MATH. SCAND. 43 (1978), 157–168

# DISINTEGRATION AND COMPACT MEASURES

JAN K. PACHL

SUMMARY. A probability space is compact in the sense of Marczewski if and only if it admits countably additive disintegrations. It follows that the restriction of a compact measure to a sub-$\sigma$-algebra is compact.

## 1. Introduction.

This paper is concerned with countably additive disintegrations of countably additive measures (or, in another language, with regular conditional probabilities). Chatterji [2], Hoffmann-Jørgensen [5] and Pellaumail [14] have shown that the lifting theorem permits us to remove the traditional countability assumptions in existence theorems. Our present task is to find out what can be done about the other assumption, that the measure which is being disintegrated is Radon.

In section 3 we prove that a weaker condition, the compactness in the sense of Marczewski [9], is sufficient for the existence of disintegration as defined below; moreover, this condition is also necessary (section 2). This characterization of compact probabilities may be used to prove that the restriction of a compact probability to a sub-$\sigma$-algebra is compact (section 4).

A *probability space* $(X, \mathscr{A}, P)$ is a nonempty set $X$ together with a $\sigma$-algebra $\mathscr{A}$ on $X$ and a probability $P$ on $\mathscr{A}$ (that is, a nonnegative countably additive measure with $PX=1$). When $\mathscr{L}$ is a class of subsets of a (fixed) set $X$, we denote by $\alpha(\mathscr{L})$ and $\sigma(\mathscr{L})$ the algebra and the $\sigma$-algebra generated by $\mathscr{L}$. When $\mathscr{A}$ and $\mathscr{B}$ are $\sigma$-algebras on $X$ and $Y$, respectively, the symbol $\mathscr{A}\otimes\mathscr{B}$ stands for the system of the sets $E\times F$, where $E\in\mathscr{A}$ and $F\in\mathscr{B}$. Thus $\sigma(\mathscr{A}\otimes\mathscr{B})$ is the smallest $\sigma$-algebra making both the projections $\pi_X\colon X\times Y\to X$ and $\pi_Y\colon X\times Y\to Y$ measurable. If $(X,\mathscr{A},P)$ and $(Y,\mathscr{B},Q)$ are two probability spaces then a probability $R$ on $\sigma(\mathscr{A}\otimes\mathscr{B})$ is called a *joint probability* when $R(E\times Y)=PE$ for every $E\in\mathscr{A}$ (denoted $\pi_X[R]=P$) and $R(X\times F)=QF$ for every $F\in\mathscr{B}$ (denoted $\pi_Y[R]=Q$). If $\mathscr{A}$ is an algebra then $\mathscr{S}(\mathscr{A})$ denotes the space of (real-valued) $\mathscr{A}$-simple functions.

We will find it convenient to follow Valadier [17] in stating our results for measures on products (see also [4], [8]).

Research supported in part by the Danish Natural Science Research Council.
Received September 1, 1977.

1.1. DEFINITION. Let $\mathscr{A}$ and $\mathscr{B}$ be $\sigma$-algebras on $X$ and $Y$, let $R$ be a probability on $\sigma(\mathscr{A}\otimes\mathscr{B})$. Put $Q=\pi_Y[R]$; in other words, $QF=R(X\times F)$ for $F\in\mathscr{B}$. Suppose that there is, for every $y\in Y$, a $\sigma$-algebra $\mathscr{A}_y$ on $X$ and a probability $P_y$ on $\mathscr{A}_y$ and that

(a) for each $E\in\mathscr{A}$ there exists a set $N\in\mathscr{B}$ such that $QN=0$, $E\in\mathscr{A}_y$ for all $y\in Y\setminus N$ and the function

$$y\mapsto P_yE;\qquad y\in Y\setminus N\,,$$

is $(\mathscr{B}\,|\,Y\setminus N)$-measurable; and

(b) if $E\in\mathscr{A}$ and $F\in\mathscr{B}$ then

$$\int_F P_yE\,dQ(y) \;=\; R(E\times F)$$

(in view of (a), the integral is well defined).

The family $\{(\mathscr{A}_y,P_y)\}_{y\in Y}$ is then called a *Q-disintegration of R*.

