MATH. SCAND. 43 (1978), 169–176

SUB-ELLIPTIC ESTIMATES FOR THE OBLIQUE DERIVATIVE PROBLEM

BENGT WINZELL

0. Introduction.

The oblique derivative problem is usually posed in the following way: Given g in Ω and f on $\partial\Omega$, find a function u which satisfies $\mathscr{L}u=g$ in Ω and $\partial u/\partial l=f$ on $\partial\Omega$. Here Ω is a domain in R^n, $n\geqq 3$, $\mathscr{L}$ an elliptic partial differential operator of the second order and l a unit vector field on $\partial\Omega$.

If l is never tangential to $\partial\Omega$ and certain smoothness conditions on l, $\mathscr{L}$ and Ω are fulfilled, then the problem is elliptic, which among other things means that a solution is in $C^{2+\lambda}(\bar{\Omega})$ if $g\in C^{\lambda}(\bar{\Omega})$ and $f\in C^{1+\lambda}(\partial\Omega)$. There is a corresponding formulation in the Sobolev norms.

The degenerating problem, i.e. when l now and then becomes tangential, has been examined by many authors during the last ten years. The main steps, also including regularity investigations were Hörmander [4], Egorov and Kondrat'ev [3], Maz'ja [5], Winzell [8], Melin and Sjöstrand [6] and Taira [7] (with a slightly different boundary condition). From these it was evident that there must be a loss of regularity such that u in general has one derivative less than what an elliptic estimate would give. However, Egorov [2] indicated that a loss of one derivative is too much in certain cases and that in fact the amount of regularity that the solutions gain depend on the order of contact between l and $\partial\Omega$. Egorov's sub-elliptic estimates were stated in Sobolev space language. Our aim is to derive the corresponding estimates in Hölder classes.

1. Notations and basic assumptions.

We assume that the field l is the sum of the normal component $\alpha\hat{n}$ and the tangent vector field X. Here $\hat{n}$ is the outer normal to the boundary. We also assume that l and Ω are of class C^3 and thus it follows that there exist integral curves to X through every point $p\in\partial\Omega$. Such curves are called X-*curves* and we denote the maximal X-curve through p by γ_p. It will be convenient to use the following standard parametrization of γ_p:

Received October 24, 1977.

$$s \to \tilde{x}_p(s) \text{ where } \tilde{x}_p(0) = p \text{ and } \frac{d}{ds}\tilde{x}_p(s) = X \circ \tilde{x}_p(s) .$$

Extensions of l to $\bar{\Omega}$ will be denoted by L and we always assume that L is of unit length near $\partial\Omega$. Integral curves to L will be called *L-curves*. The maximal L-curve through p will be denoted by Γ_p and we use a standard parametrization $s \to x_p(s)$ analogous to that for γ_p.

The operator $\mathscr{L}$ has the form $a_{ij}D_iD_j+b_iD_i+c$ where D_i represents differentiation with respect to the variable x_i and the summation convention is used. We will assume that the a_{ij}-s belong to C^3, that the b_i-s belong to C^2, that c is in C^1 in $\bar{\Omega}$, and that $\mathscr{L}$ is elliptic.

Define the *set of tangency* for l by

$$H = \{p \in \partial\Omega : \alpha(p)=0\} .$$

2. Statement of results.

Assume that there is a real number $m>0$ and two positive constants s_0 and α_0 such that for any $p \in H$:

$$|\alpha(\tilde{x}_p(s))| \geqq \alpha_0 \cdot |s|^m \qquad |s| \leqq s_0 . \tag{2.1}$$

For example, if H contains a point where γ_p has contact of order k_1 with H and if α has a zero of order k_2 considered as a function on $\partial\Omega$ then $m \geqq k_1 \cdot k_2$.

