If $x \in \operatorname{supp} f$ then

$$\begin{aligned} N * f(x) &\leqq \int_\omega g_1(x-y)\, d\xi(y) + \int_\omega g_2(x-y)\, d\eta(y) \\ &= \int g_1(x-y)\, dN * \mu_\omega(y) + \int g_2(x-y)\, dN * \nu_\omega(y) \\ &= N * (\mu_\omega * g_1 + \nu_\omega * g_2) \end{aligned}$$

Now N satisfies the domination principle and hence

$$\begin{aligned} N * f &\leqq N * \mu_\omega * g_1 + N * \nu_\omega * g_2 \\ &\leqq \xi * g_1 + \eta * g_2 \end{aligned}$$

and finally

$$N * f \leqq (\xi \wedge \eta) * g .$$

Let $A \subseteq \varepsilon(N)$ be arbitrary and let A^* be the set of all infimums of finitely many measures from A. In the first part we proved that $A^* \subseteq \varepsilon(N)$, but since $\varepsilon(N)$ is vaguely closed and A^* is downward filtering we get

$$\inf A = \inf A^* \in \varepsilon(N) .$$

DEFINITION. If ξ is a N-excessive measure and $\omega \subseteq G$ and open set, then

$$R_\xi^\omega = \inf\{\eta \in \varepsilon(N) \mid \eta \geqq \xi \text{ in } \omega\}$$

is called the reduced measure of ξ on ω (with respect to N).

The following properties of R_ξ^ω are immediate from the definition and Lemma 1.3:

$$R_\xi^\omega \in \varepsilon(N), \quad R_\xi^\omega \leqq \xi, \quad R_\xi^\omega = \xi \text{ in } \omega$$

$$R_\xi^\omega \text{ is increasing in } \xi \text{ and } \omega$$

and for $\eta \in \varepsilon(N)$ the following implication holds

$$R_\xi^\omega \leqq \eta \text{ in } \omega \Rightarrow R_\xi^\omega \leqq \eta$$

REMARK. The reduced measure is the same as the balayaged pseudo-potential or balayaged convolution kernel considered by M. Itô (cf. e.g. [3, p. 305]). This treatment of these measures was suggested by C. Berg.

The next five lemmas will give us the tools, which are necessary in handling the reduced measures.

LEMMA 1.4. *Let $(\xi_i)_{i\in I}$ be an increasing net of N-excessive measures and ω an open set and suppose that $\xi = \lim_I \xi_i$ exists. Then ξ is N-excessive, $(R_{\xi_i}^\omega)_{i\in I}$ is increasing and*

$$\lim_I R_{\xi_i}^\omega = R_\xi^\omega .$$

If $(\omega_\alpha)_{\alpha\in A}$ is an increasing net of open sets with $\omega_\alpha \uparrow \omega$, then

$$R_\xi^{\omega_\alpha} \uparrow R_\xi^\omega .$$

PROOF. It is immediate that $\xi \in \varepsilon(N)$ and that $(R_{\xi_i}^\omega)_{i\in I}$ is increasing. Hence $\lim_I R_{\xi_i}^\omega$ exists and

$$\lim_I R_{\xi_i}^\omega \leqq R_\xi^\omega .$$

The measure $\lim_I R^\omega_{\xi_i}$ is N-excessive, and

$$\lim_I R^\omega_{\xi_i} = \lim_I \xi_i = \xi \quad \text{in } \omega$$

which implies $R^\omega_\xi \leqq \lim_I R^\omega_{\xi_i}$.

In order to prove the second part of the lemma it is easily seen, that $(R^{\omega_\alpha}_\xi)_{\alpha\in A}$ is increasing and that

$$\lim_A R^{\omega_\alpha}_\xi \leqq R^\omega_\xi .$$

If $\varphi \in C^+_K$ and $\operatorname{supp}\varphi \subseteqq \omega$ we can choose $\alpha_0 \in A$ such that $\operatorname{supp}\varphi \subseteqq \omega_{\alpha_0}$. Then

$$\lim_A \langle R^{\omega_\alpha}_\xi, \varphi\rangle \geqq \langle R^{\omega_{\alpha_0}}_\xi, \varphi\rangle = \langle \xi, \varphi\rangle$$

i.e. $\lim_A R^{\omega_\alpha}_\xi \geqq \xi$ in ω and hence $\lim_A R^{\omega_\alpha}_\xi \geqq R^\omega_\xi$.