Two remarks are in order. Firstly, we do not assume that $\mathscr{A}_y\subset\mathscr{A}$; but this is only a formal matter — $\mathscr{A}_y$ can be replaced by $\mathscr{A}_y\cap\mathscr{A}$. Secondly, and what is more important, we do not assume that $\mathscr{A}_y\supset\mathscr{A}$. The reason is that we want the complete Lebesgue probability to be disintegrable:

1.2. EXAMPLE. Assume the continuum hypothesis. Put $X=[0,1]^2$, $Y=[0,1]$. Denote by $P$ and $Q$ the ordinary (complete) Lebesgue probabilities in $[0,1]^2$ and $[0,1]$; these are defined on the $\sigma$-algebras $\mathscr{A}$ and $\mathscr{B}$ of Lebesgue measurable subsets of $X$ and $Y$, respectively. Define a probability $R$ on $\sigma(\mathscr{A}\otimes\mathscr{B})$ by

$$RG \;=\; P\{(x_1,x_2)\in X \;\big|\; ((x_1,x_2),x_2)\in G\} \qquad \text{for } G\in\sigma(\mathscr{A}\otimes\mathscr{B})\,.$$

Then there is no $Q$-disintegration $\{(\mathscr{A}_y,P_y)\}_{y\in Y}$ of $R$ satisfying $\mathscr{A}_y\supset\mathscr{A}$ for all $y\in Y$.

In fact, if $\{(\mathscr{A}_y,P_y)\}_y$ is a $Q$-disintegration of $R$ then for $Q$-almost all $y\in Y$ the restriction of $P_y$ to the $\sigma$-algebra of Borel subsets of $[0,1]^2$ coincides with the "natural" disintegration (because the disintegration is essentially unique — see [13, Ch. V, Th. 8.1]). Hence for some $y\in Y$ (in fact for $Q$-almost all $y\in Y$), $P_y$ extends the one-dimensional Lebesgue probability in $[0,1]\times\{y\}\subset X$ to $\mathscr{A}_y$. The $\sigma$-algebra $\mathscr{A}$ contains all the subsets of $[0,1]\times\{y\}$; thus if $\mathscr{A}_y\supset\mathscr{A}$, we obtain an extension of the Lebesgue probability in $[0,1]$ to the $\sigma$-algebra of all subsets of $[0,1]$. By Ulam's theorem [12, 5.6], this contradicts the continuum hypothesis.

The image of $R$ in $X$ has not been mentioned in 1.1; yet this probability has its say in existence results.

1.3. DEFINITION. Let $(X,\mathscr{A},P)$ and $(Y,\mathscr{B},Q)$ be two probability spaces. We say that $P$ *endorses $Q$-disintegration* if every joint probability on $\sigma(\mathscr{A}\otimes\mathscr{B})$ has a $Q$-disintegration.

1.4. FURTHER TERMINOLOGY. (a) A *lattice on $X$* is a class of subsets of $X$ that contains $\varnothing$ and $X$ and is closed under finite unions and finite intersections.

(b) A lattice $\mathscr{K}$ of sets is *semicompact* if every countable class $\mathscr{K}_0\subset\mathscr{K}$ with $\bigcap\mathscr{K}_0=\varnothing$ contains a finite class $\mathscr{K}_{00}\subset\mathscr{K}_0$ such that $\bigcap\mathscr{K}_{00}=\varnothing$.

(c) Let $(X,\mathscr{A},P)$ be a probability space and $\mathscr{K}\subset\mathscr{A}$; we say that $\mathscr{K}$ *approximates $P$* if for any $E\in\mathscr{A}$ and $\varepsilon>0$ there is a $K\in\mathscr{K}$ such that $K\subset E$ and $P(E\setminus K)<\varepsilon$.

(d) A probability space $(X,\mathscr{A},P)$ (and the probability $P$) is *compact* if there is a semicompact lattice $\mathscr{K}\subset\mathscr{A}$ that approximates $P$. (Perhaps it would be more consistent to call such a probability semicompact.) Recall [9] that, in this definition, $\mathscr{K}$ may be assumed to be closed under countable intersections.

## 2. Compactness and the Stone space.

Every measurable space can be represented in a subset of its Stone space; accordingly, every measure space can be represented in a subset of a compact measure space. It is therefore useful to know when compactness is inherited from a measure in a large set to the one induced in a subset.

Vinokurov [18] and Musiał [11] investigated this question for $\aleph_1$-generated measures. For general measures, valuable information can be obtained with the help of disintegration.

In the following theorem, the symbol $(X,\mathscr{A},P)\subset(Y,\mathscr{B},Q)$ means that $X$ is a $Q$-thick subset of $Y$ (i.e., $QF=0$ whenever $F\in\mathscr{B}$ and $F\cap X=\varnothing$), $\mathscr{A}\subset\mathscr{B}\mid X$, and $P(F\cap X)=QF$ when $F\in\mathscr{B}$ and $F\cap X\in\mathscr{A}$. The completion of $(X,\mathscr{A},P)$ is denoted by $(X,\hat{\mathscr{A}},\hat{P})$.