The first result is

THEOREM 1. *Assume that α does not change its sign from minus to plus along any γ_p. Let $u \in C^{1+\lambda}(\bar{\Omega}) \cap C^2(\Omega)$ be a solution of $\mathscr{L}u=g$ in Ω with $\partial u/\partial l=f$ on $\partial\Omega$ such that $g \in C^\lambda(\bar{\Omega})$ and $f \in C^{1+\lambda}(\partial\Omega)$. Then for any ε satisfying $0<\varepsilon<(m+1)^{-1}$ and $\lambda+\varepsilon<1$ the function u belongs to $C^{1+\lambda+\varepsilon}(\bar{\Omega}) \cap C^{2+\lambda}_{\text{loc}}(\bar{\Omega} \setminus H)$ and there is a sub-elliptic estimate*

$$\|u\|^{\Omega}_{1+\lambda+\varepsilon} \leqq C(\lambda,\varepsilon,l,\Omega,\mathscr{L}) \cdot \{\|g\|^{\Omega}_{\lambda} + \|f\|^{\partial\Omega}_{1+\lambda}\} .$$

For the case when α has the opposite behavior we have

THEOREM 2. *Assume that H is a sub-manifold of $\partial\Omega$ with dimension $n-2$ and of class C^3, that X makes a strictly positive angle with H and that α changes its sign from minus to plus on every γ_p with $p \in H$. Let $u \in C^{1+\lambda}(\bar{\Omega}) \cap C^2(\Omega)$ be a solution of $\mathscr{L}u=g$ in Ω with $\partial u/\partial l=f$ on $\partial\Omega$ such that $g \in C^\lambda(\bar{\Omega})$ and $f \in C^{1+\lambda}(\partial\Omega)$.*

Then for any ε satisfying $0<\varepsilon<(m+1)^{-1}$ and $\lambda+\varepsilon<1$ the function u belongs to $C^{1+\lambda+\varepsilon}(\bar{\Omega}) \cap C^{2+\lambda}_{\text{loc}}(\bar{\Omega} \setminus H)$ if and only if $u|_H \in C^{1+\lambda+\varepsilon}(H)$. There is the sub-elliptic estimate

$$\|u\|^{\Omega}_{1+\lambda+\varepsilon} \leqq C(\lambda, \varepsilon, l, \Omega, \mathscr{L})\cdot\{\|g\|^{\Omega}_{\lambda}+\|f\|^{\partial\Omega}_{1+\lambda}+\|u\|^{H}_{1+\lambda+\varepsilon}\} .$$

REMARKS. 1). The two theorems can be combined to give a regularity result when $H=H_1 \cup H_2$ where $H_1 \cap H_2$ is empty and α behaves according to the hypothesis of Theorem 1 on H_1 and according to Theorem 2 on H_2.

2). At present it is not clear whether or not the inequality $\varepsilon<(m+1)^{-1}$ can be replaced by the corresponding equality. It is worth noticing that Sobolev space technique gave the result with equality.

3). We have required more regularity of $\mathscr{L}$, l and Ω than is necessary. In fact it is sufficient to impose the strong conditions only in an arbitrarily small neighbourhood of H.

3. Some lemmas.

The following Schauder type estimates are probably well known. However, we have not been able to find a direct reference and hence we will sketch the proofs.

LEMMA 1. *Let* $\delta>0$ *and put* $\Omega^{\delta}=\{x \in \Omega : \text{dist}\,(x, \partial\Omega)>\delta\}$. *Then for every* $0<\lambda\leqq\lambda'<1$ *there is a constant C which does not depend on* δ *such that for all* $u \in C^2(\Omega)\cap C^{1+\lambda}(\bar{\Omega})$ *there is the estimate*

$$[u]^{\Omega^{\delta}}_{1+\lambda'} \leqq C\cdot\{\|\mathscr{L}u\|^{\Omega}_{0}+\delta^{\lambda-\lambda'}\cdot[u]^{\Omega}_{1+\lambda}\} \tag{3.1}$$