LEMMA 1.5. *The reduced measure of $\xi \in \varepsilon(N)$ on an open relatively compact set ω is a N-potential, i.e., $R^\omega_\xi = N*\mu$, with* $\operatorname{supp}\mu \subseteqq \bar\omega$.

PROOF. Let $(\omega_\alpha)_{\alpha\in A}$ be the family of all open relatively compact sets satisfying $\bar\omega_\alpha \subseteqq \omega$ and order A by inclusion of the sets. For $\alpha \in A$ we denote by ε'_α a balayaged measure of ε_0 on ω_α relative to (N,ξ), i.e., ε'_α satisfies

$$N*\varepsilon'_\alpha \leqq \xi$$
$$N*\varepsilon'_\alpha = \xi \quad \text{in } \omega_\alpha$$

and

$$\operatorname{supp}\varepsilon'_\alpha \subseteqq \bar\omega_\alpha \subseteqq \omega .$$

Lemma 1.2 and the fact, that $N*\varepsilon'_\alpha \in \varepsilon(N)$ now implies

$$R^{\omega_\alpha}_\xi \leqq N*\varepsilon'_\alpha \leqq R^\omega_\xi$$

so by Lemma 1.4, $\lim_A N*\varepsilon'_\alpha = R^\omega_\xi$.

As $(N*\varepsilon'_\alpha)_{\alpha\in A}$ is vaguely bounded $(\varepsilon'_\alpha)_{\alpha\in A}$ is vaguely bounded and then a vague cluster-point μ exists, which satisfies $\operatorname{supp}\mu \subseteqq \bar\omega$ and

$$N*\mu = \lim_A N*\varepsilon'_\alpha = R^\omega_\xi .$$

LEMMA 1.6. *A positive measure ξ is N-excessive if and only if ξ is vague limit of an increasing net of N-potentials (of measures with compact support).*

PROOF. Let Ω be the set of all open relatively compact subsets of G. By the above lemma $(R_\xi^\omega)_{\omega\in\Omega}$ is an increasing net of N-potentials and by Lemma 1.4

$$R_\xi^\omega \uparrow R_\xi^G = \xi \quad \text{as } \omega \uparrow G .$$

The converse statement is an immediate consequence of the facts, that $\varepsilon(N)$ is vaguely closed and that a N-potential is N-excessive.

LEMMA 1.7. *If $\xi, \eta \in \varepsilon(N)$ and ω is an open set then*

$$R_{\xi+\eta}^\omega = R_\xi^\omega + R_\eta^\omega .$$

PROOF. From the definition of reduced measures follows

$$R_\xi^\omega + R_\eta^\omega = \xi + \eta \quad \text{in } \omega$$

and hence $R_{\xi+\eta}^\omega \leqq R_\xi^\omega + R_\eta^\omega$.

Let $(\omega_\alpha)_{\alpha\in A}$ be an increasing net of open relatively compact sets satisfying $\bar{\omega}_\alpha \subseteqq \omega$ and $\omega_\alpha \uparrow \omega$. Then by Lemma 1.5 follows that μ_α, ν_α exist such that

$$R_\xi^{\omega_\alpha} = N * \mu_\alpha, \quad R_\eta^{\omega_\alpha} = N * \nu_\alpha$$

where μ_α and ν_α are balayaged measures of ε_0 on ω_α relative to (N,ξ) and (N,η) respectively. The measure $\mu_\alpha + \nu_\alpha$ is a balayaged measure of ε_0 on ω_α relative to $(N, \xi+\eta)$ and then Lemma 1.2 implies

$$\begin{aligned} R_{\xi+\eta}^\omega &\geqq \lim_A N * (\mu_\alpha + \nu_\alpha) \\ &= \lim_A R_\xi^{\omega_\alpha} + \lim_A R_\eta^{\omega_\alpha} \\ &= R_\xi^\omega + R_\eta^\omega . \end{aligned}$$

Let $\mathscr{V}$ be the family of compact neighbourhoods of the neutral element of G. For a N-excessive measure ξ the net $(R_\xi^{\complement V})_{V\in\mathscr{V}}$ is decreasing as V increases towards G. Moreover if $\mu \in D^+(\xi)$ then $\xi * \mu \in \varepsilon(N)$ and the following lemma holds.