2.1. THEOREM. *Let $(X,\mathscr{A},P)$ and $(Y,\mathscr{B},Q)$ be two probability spaces such that $(X,\mathscr{A},P)\subset(Y,\mathscr{B},Q)$ and $\mathscr{B}\mid X\subset\hat{\mathscr{A}}$. Define a joint probability $R$ on $\sigma(\mathscr{A}\otimes\mathscr{B})$ by*

$$RG=\hat{P}\{x\in X \mid (x,x)\in G\} \quad \text{for } G\in\sigma(\mathscr{A}\otimes\mathscr{B}).$$

*Suppose that $Q$ is compact and that there exists a $Q$-disintegration of $R$.*

*Then $P$ is compact.*

PROOF. Take a semicompact lattice $\mathscr{L}\subset\mathscr{B}$ that approximates $Q$ and is closed under countable intersections, and a $Q$-disintegration $\{(\mathscr{A}_y, P_y)\}_{y\in Y}$ of $R$. To define a semicompact class approximating $P$, put

$$\mathscr{K} = \{E \in \mathscr{A} \mid \text{there is an } F \in \mathscr{L} \text{ such that } F\cap X = E \text{ and } E \in \mathscr{A}_y,\ P_yE=1 \text{ for each } y \in F\}\,.$$

Obviously, $\mathscr{K}$ is closed under finite intersections.

To check that $\mathscr{K}$ is semicompact, pick a sequence of sets $K_1, K_2, \ldots \in \mathscr{K}$ decreasing to the empty set: $K_n \searrow \varnothing$. For each $n$ find a set $L_n \in \mathscr{L}$ such that $L_n \cap X = K_n$ and

$$K_n \in \mathscr{A}_y \quad \text{and} \quad P_yK_n = 1 \quad \text{whenever } y \in L_n\,.$$

Since $\mathscr{L}$ is a lattice, we may assume that the sequence $\{L_n\}_n$ is decreasing: $L_n \searrow L$. It is easy to see that $L=\varnothing$; if $y \in L$ then $K_n \in \mathscr{A}_y$ and $P_yK_n=1$ for every $n$, hence

$$P_y\varnothing = P_y\left(\bigcap_n K_n\right) = \lim_n P_yK_n = 1\,,$$

which holds for no $y \in Y$. As $\mathscr{L}$ is semicompact, there is an $n$ such that $L_n=\varnothing$, hence also $K_n=\varnothing$.

It remains to be proved that $\mathscr{K}$ approximates $P$. Start with any $E \in \mathscr{A}$ and $\varepsilon>0$; there is an $F \in \mathscr{B}$ such that $F\cap X=E$. Now find an $L_1 \in \mathscr{L}$ with $L_1 \subset F$ and $Q(F\setminus L_1)<\varepsilon/2$ and use the assumption $\mathscr{A} \subset \mathscr{B}\,|\,X \subset \hat{\mathscr{A}}$ to find an $H_1 \in \mathscr{B}$ such that

$$H_1 \subset L_1, \quad H_1 \cap X \in \mathscr{A} \quad \text{and} \quad Q(L_1\setminus H_1) = 0\,.$$

By the definition of disintegration, $H_1 \cap X \in \mathscr{A}_y$ for $Q$-almost all $y \in H_1$ and

$$\int_{H_1} P_y(H_1\cap X)\,dQ(y) = R((H_1\cap X)\times H_1) = P(H_1\cap X) = QH_1\,;$$

hence there is an $F_1 \in \mathscr{B}$ such that $F_1 \subset H_1$, $Q(H_1\setminus F_1)=0$ and

$$H_1 \cap X \in \mathscr{A}_y \quad \text{and} \quad P_y(H_1\cap X) = 1 \quad \text{for every } y \in F_1\,.$$

This $F_1$ is in turn approximated by an $L_2 \in \mathscr{L}$ with $Q(F_1\setminus L_2)<\varepsilon/2^2$. Again, there are $H_2, F_2 \in \mathscr{B}$ such that

$$F_2 \subset H_2 \subset L_2, \quad H_2\cap X \in \mathscr{A}, \quad Q(L_2\setminus F_2) = 0\,,$$

and

$$H_2 \cap X \in \mathscr{A}_y \quad \text{and} \quad P_y(H_2\cap X) = 1 \quad \text{for every } y \in F_2\,.$$