PROOF. Let $g=\mathscr{L}u$ and note that the function v defined by $v(x)=u(\delta x)$ for $x \in \delta^{-1}\cdot\Omega$ satisfies the equation

$$\mathscr{L}'v = A_{ij}v_{ij}+\delta B_i v_i+\delta^2 Cv = \delta^2\cdot g(\delta x)$$

where $A_{ij}(x)=a_{ij}(\delta x)$, $B_i(x)=b_i(\delta x)$ and $C(x)=c(\delta x)$. Now let $\mathscr{A}$ be the intersection of $\delta^{-1}\Omega$ with a ball of radius r_0 and let $\mathscr{A}'=\{x \in \mathscr{A} : \text{dist}\,(x, \partial\mathscr{A}) \geqq 1\}$.

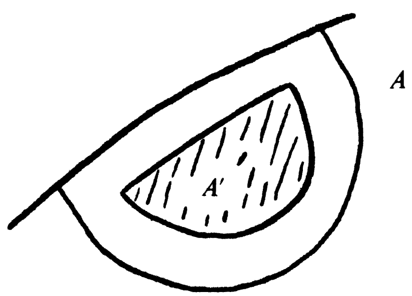

Fig. 1.

Subtract from v a linear function such that at a point $x_0 \in \mathscr{A}$, the new function w satisfies $w(x_0)=0$, $\operatorname{grad} w(x_0)=0$, and

$$\mathscr{L}'w = -\delta B_i \cdot v_i(x_0) - \delta^2 C \cdot v(x_0) + \delta^2 \cdot g(\delta x) .$$

Because of Theorem 9.3 in Agmon–Douglis–Nirenberg [1] we get

$$[w]_{1+\lambda'}^{\mathscr{A}'} \leqq C(r_0, \Omega, \mathscr{L}, \lambda') \cdot \{\delta \cdot \|\operatorname{grad} v\|_0^{\Omega} + \delta^2 \|v\|_0^{\Omega} + \delta^2 \cdot \|g\|_0^{\Omega} + \|w\|_0^{\mathscr{A}}\} .$$

Since $w(x_0)=0$ and $\operatorname{grad} w(x_0)=0$ it follows that

$$\|w\|_0^{\mathscr{A}} \leqq (1+\lambda)^{-1} r_0^{1+\lambda} [w]_{1+\lambda} .$$

Fix $r_0 \geqq 1$ and cover $\delta^{-1}\partial\Omega$ by finitely many sets of the type $\mathscr{A}$ such that $\delta^{-1}\Omega^{\delta}$ is covered by the corresponding $\mathscr{A}'$-s. The seminorm $[\cdot]_{1+\lambda}$ does not notice linear functions and hence we get

$$[v]_{1+\lambda'}^{\delta^{-1}\Omega^{\delta}} \leqq C \cdot \{\delta \|\operatorname{grad} v\|_0 + \delta^2 \|v\|_0 + [v]_{1+\lambda} + \delta^2 \|g\|_0\}$$

where on the right hand side the norms are to be taken over $\delta^{-1}\Omega$. This transforms into

$$\delta^{1+\lambda'} \cdot [u]_{1+\lambda'}^{\Omega^{\delta}} \leqq C \cdot \{\delta^2 \|u\|_1^{\Omega} + \delta^{1+\lambda} [u]_{1+\lambda}^{\Omega} + \delta^2 \|g\|_0^{\Omega}\}$$

which leads to (3.1).

LEMMA 2. *Assume that L is an extension of l to $\bar{\Omega}$ of class C^3 and take $0 < \lambda \leqq \lambda' < 1$. Let $u \in C^{1+\lambda'}(\bar{\Omega}) \cap C^2(\Omega)$ be a solution of $\mathscr{L}u=0$ in Ω such that $\partial u/\partial l = f$ belongs to $C^{1+\lambda}$ on the boundary. Then $\partial u/\partial L = (L \cdot \operatorname{grad})\, u$ belongs to $C^{1+\lambda}(\bar{\Omega}) \cap C^{1+\lambda'}(\Omega)$ and*

$$\left[\frac{\partial u}{\partial L}\right]_{1+\lambda'}^{\Omega^{\delta}} \leqq C \cdot \{\delta^{\lambda-\lambda'} \|f\|_{1+\lambda}^{\partial\Omega} + \|u\|_{1+\lambda'}^{\Omega}\} \tag{3.2}$$

where C does not depend on δ.