LEMMA 1.8. *Let $\xi \in \varepsilon(N)$ and $\mu \in D^+(\xi)$ then*

$$\lim_{V\uparrow G} R_{\xi*\mu}^{\complement V} = \left(\lim_{V\uparrow G} R_\xi^{\complement V}\right) * \mu .$$

PROOF. First we will suppose that the support of μ is compact.
Let $V \in \mathscr{V}$. Since $R_\xi^{\complement V} = \xi$ in $\complement V$ it follows that

$$R_\xi^{\complement V} * \mu = \xi * \mu \quad \text{in } \complement(V + \operatorname{supp} \mu) .$$

The measure $R_\xi^{\complement V} * \mu$ is N-excessive and hence

$$R_{\xi*\mu}^{\complement(V+\operatorname{supp}\mu)} \leqq R_\xi^{\complement V} * \mu$$

and as V increases towards G, we obtain

$$\begin{aligned} \lim_{V \uparrow G} R_{\xi*\mu}^{\complement V} &= \lim_{V \uparrow G} R_{\xi*\mu}^{\complement(V+\operatorname{supp}\mu)} \\ &\leqq \lim_{V \uparrow G} (R_\xi^{\complement V} * \mu) \\ &= \left(\lim_{V \uparrow G} R_\xi^{\complement V}\right) * \mu . \end{aligned}$$

Conversely let $W \in \mathscr{V}$ be given and choose $V \in \mathscr{V}$ such that

$$W - \operatorname{supp} \mu \subseteqq V .$$

According to Lemma 1.5 we have, that $R_\xi^{\complement V}$ is the vague limit of an increasing net $(N * \nu_\alpha)_{\alpha \in A}$ of N-potentials satisfying $\operatorname{supp} \nu_\alpha \subseteqq \complement V$, $\forall \alpha \in A$. Then $R_\xi^{\complement V} * \mu$ is the vague limit of increasing net of N-potentials $(N * \nu_\alpha * \mu)_{\alpha \in A}$ with

$$\operatorname{supp} (\nu_\alpha * \mu) \subseteqq \complement V + \operatorname{supp} \mu \subseteqq \complement W .$$

This implies by Lemma 1.2 that

$$N * \nu_\alpha * \mu \leqq R_{\xi*\mu}^{\complement W} \quad \text{for all } \alpha \in A$$

and hence

$$\left(\lim_{V \uparrow G} R_\xi^{\complement V}\right) * \mu \leqq R_\xi^{\complement V} * \mu \leqq R_{\xi*\mu}^{\complement W} .$$

Finally when W increases

$$\left(\lim_{V \uparrow G} R_\xi^{\complement V}\right) * \mu \leqq \lim_{W \uparrow G} R_{\xi*\mu}^{\complement W} .$$

Now let $\mu \in D^+(\xi)$ be arbitrary. For a compact subset K of G, $\mu|_K$ denotes the restriction of the positive measure μ to K. Let $V \in \mathscr{V}$, then

$$R_{\xi*\mu|K}^{\complement V} \leqq R_{\xi*\mu}^{\complement V}$$

and using the first part of the proof and then letting K increase towards G, we obtain

$$\left(\lim_{V \uparrow G} R_\xi^{\complement V}\right) * \mu \leqq \lim_{V \uparrow G} R_{\xi*\mu}^{\complement V} .$$