Having constructed, in this way, the sets $L_n$, $H_n$ and $F_n$ for $n=1,2,\ldots$, we put

$L = \bigcap_n L_n$ and $K = L \cap X$. Then $K \in \mathscr{A}$, because $K = \bigcap_n (H_n \cap X)$; obviously $K \subset E$ and

$$P(E \setminus K) = Q(F \setminus L) = Q(F \setminus L_1) + \sum_{n=1}^{\infty} Q(L_n \setminus L_{n+1}) < \varepsilon .$$

Finally, to verify that $K \in \mathscr{K}$, note that $(H_n \cap X) \searrow K$ while $(H_n \cap X) \in \mathscr{A}_y$ and $P_y(H_n \cap X) = 1$ for every $y \in L$. Hence $K \in \mathscr{A}_y$ and $P_y K = 1$ for every $y \in L$.

Our aim in this section is to show that the probabilities endorsing disintegration are compact. Thus it would be desirable, in view of the preceding theorem, to have every probability space embedded into a compact one. Fortunately, this is not difficult to achieve: When $(X, \mathscr{A}, P)$ is a probability space, we employ the Stone space of $\mathscr{A}$ to find a set $S(\mathscr{A})$ and an algebra $\mathscr{B}_0$ on $S(\mathscr{A})$ such that (see [15, § 8.G])

(a) $X \subset S(\mathscr{A})$ and $\mathscr{A} = \{F \cap X \mid F \in \mathscr{B}_0\}$;

(b) the algebra $\mathscr{B}_0$ is semicompact (in fact, $\mathscr{B}_0$ is even compact, in the sense that any decreasing net of nonempty sets in $\mathscr{B}_0$ has nonempty intersection; but we do not need this stronger property).

Put $\mathscr{B} = \sigma(\mathscr{B}_0)$ and

$$QF = P(F \cap X) \quad \text{for } F \in \mathscr{B} .$$

The probability $Q$ on $\mathscr{B}$ is approximately by the semicompact class $(\mathscr{B}_0)_\delta$ (the class of countable intersections of sets in $\mathscr{B}_0$).

If $(X, \mathscr{A}, P)$ endorses $Q$-disintegration (or, at least, if the joint probability $R$ constructed in 2.1 can be disintegrated), then $P$ is compact, by virtue of 2.1. However, we are after a condition that is not only sufficient but also necessary for compactness. As the completeness of $Q$ seems to be indispensable for the way back, it is advisable to replace $(S(\mathscr{A}), \mathscr{B}, Q)$ by its completion $(S(\mathscr{A}), \hat{\mathscr{B}}, \hat{Q})$; the assumptions in 2.1 remain valid, because $\hat{\mathscr{B}} \mid X = \hat{\mathscr{A}}$. Hence if $P$ endorses $Q$-disintegration then $P$ is compact. In particular:

2.2. Theorem. *If a probability $P$ endorses $Q$-disintegration relative to every complete probability $Q$, then $P$ is compact.*

This is the first half of our characterization of compactness; the converse will be proved in the next section. At this time we can already prove the following permanence property:

2.3. Proposition. *Suppose that a probability space $(X, \mathscr{A}, P)$ endorses*

*$Q$-disintegration relative to every complete probability $Q$. Then the restriction of $P$ to any sub-$\sigma$ algebra of $\mathscr{A}$ also does so.*

PROOF. Let $\mathscr{E}$ be a sub-$\sigma$-algebra of $\mathscr{A}$, and $(Y, \mathscr{B}, Q)$ any complete probability space. The assertion will become evident as soon as we know that any joint probability on $\sigma(\mathscr{E} \otimes \mathscr{B})$ extends to a joint probability on $\sigma(\mathscr{A} \otimes \mathscr{B})$. Thus let $R$ be a probability on $\sigma(\mathscr{E} \otimes \mathscr{B})$ such that $\pi_X[R] = P \mid \mathscr{E}$ and $\pi_Y[R] = Q$. For any function $g \in \mathscr{S}(\alpha(\mathscr{E} \otimes \mathscr{B}))$ put

$$\tilde{R}(g) = \int_{X \times Y} g \, dR .$$

For any $g \in \mathscr{S}(\alpha(\mathscr{A} \otimes \mathscr{B}))$ put

$$p(g) = \inf\left\{ \int_X h \, dP \;\middle|\; h \in \mathscr{S}(\mathscr{A}) \text{ and } h \circ \pi_X \geqq g \right\} .$$