PROOF. First we note that $\partial u/\partial L$ weakly satisfies the identity

$$\mathscr{L}\left(\frac{\partial u}{\partial L}\right) = D_j(a'_{ij} u_i) + b'_i u_i + D_i(c'_i u) + c' u$$

where the coefficients a'_{ij}, b'_i, c'_i and c' belong to C^1. Hence the lemma follows from Agmon–Douglis–Nirenberg [1] as in the previous proof.

4. Proof of Theorem 1.

Fix an extension L of l. According to (2.1) there is a positive constant α_1 such that

$$\text{dist}\,(x_p(s), \partial\Omega) \geqq \alpha_1 \cdot |s|^{m+1} \tag{4.1}$$

for those s between $-s_0$ and s_0 for which $x_p(s) \in \bar{\Omega}$.

Since we always can subtract a solution $v \in C^{2+\lambda}(\bar{\Omega})$ of $\mathscr{L}v = g$ from u, we can assume that $\mathscr{L}u = 0$. Another simplification is to assume that $\alpha \geqq 0$ on $\partial\Omega$. We return to the general case later.

We will make extensive use of the following identity

$$u(p) = u(x_p(-t)) + \int_{-t}^{0} \frac{\partial u}{\partial L} \circ x_p(s)\, ds \tag{4.2}$$

which is true as long as L has unit length along $\{x_p(s) : -t \leqq s \leqq 0\}$.

We will define a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of exponents in Hölder estimates. Here $\lambda_0 = \lambda$ and the subsequent exponent are defined recursively. Let us introduce the notation

$$\delta(s) = \alpha_1 \cdot |s|^{m+1} . \tag{4.3}$$

Because of the C^2-dependence of $x_p(s)$ on p, (4.1), and (4.2) we find that if $u \in C^{1+\lambda_k}(\bar{\Omega})$ and $\lambda \leqq \lambda_k \leqq \lambda' < 1$ then

$$|Du(p) - Du(q)| \leqq C \cdot \|u\|_{1+\lambda'}^{\Omega^{\delta(t)}} \cdot |p-q|^{\lambda'} +$$

$$+ C \cdot |p-q|^{\lambda_k} \cdot \int_0^t \left\| \frac{\partial u}{\partial L} \right\|_{1+\lambda_k}^{\Omega^{\delta(s)}} ds + C \cdot \left\{ \|u\|_1 + \left\| \frac{\partial u}{\partial L} \right\|_1 \right\} \cdot |p-q|^{\lambda'} .$$

Here the last term is of smaller order and can be absorbed by the others. From Lemma 1 and Lemma 2 it follows that

$$|Du(p) - Du(q)| \leqq C_1 \cdot |p-q|^{\lambda'} \cdot \delta(t)^{\lambda_k - \lambda'} + C_2 \cdot |p-q|^{\lambda_k} \cdot \int_0^t \delta(s)^{\lambda - \lambda_k} ds .$$