Let $\varphi \in C_K^+$ and $\delta > 0$ be given and choose K compact such that

$$\langle \xi * (\mu - \mu|_K), \varphi \rangle < \delta .$$

By Lemma 1.7 we get

$$\begin{aligned}\langle R^{CV}_{\xi*\mu}, \varphi \rangle &= \langle R^{CV}_{\xi*\mu|_K}, \varphi \rangle + \langle R^{CV}_{\xi*(\mu-\mu|_K)}, \varphi \rangle \\ &< \langle R^{CV}_{\xi*\mu|_K}, \varphi \rangle + \delta\end{aligned}$$

and therefore

$$\begin{aligned}\langle \lim_{V\uparrow G} R^{CV}_{\xi*\mu}, \varphi \rangle &\leqq \langle (\lim_{V\uparrow G} R^{CV}_{\xi}) * \mu|_K, \varphi \rangle + \delta \\ &\leqq \langle (\lim_{V\uparrow G} R^{CV}_{\xi}) * \mu, \varphi \rangle + \delta .\end{aligned}$$

Finally letting $\delta \downarrow 0$ the remaining inequality is obtained.

2. The Riesz' Decomposition Theorem.

The net $(R_N^{CV})_{V \in \mathscr{V}}$ is decreasing as V increases towards G, so let

$$N' = \lim_{V\uparrow G} R_N^{CV}, \quad N_0 = N - N' .$$

The convolution kernels N_0, N' are called the regular respectively the singular part of N. Note that $N' \in \varepsilon(N)$ and by Lemma 1.8 we have for $\mu \in D^+(N)$

$$R^{CV}_{N*\mu} \downarrow N' * \mu .$$

The convolution kernel N is called non-singular if $N_0 \neq 0$.

Let Ω denote the set of all open relatively compact subsets of G.

LEMMA 2.1. *Let $\xi \in \varepsilon(N)$ and let μ be a positive measure with compact support. There exist balayaged measures μ_ω of μ on $\omega \in \Omega$ relative to (N, ξ) with the additional property, that*

$$\omega_1 \supseteqq \omega_2 \Rightarrow \mu_{\omega_1} \leqq \mu_{\omega_2} \text{ in } \omega_2 .$$

PROOF. It can be proved (cf. e.g. [2]) that

$$N + c\varepsilon_0 \prec N + c\varepsilon_0 \quad \text{for all } c > 0$$

and similarly it can be seen, that

$$N + c\varepsilon_0 \sqsubset N \quad \text{for all } c > 0 .$$

Moreover it is easily seen that $N + c\varepsilon_0 \prec \xi$. For $c > 0$ and $\omega \in \Omega$ let μ_ω^c be a balayaged measure of μ on ω relative to $(N + c\varepsilon_0, \xi)$ such that $(N + c\varepsilon_0) * \mu_\omega^c$

equals the reduced measure of $\xi * \mu$ on ω with respect to $N + c\varepsilon_0$ (cf. Lemma 1.5).

For $\omega_1 \supseteqq \omega_2$ we have

$$(N + c\varepsilon_0) * \mu^c_{\omega_2} \leqq (N + c\varepsilon_0) * \mu^c_{\omega_1} ,$$

which by Lemma 1.2 implies

$$N * \mu^c_{\omega_2} \leqq N * \mu^c_{\omega_1} .$$

But since

$$(N + c\varepsilon_0) * \mu^c_{\omega_2} = (N + c\varepsilon_0) * \mu^c_{\omega_1} = \xi * \mu \quad \text{in } \omega_2$$

we obtain

$$\mu^c_{\omega_1} \leqq \mu^c_{\omega_2} \quad \text{in } \omega_2 .$$

As $N * \mu^c_\omega \leqq \xi * \mu$ holds for all $c > 0$, the set $\{\mu^c_\omega \mid c > 0\}$ is contained in a vaguely compact set K_ω of positive measures on $\bar{\omega}$, and hence

$$\forall c > 0, \ (\mu^c_\omega)_{\omega \in \Omega} \in \prod_{\omega \in \Omega} K_\omega$$

which by the Tychonoff theorem is compact. Any vague cluster point $(\mu_\omega)_{\omega \in \Omega}$ of $(\mu^c_\omega)_{\omega \in \Omega}$ as $c \downarrow 0$ is easily seen to have the desired property.

We will now define a class of N-excessive measures, which play an important role in the Riesz' decomposition theorem. Further explanation of the terminology will follow later in the paper.