Then $\tilde{R}(g) \leqq p(g)$ for each $g \in \mathscr{S}(\alpha(\mathscr{E} \otimes \mathscr{B}))$, while $p$ is positively homogeneous and subadditive:

$$p(\lambda g) = \lambda p(g) \quad \text{for any real number } \lambda > 0 \text{ and any } g \in \mathscr{S}(\alpha(\mathscr{A} \otimes \mathscr{B})) ,$$

$$p(g_1 + g_2) \leqq p(g_1) + p(g_2) \quad \text{for any } g_1, g_2 \in \mathscr{S}(\alpha(\mathscr{A} \otimes \mathscr{B})) .$$

By the Hahn-Banach theorem [3, II.3.10], the linear map $\tilde{R}$ can be extended to a linear map $\tilde{R}_1$ on $\mathscr{S}(\alpha(\mathscr{A} \otimes \mathscr{B}))$ satisfying $\tilde{R}_1 \leqq p$. The formula $R_1 G = \tilde{R}_1(I_G)$ defines a finitely additive function on $\alpha(\mathscr{A} \otimes \mathscr{B})$, which is nonnegative because for any $G \in \alpha(\mathscr{A} \otimes \mathscr{B})$ we have

$$-R_1 G = \tilde{R}_1(-I_G) \leqq p(-I_G) \leqq 0 . \; .$$

Now $P$ is compact (this is where 2.2 is used), hence $R_1$ is countably additive by [10, 1.i]. Thus $R_1$ extends to a probability on $\sigma(\mathscr{A} \otimes \mathscr{B})$, denoted again by $R_1$. This $R_1$ agrees with $R$ on $\sigma(\mathscr{E} \otimes \mathscr{B})$, because $R_1$ and $R$ coincide on $\alpha(\mathscr{E} \otimes \mathscr{B})$ and both are countably additive.

Finally, $R_1$ is a joint probability: for $E \in \mathscr{A}$ we have

$$R_1(E \times Y) \leqq p(I_{E \times Y}) \leqq \int_X I_E \, dP = PE ,$$

$$-R_1(E \times Y) = \tilde{R}_1(-I_{E \times Y}) \leqq p(-I_{E \times Y}) \leqq -\int_X I_E \, dP = -PE ,$$

whence $R_1(E \times Y) = PE$.

## 3. Existence of disintegration.

The purpose of this section is to prove that any compact probability endorses $Q$-disintegration whenever $Q$ is complete. Since the lifting theorem automatically produces a "finitely additive disintegration", the only difficulty is to pass from the finitely additive function to a countably additive one; this is where the compactness comes into play.

If the semicompact system in question is separated by the corresponding class of "open" sets then a suitable countably additive measure can be described constructively (see (e.g. 3.1 in [7], or section 5 in [16]). No such separation property is assumed in the sequel; we find a countably additive disintegration by means of the axiom of choice (disguised as the Hahn–Banach theorem and Zorn's lemma). The price for the generality is that any trace of uniqueness or naturalness is hopelessly lost.

Recall that a nonnegative (finite) real-valued function $\beta$ on a lattice is called

(a) *submodular* when $\beta\varnothing=0$ and

$$\beta(K_1\cup K_2)+\beta(K_1\cap K_2) \leqq \beta K_1+\beta K_2$$

for all $K_1,K_2\in\mathscr{K}$;

(b) *supermodular* when $\beta\varnothing=0$ and

$$\beta(K_1\cup K_2)+\beta(K_1\cap K_2) \geqq \beta K_1+\beta K_2$$

for all $K_1,K_2\in\mathscr{K}$;

(c) *modular* when it is both submodular and supermodular;

(d) *monotone* when $\beta K_1\leqq\beta K_2$ for $K_1,K_2\in\mathscr{K}$, $K_1\subset K_2$.

Our interest in these properties comes from the fact that any monotone modular function on a lattice $\mathscr{K}$ extends to a nonnegative additive function on $\alpha(\mathscr{K})$ (see e.g. [16, section 8]).

If $\mathscr{K}$ is a lattice on $X$ and $\beta$ is a real-valued function on $\mathscr{K}$ then the "inner measure" $\beta_*$ is defined by

$$\beta_*E = \sup\{\beta K \mid K\in\mathscr{K} \text{ and } K\subset E\}$$

for every $E\subset X$.

The following result is proved, though not stated exactly in this form, in section 8 of [16].