DEFINITION. A positive measure $\eta \in \varepsilon(N)$ is called N-invariant if for all positive measures $\nu \in D^+(N)$ we have

$$N * \nu \leqq \eta \ \wedge \ \eta - N * \nu \in \varepsilon(N) \Rightarrow \nu = 0 .$$

We are now able to prove the Riesz' decomposition theorem, which is well-known for Hunt-kernels, and it was proved for non-singular kernels satisfying the domination principle by M. Itô [3] for σ-compact groups.

The proof given below is mainly due to Itô.

THEOREM 2.2. *Let N be a non-singular convolution kernel satisfying the domination principle. Then the following are equivalent for a positive measure ξ:*

(i) $\xi \in \varepsilon(N)$.

(ii) *There exists a positive measure ν and a N-invariant measure η such that*

$$\xi = N * \nu + \eta .$$

PROOF. (ii) $\Rightarrow$ (i) is immediate.

(i) $\Rightarrow$ (ii). Suppose $\xi \in \varepsilon(N)$ and choose the balayaged measures $(\mu_\omega)_{\omega \in \Omega}$ of ε_0 on $\omega \in \Omega$ relative to (N, ξ) introduced in the previous lemma. It is easily seen that

$$\nu = \lim_{\Omega} \mu_\omega$$

exists as ω increases. If $\nu|_\omega$ denotes the restriction of ν to ω, then

$$\nu|_\omega \leqq \mu_\omega, \qquad \mu_\omega - \nu|_\omega \to 0 .$$

Now define the N-excessive measure η by

$$\eta = \xi - N * \nu = \lim_{\Omega} N * (\mu_\omega - \nu|_\omega)$$

and we have to prove that η is N-invariant in order to obtain the desired decomposition.

Because N was supposed non-singular we can choose $V \in \mathscr{V}$ such that $R_N^{\complement V} \neq N$, and then

$$(N - R_N^{\complement V}) * (\mu_\omega - \nu|_\omega) \to 0 \quad \text{as } \omega \uparrow G ,$$

which implies

$$\lim_{\Omega} R_N^{\complement V} * (\mu_\omega - \nu|_\omega) = \eta .$$

According to Lemma 1.6 a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures with compact support exists such that

$$N * \lambda_\alpha \uparrow R_N^{\complement V} .$$

So for every $\alpha \in A$ we have

$$\eta * \lambda_\alpha = \lim_{\Omega} N * \lambda_\alpha * (\mu_\omega - \nu|_\omega) \leqq \lim_{\Omega} N * (\mu_\omega - \nu|_\omega) = \eta$$

and hence

$$\limsup_{A} \eta * \lambda_\alpha \leqq \eta .$$

Moreover we get

$$\begin{aligned}
\eta &= \lim_{\Omega} R_N^{\complement V} * (\mu_\omega - \nu|_\omega) \\
&= \lim_{\Omega} \lim_{A} N * \lambda_\alpha * \mu_\omega - \lim_{\Omega} \lim_{A} N * \lambda_\alpha * \nu|_\omega \\
&\leqq \liminf_{A} \xi * \lambda_\alpha - \lim_{A} \lim_{\Omega} N * \lambda_\alpha * \nu|_\omega \\
&= \liminf_{A} \eta * \lambda_\alpha
\end{aligned}$$

where the interchanging of limits is justified by monotonicity. Now suppose that

$$\eta = N*\lambda+\zeta \quad \text{for } \lambda \in D^+(N) \text{ and } \zeta \in \varepsilon(N).$$

As for η we have $\lim_A \sup \zeta*\lambda_\alpha \leqq \zeta$ and therefore

$$\begin{aligned} N*\lambda+\zeta &= \lim_A \eta*\lambda_\alpha \\ &= \lim_A N*\lambda*\lambda_\alpha + \lim_A \zeta*\lambda_\alpha \\ &\leqq R_N^{\complement V}*\lambda+\zeta \end{aligned}$$

which implies

$$(N-R_N^{\complement V})*\lambda \leqq 0$$

and hence $\lambda=0$, which states that η is N-invariant.

PROPOSITION 2.3. *Let N be a non-singular convolution kernel satisfying the domination principle. For a positive measure η the following are equivalent*:

(i) *η is N-invariant.*

(ii) *There exists a net $(\lambda_\alpha)_{\alpha\in A}$ of positive measures in $D^+(N)$ such that $(N*\lambda_\alpha)_{\alpha\in A}$ is increasing and*

$$\lim_A N*\lambda_\alpha = \eta, \quad \lim_A \lambda_\alpha = 0.$$

(iii) *$\eta \in \varepsilon(N)$ and for every compact set $K \subseteqq G$*

$$R_\eta^{\complement K} = \eta.$$

PROOF. (i) $\Rightarrow$ (ii). Let Ω be the set of open relatively compact subsets of G as before and order Ω by

$$\omega_2 \leqq \omega_1 \Leftrightarrow \bar{\omega}_2 \subseteqq \omega_1 \ \vee \ \omega_1 = \omega_2.$$

Choose the balayaged measures $(\lambda_\omega)_{\omega\in\Omega}$ of ε_0 on $\omega\in\Omega$ relative to (N,η) introduced in Lemmma 2.2.

For $\omega_1,\omega_2 \in \Omega$ and $\bar{\omega}_2 \subseteqq \omega_1$ we have

$$N*\lambda_{\omega_2} \leqq \eta = N*\lambda_{\omega_1} \quad \text{in } \omega_1 \supseteqq \operatorname{supp}\lambda_{\omega_2}$$

and hence by Lemma 1.2, $N*\lambda_{\omega_2} \leqq N*\lambda_{\omega_1}$. The net $(N*\lambda_\omega)_{\omega\in\Omega}$ is therefore increasing and satisfies

$$\lim_\Omega N*\lambda_\omega = \eta.$$

From the additional property of $(\lambda_\omega)_{\omega\in\Omega}$ in Lemma 2.2 follows that $(\lambda_\omega)_{\omega\in\Omega}$ converges vaguely towards a positive measure ν when ω increases to G. Then we have

$$\nu|_\omega \leqq \lambda_\omega$$

which implies that $\nu \in D^+(N)$ and moreover

$$0 \leqq \eta - N*\nu = \lim_\Omega N*(\lambda_\omega - \nu|_\omega) \in \varepsilon(N)\,.$$

But as η was assumed N-invariant

$$\lim_\Omega \lambda_\omega = \nu = 0\,.$$

(ii) $\Rightarrow$ (iii). Let K be a compact subset of G and suppose that (ii) is fulfilled and then η is clearly N-excessive. Choose $V \in \mathscr{V}$ such that the interior of V contains K. For each $\alpha \in A$ λ_α^V and $\lambda_\alpha^{\complement V}$ denotes the restriction of λ_α to V and $\complement V$ respectively. We now have

$$\lambda_\alpha = \lambda_\alpha^V + \lambda_\alpha^{\complement V}, \qquad \lim_A \lambda_\alpha^V = 0, \qquad \operatorname{supp} \lambda_\alpha^V \subseteq V$$

which implies

$$\lim_A N*\lambda_\alpha^V = 0\,.$$

But as $N*\lambda_\alpha^{\complement V} \leqq N*\lambda_\alpha \leqq \eta$ and $\operatorname{supp}\lambda_\alpha^{\complement V} \subseteq \overline{\complement V} \subseteq K$,

$$\eta = \lim_A N*\lambda_\alpha^{\complement V} \leqq R_\eta^{\complement K} \leqq \eta$$

i.e., $R_\eta^{\complement K} = \eta$.

(iii) $\Rightarrow$ (i). Let $\eta = N*\nu + \zeta$ be a Riesz decomposition of the N-excessive measure η. For $V \in \mathscr{V}$ we have by Lemma 1.7

$$\eta = R_\eta^{\complement V} = R_{N*\nu}^{\complement V} + R_\zeta^{\complement V} = R_{N*\nu}^{\complement V} + \zeta$$

and as V increases we obtain by Lemma 1.8

$$N*\nu + \zeta = \eta = N'*\nu + \zeta\,.$$

Then $N \neq N'$ implies that $\nu = 0$ and hence $\eta = \zeta$ is N-invariant.