3.1. Lemma. *Suppose that $\mathscr{K}$ is a lattice on $X$, while $\beta$ is a monotone submodular function on $\mathscr{K}$. Then there is a monotone modular function $\gamma$ on $\mathscr{K}$ such that $\gamma\leqq\beta$ and $\gamma X=\beta X$.*

It will be more convenient for us to work with the dual statement:

3.2. LEMMA. *Suppose that $\mathscr{K}$ is a lattice on $X$, while $\delta$ is a monotone supermodular function on $\mathscr{K}$. Then there is a monotone modular function $\gamma$ on $\mathscr{K}$ such that $\gamma \geqq \delta$ and $\gamma X = \delta X$.*

PROOF. Apply 3.1 to the function $\beta C = \delta X - \delta(X \setminus C)$ on the dual lattice

$$\mathscr{K}^{c} = \{C \subset X \mid X \setminus C \in \mathscr{K}\} .$$

The next result says that maximal elements in the set of normalized monotone modular functions are tight (in the sense of [16]). The reader may compare this result with [1, 3.4].

3.3. LEMMA. *Suppose that $\mathscr{K}$ is a lattice on $X$ and $\beta$ is a monotone modular function on $\mathscr{K}$ with $\beta X = 1$. Fix a set $K_0 \in \mathscr{K}$. Then there is a monotone modular function $\gamma$ on $\mathscr{K}$ such that*

$$\gamma \geqq \beta, \quad \gamma X = 1 \quad \textit{and} \quad \gamma K_0 + \gamma_*(X \setminus K_0) = 1 .$$

PROOF. First put

$$\gamma_1 K = \sup\{\beta(K \cap L) \mid L \in \mathscr{K} \text{ and } L \cap K_0 = \varnothing\}$$

for $K \in \mathscr{K}$. Then $\beta - \gamma_1$ is monotone and modular. Now put

$$\delta K = \sup\{\beta L - \gamma_1 L \mid L \in \mathscr{K} \text{ and } L \cap K_0 \subset K\}$$

for $K \in \mathscr{K}$. It is easy to see that $\delta$ is monotone and supermodular. By 3.2, there is a monotone modular function $\gamma_2$ on $\mathscr{K}$ such that $\gamma_2 \geqq \delta$ and

$$\gamma_2 X = \delta X = \beta X - \gamma_1 X = 1 - \gamma_1 X .$$

Set $\gamma = \gamma_1 + \gamma_2$. For each $K \in \mathscr{K}$ we have

$$\gamma K = \gamma_1 K + \gamma_2 K \geqq \gamma_1 K + \delta K \geqq \beta K .$$

Further,

$$\gamma K_0 \geqq \delta K_0 = \beta X - \gamma_1 X = 1 - \beta_*(X \setminus K_0) \geqq 1 - \gamma_*(X \setminus K_0) ,$$

3.4. PROPOSITION. *Suppose that $\mathscr{K}$ is a lattice on $X$, while $\beta$ is a monotone modular function on $\mathscr{K}$ with $\beta X = 1$. Then there is a monotone modular function $\gamma$ on $\mathscr{K}$ such that $\gamma \geqq \beta$, $\gamma X = 1$ and*

$$\gamma K_0 + \gamma_*(X \setminus K_0) = 1 \quad \textit{for every } K_0 \in \mathscr{K} .$$

PROOF. The set of monotone modular functions $\delta$ on $\mathscr{K}$ satisfying $\delta X = 1$ and $\beta \leqq \delta$ is partially ordered by $\leqq$, and every totally ordered subset has an upper bound. By Zorn's Lemma [3, I.2.7], this set contains a maximal element $\gamma$, and 3.3 implies that $\gamma K_0 + \gamma_*(X \setminus K_0) = 1$ for every $K_0 \in \mathscr{K}$.

Now we are ready for the main result.

3.5. THEOREM. *Let $(X, \mathscr{A}, P)$ and $(Y, \mathscr{B}, Q)$ be two probability spaces and let $R$ be a joint probability on $\sigma(\mathscr{A} \otimes \mathscr{B})$. Suppose that $Q$ is complete and $P$ is approximated by a semicompact lattice $\mathscr{K} \subset \mathscr{A}$ which is closed under countable intersections.*

*Then there is a $Q$-disintegration $\{(\mathscr{A}_y, P_y)\}_{y \in Y}$ of $R$ such that $\mathscr{A}_y \supset \mathscr{K}$ and $\mathscr{K}$ approximates $P_y$ for each $y \in Y$.*