The last characterization shows that invariant measures are "invariant" under reduction on complements of compact sets.

It is of course sufficient to consider reduced measures on complement of compact neighbourhoods of the neutral elements.

The set of N-invariant measures has some nice properties which are stated in the next corollary.

COROLLARY 2.4. *The set of N-invariant measures is a convex cone, which is closed under an increasing limit process, and for positive measures $\eta \in \varepsilon(N)$ and $\mu \in D^+(\eta)$, $\mu \neq 0$ we have*

$$\eta \text{ is } N\text{-invariant} \Leftrightarrow \eta * \mu \text{ is } N\text{-invariant} .$$

PROOF. The first two statements are immediate from Proposition 2.3, Lemma 1.4 and Lemma 1.7.

Now suppose η is N-invariant, then Lemma 1.8 implies

$$\eta * \mu = \left(\lim_{V \uparrow G} R_\eta^{\complement V}\right) * \mu = \lim_{V \uparrow G} R_{\eta * \mu}^{\complement V} \leqq \eta * \mu$$

and hence

$$R_{\eta * \mu}^{\complement V} = \eta * \mu \quad \text{for all } V \in \mathscr{V} .$$

Conversely if $\eta * \mu$ is N-invariant, then

$$\eta * \mu = \lim_{V \uparrow G} R_{\eta * \mu}^{\complement V} = \left(\lim_{V \uparrow G} R_\eta^{\complement V}\right) * \mu .$$

But as $R_\eta^{\complement V} \leqq \eta$ and $\mu \neq 0$ we have

$$R_\eta^{\complement V} = \eta \quad \text{for all } V \in \mathscr{V} .$$

PROPOSITION 2.5. *Let N be non-singular, then the regular part N_0 satisfies the relative balayage principle with respect to the singular part N'.*

PROOF. Let μ be a positive measure with compact support and ω an open relatively compact set.

Define $\mu_0 = \mu$ and then by recursion for each positive integer n the measure μ_n to be a balayaged measure of μ_{n-1} on ω relative to (N, N') (cf. Proposition 1.1). Therefore $\operatorname{supp} \mu_n \subseteq \bar{\omega}$ for all $n \geqq 1$ and

$$N * \mu_n \leqq N' * \mu_{n-1}$$

$$N * \mu_n = N' * \mu_{n-1} \quad \text{in } \omega .$$

Now by adding the first K inequalities we obtain

$$N * \left(\sum_{n=1}^{K} \mu_n\right) \leqq N' * \left(\sum_{n=0}^{K-1} \mu_n\right)$$

with equality in ω. If we split the left-hand side into the convolutions with the regular and singular parts of N, we then get

$$N' * \mu_n + N_0 * \left(\sum_{n=1}^{K} \mu_n\right) \leqq N' * \mu_0$$

with equality in ω. The inequality implies that $\sum_{n=1}^{\infty} \mu_n$ converges to a measure in $D^+(N_0)$ and in particular $\mu_n \to 0$. Hence

$$N' * \mu_n \to 0$$

and

$$N_0 * \left(\sum_{n=1}^{\infty} \mu_n \right) \leqq N' * \mu$$

$$N_0 * \left(\sum_{n=1}^{\infty} \mu_n \right) = N' * \mu \quad \text{in } \omega ,$$

i.e., $\sum_{n=1}^{\infty} \mu_n$ is a balayaged measure of μ on ω relative to (N_0, N').

REMARK. It is easy to prove that $N_0 \prec N'$. Suppose that $f, g \in C_K^+$ and that

$$N_0 * f \leqq N' * g \quad \text{on } \operatorname{supp} f .$$

Then

$$N * f \leqq N' * (f+g) \quad \text{on } \operatorname{supp} f$$

and using $N' \in \varepsilon(N)$ this implies

$$N * f \leqq N' * (f+g) \quad \text{on } G .$$

Hence

$$N_0 * f \leqq N' * g \quad \text{on } G .$$

COROLLARY 2.6. *The regular part N_0 of N satisfies the transitive domination principle with respect to N.*

PROOF. Let $f, g \in C_K^+$ and suppose that

$$N_0 * f \leqq N_0 * g \quad \text{in } \operatorname{supp} f .$$

If for a positive measure μ we define the reflected measure $\check{\mu}$ by

$$\langle \check{\mu}, \varphi \rangle = \int \varphi(-x) \, d\mu(x)$$

for all $\varphi \in C_K^+$, then it is easily seen that $\check{N}$ satisfies the domination principle, whenever N does and likewise $\check{N}_0$ satisfies the relative balayage principle with respect to $\check{N}'$ by Proposition 2.5.