PROOF. Choose a lifting $\varrho$ on $(Y, \mathscr{B}, Q)$ (see [6, IV–Th. 3]). By the Radon–Nikodým theorem, for each $E \in \mathscr{A}$ there exists a $\mathscr{B}$-measurable function $h_E$ such that

$$\int_F h_E \, dQ = R(E \times F)$$

for every $F \in \mathscr{B}$. For each $y \in Y$ define a function $\beta_y$ on $\mathscr{K}$ by

$$\beta_y K = \varrho h_K(y), \qquad K \in \mathscr{K}.$$

From the properties of lifting it follows that $\beta_y$ is monotone and modular and $\beta_y X = 1$. Apply 3.4 to obtain a monotone modular function $\gamma_y$ on $\mathscr{K}$ such that $\gamma_y \geqq \beta_y$, $\gamma_y X = 1$ and

$$\gamma_y K_0 + (\gamma_y)_*(X \setminus K_0) = 1 \qquad \text{for every } K_0 \in \mathscr{K}.$$

Let $P_y$, defined on a $\sigma$-algebra $\mathscr{A}_y$, be the maximal additive extension of $\gamma_y$ that is approximated by $\mathscr{K}$. In other words, extend $\gamma_y$ to the countably additive measure on $\sigma(\mathscr{K})$ ([16, 4–Th. 1(ii)]) and complete this measure to get $P_y$ on $\mathscr{A}_y$.

To say the same still differently, a set $E \subset X$ belongs to $\mathscr{A}_y$ if and only if

$$(\gamma_y)_* E + (\gamma_y)_*(X \setminus E) = 1$$

and $P_y$ is the restriction of $(\gamma_y)_*$ to $\mathscr{A}_y$.

For a fixed $E \in \mathscr{A}$, choose an increasing sequence of sets $K_1, K_2, \ldots \in \mathscr{K}$ such that $K_n \subset E$ and $\sup_n PK_n = PE$, and an increasing sequence of sets $L_1, L_2, \ldots \in \mathscr{K}$ such that $L_n \subset X \setminus E$ and $\sup_n PL_n = P(X \setminus E)$. Then

$$\sup_n \beta_y K_n \leqq (\gamma_y)_* E \leqq 1 - (\gamma_y)_*(X \setminus E) \leqq 1 - \sup_n \beta_y L_n$$

for every $y \in Y$. At the same time, for every $F \in \mathscr{B}$ we have

$$\int_F \left(\sup_n \beta_y K_n\right) dQ(y) = \sup_n \int_F \beta_y K_n \, dQ(y) = \sup_n R(K_n \times F) = R(E \times F),$$

and similarly

$$\int_F \left(1 - \sup_n \beta_y L_n\right) dQ(y) = R(E \times F).$$

It follows that

$$(\gamma_y)_* E = 1 - (\gamma_y)_*(X \setminus E)$$

for $Q$-almost all $y \in Y$ and

$$\int_F (\gamma_y)_* E \, dQ(y) = R(E \times F)$$

for every $F \in \mathscr{B}$; in other words, $E \in \mathscr{A}_y$ for $Q$-almost all $y \in Y$ and

$$\int_F P_y E \, dQ(y) = R(E \times F)$$

for every $F \in \mathscr{B}$. The theorem is proved.

3.6. Corollary. *Every compact probability endorses $Q$-disintegration for every complete probability $Q$.*

## 4. Concluding remarks.

From 2.2, 2.3 and 3.6 we obtain immediately:

4.1. Proposition. *The restriction of a compact probability to a sub-$\sigma$-algebra is compact.*

As a matter of fact, this result can be proved without going through the extension procedure in the proof of 2.3: In view of 2.1, it is not necessary to extend *every* joint probability on $\sigma(\mathscr{E} \otimes \mathscr{B})$ (where $\mathscr{E} \subset \mathscr{A}$) to a joint probability on $\sigma(\mathscr{A} \otimes \mathscr{B})$. It is enough to extend the measure induced by the canonical map from $(X, \mathscr{E}, \hat{P} \mid \mathscr{E})$ onto the "diagonal" in $X \times S(\mathscr{E})$; and this measure can be extended in the obvious way, without invoking the Hahn–Banach theorem. Nevertheless, the axiom of choice is still hidden in the proof of both 2.1 and 3.5, and it is doubtful whether anything like 4.1 can be proved constructively.