Define

$$\omega = \{ y \in G \mid f(y) > 0 \} \subseteqq \operatorname{supp} f$$

and let $x \in G$ be given. If ε_x denotes the Dirac-measure concentrated at x, we can find a balayaged measure ε_1 of ε_x on ω relative to $(\check{N}, \check{N})$ and a balayaged measure ε_2 of ε_1 on ω relative to $(\check{N}_0, \check{N}')$. Then we have

$$\begin{aligned} N*f(x) &= \langle \check{N}*\varepsilon_x, f\rangle = \langle \check{N}*\varepsilon_1, f\rangle = \langle \check{N}_0*\varepsilon_1+\check{N}_0*\varepsilon_2, f\rangle \\ &= \langle \varepsilon_1+\varepsilon_2, N_0*f\rangle \leqq \langle \varepsilon_1+\varepsilon_2, N_0*g\rangle \leqq \langle \check{N}_0*\varepsilon_1+\check{N}'*\varepsilon_1, g\rangle \\ &\leqq N*g(x) \end{aligned}$$

which was to be proved.

DEFINITION. A convolution kernel N is said to satisfy the principle of unicity of mass if for all $\mu, \nu \in D^+(N)$ we have

$$N*\mu = N*\nu \Rightarrow \mu = \nu .$$

Finally we will prove the desired form of the Riesz' decomposition theorem with unique decomposition.

THEOREM 2.7. *Let N be a non-singular convolution kernel satisfying the domination principle and ξ a N-excessive measure. Then there exist a measure $\nu \in D^+(N)$ and a N-invariant measure η such that*

$$\xi = N*\mu+\eta .$$

The potential $N*\nu$ and the N-invariant measure η are uniquely determined. Moreover ν is uniquely determined if (and only if) N satisfies the principle of unicity of mass.

PROOF. The last statement is trivial. Thus we suppose that $\mu, \nu \in D^+(N)$ and η, ζ are N-invariant measures such that

$$(*) N*\nu+\eta = N*\mu+\zeta .$$

For $V \in \mathscr{V}$ we obtain by Lemma 1.7 and Proposition 2.3

$$R^{\complement V}_{N*\nu}+\eta = R^{\complement V}_{N*\mu}+\zeta .$$

Then we can use Lemma 1.8 letting $V \uparrow G$ and obtain

$$N'*\nu+\eta = N'*\mu+\zeta$$

which compared to (*) gives

$$N_0*\nu = N_0*\mu$$

But as $N_0 \sqsubset N$ Lemma 1.2 implies

$$N*\nu = N*\mu$$

and hence $\eta=\zeta$.

BIBLIOGRAPHY

1. C. Berg and G. Forst, *Potential theory on locally compact abelian groups* (Ergebnisse Math. 87) Springer-Verlag, Berlin - Heidelberg - New York, 1975.
2. M. Itô, *Sur le principe de domination pour les noyaux de convolution*, Nagoya Math. J. 50 (1973), 149–173.
3. M. Itô, *Sur le principe relatif de domination pour les noyaux de convolution*, Hiroshima Math. J. 5 (1975), 293–350.
4. M. Itô, *Sur le principe de domination relatif et le balayage* (manuscript).
5. M. Itô, *Caractérisation du principe de domination pour les noyaux de convolution non-bornés*, Nagoya Math. J. 57 (1975), 167–197.

KØBENHAVNS UNIVERSITETS
MATEMATISKE INSTITUT
UNIVERSITETSPARKEN 5
DK-2100 KØBENHAVN Ø
DANMARK