4.2. *Existence of decent disintegration.* When considering, perhaps more sensibly, only the disintegrations $\{(\mathscr{A}_y, P_y)\}_{y\in Y}$ such that $\mathscr{A}_y \supset \mathscr{A}$ for all $y$, the property of endorsing $Q$-disintegration for every complete $Q$ is no longer necessary for compactness. The complete Lebesgue probability in $[0, 1]^2$ is a counterexample (see 1.2 above). On the other hand, 2.3 remains true (with the same proof). Moreover, from 3.5 we infer that a probability $P$ endorses $Q$-disintegration in this sense (i.e. with $\mathscr{A}_y \supset \mathscr{A}$ for all $y$) for every complete $Q$, whenever $P$ is approximated by a semicompact lattice $\mathscr{K}$ such that $\mathscr{A} \subset \hat{\sigma}(\mathscr{K})$. (Here $\hat{\sigma}(\mathscr{K})$ denotes the "$\mathscr{K}$-universal completion" of $\sigma(\mathscr{K})$, i.e. the $\sigma$-algebra of those sets which are measurable relative to every $\mathscr{K}$-approximated probability on $\sigma(\mathscr{K})$.) For example, if $X$ is an arbitrary topological space (no separation axioms are assumed) and $\mathscr{K}$ is the class of closed compact subsets (together with $X$, to make $\mathscr{K}$ a lattice), then $\mathscr{K}$ is semicompact and every closed subset of $X$ belongs to $\hat{\sigma}(\mathscr{K})$. Thus our result contains a disintegration theorem for Radon Borel measures in arbitrary topological spaces.

ACKNOWLEDGEMENT. The author is grateful to all who were helping him in 1976/77. Without their help this paper would not have been written.

## REFERENCES

1. W. Adamski, *Capacitylike set functions and upper envelopes of measures*, Math. Ann. 229 (1977), 237–244.
2. S. D. Chatterji, *Disintegration of measures and lifting*, in *Vector and operator valued measures and applications*, eds. D. H. Tucker and H. B. Maynard (Proc. Symp. Alta, Utah, 1972,pp. 69–83), Academic Press, New York 1973. MR 49 #5837.
3. N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory* (Pure and Applied Math. 7), Interscience Publishers, New York 1958. MR 22 #8302.
4. G. A. Edgar, *Disintegration of measures and the vector-valued Radon–Nikodym theorem*, Duke Math. J. 42 (1973), 447–450. MR 54 #507.
5. J. Hoffmann-Jørgensen, *Existence of conditional probabilities*, Math. Scand. 28 (1971), 257–264. MR 46 #4575.
6. A. Ionescu-Tulcea and C. Ionescu-Tulcea, *Topics in the theory of lifting* (Ergebnisse Math. 48), Springer-Verlag, Berlin - Heidelberg - New York, 1969. MR 43 #2185.
7. M. Jiřina, *On regular conditional probabilities*, Czechoslovak Math. J. 9 (84) (1959), 445–451. MR 22 #6004.
8. D. Maharam, *Strict disintegration of measures*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete 32 (1975), 73–79. MR 51 #13185.
9. E. Marczewski, *On compact measures*, Fund. Math. 40 (1953), 113–124. MR 15–610.
10. E. Marczewski and C. Ryll-Nardzewski, *Remarks on the compactness and non-direct products of measures*, Fund. Math. 40 (1953), 165–170. MR 15–610.
11. K. Musiał, *Inheritness of compactness and perfectness of measures by thick subsets*, Measure Theory (Proc. Conf. Oberwolfach, 1975, pp. 31–42), Lecture Notes in Mathematics 541, Springer-Verlag, Berlin - Heidelberg - New York, 1976.

12. J. C. Oxtoby, *Measure and category* (Graduate Texts in Math. 2), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
13. K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York - London 1967. MR 37 #2271.
14. J. Pellaumail, *Application de l'existence d'un relèvement à un théorème sur la désintegration des mesures*, Ann. Inst. Henri Poincaré, Sect. B, 8 (1972), 211–215. MR 50 #7476.
15. R. Sikorski, *Boolean Algebras* (Ergebnisse Math. 25), Springer-Verlag, Berlin - Göttingen - Heidelberg, 1960. MR 23 #A 3689.
16. F. Topsøe, *On construction of measures*, Københavns Universitet Matematisk Institut Preprint Series No. 27, 1974.
17. M. Valadier, *Désintegration d'une mesure sur un produit*, C. R. Acad. Sci. Paris Sér. A, 276 (1973), 33–35. MR 47 #6990.
18. V. G. Vinokurov, *Compact measures and products of Lebesque spaces* [Russian], Mat. Sb. 74 (1967), 434–472. MR 36 #1605. English transl. = Math. USSR-Sbornik, 3 (1967), 405–433.

KØBENHAVNS UNIVERSITETS
MATEMATISKE INSTITUT
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø
DENMARK