We introduce (4.3) and carry out the integration with the result that

$$|Du(p) - Du(q)| \leqq C_3 \cdot |p-q|^{\lambda'} \cdot t^{(\lambda_k - \lambda') \cdot (m+1)} +$$

$$+ C_4 \cdot |p-q|^{\lambda_k} \cdot t^{1 - (\lambda_k - \lambda) \cdot (m+1)}$$

if $(\lambda_k - \lambda)(m+1) < 1$. With $\lambda' = \lambda + (m+1)^{-1}$ and an optimal choice of t we get

$$|Du(p) - Du(q)| \leqq C \cdot |p-q|^{\lambda_{k+1}} \tag{4.4}$$

where $\lambda_{k+1} = (\lambda_k + \lambda + (m+1)^{-1})/2$ and hence the sequence $\{\lambda_k\}$ is given by

$$\lambda_k = \lambda + (1 - 2^{-k}) \cdot (m+1)^{-1}$$

which tends to $\lambda + (m+1)^{-1}$ as k tends to infinity. By (4.4) this proves the theorem in case $\alpha \geqq 0$ on $\partial\Omega$.

To get the general result we note that with the same technique we can prove (4.4) for all p and q belonging to the same subset

$$\mathscr{A}_+ = \{p \in \partial\Omega : \alpha(p) > 0\} \quad \text{or} \quad \mathscr{A}_- = \{p \in \partial\Omega : \alpha(p) < 0\} \text{ of } \partial\Omega .$$

But if $\alpha(p) > 0$ and $\alpha(q) < 0$ then there is a $p' \in H$ on a geodesic between p and q in $\partial\Omega$ and thus in the decomposition

$$|Du(p) - Du(q)| \leqq |Du(p) - Du(p')| + |Du(p') - Du(q)|$$

we can apply (4.4) to both terms on the right hand side. Here we have used the fact that α does not change its sign from minus to plus along X-curves.

5. Proof of Theorem 2.

As in section 4 we note that it is sufficient to consider the case when $\mathscr{L}u = 0$ in Ω. The first step in the proof of Theorem 2 is to verify that the following result is true.

PROPOSITION. *Let $\tilde{H} \subset \bar{\Omega}$ be the closure of an $(n-1)$-dimensional manifold, generated by normals from H. If $u \in C^{1+\lambda}(\bar{\Omega}) \cap C^2(\Omega)$ is a solution of $\mathscr{L}u = 0$ in Ω with $\partial u/\partial l \in C^{1+\lambda}(\partial\Omega)$ and if $u|_H \in C^{1+\lambda'}(H)$, $0 < \lambda \leqq \lambda' < 1$, then $u|_{\tilde{H}} \in C^{1+\lambda'}(\tilde{H})$.*

PROOF. There are finitely many n-dimensional balls with center at points in H such that their union covers H and that in each ball, intersected by $\bar{\Omega}$, a coordinate system can be introduced in which L is the constant vector field $\hat{x}_1$, $\tilde{H}$ is given by $x_1 = 0$ and H is characterized by $x_1 = x_2 = 0$. Consider such an intersection Ω' and let $\mathscr{A} \subset \tilde{H} \cap \Omega'$ be such that $\operatorname{dist}(\mathscr{A}, \partial(\tilde{H} \cap \Omega') \setminus H) > 0$.

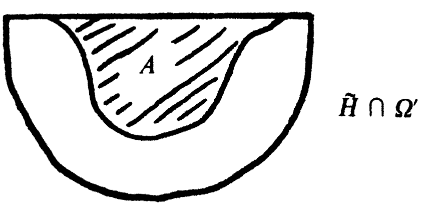

Fig. 2.

Since the relation $\mathscr{L}u = 0$ can be written

$$\mathscr{L}'u = \mathscr{P}\left(\frac{\partial u}{\partial x_1}\right)$$

where $\mathscr{L}'$ is an elliptic second order opeator in the variables $(x_2, \ldots, x_n)$ and $\mathscr{P}$ is a first order expression in $\partial u/\partial x_1$ it follows from Theorem 9.3 of [1] that $u \in C^{1+\lambda'}(\mathscr{A})$. By choosing the $\mathscr{A}$-s appropriately, their union covers a neighbourhood of H in $\tilde{H}$ and since *interior* regularity is well established the proposition is proved.

Now let p and q be two points in the closure of a component of $\partial\Omega \setminus H$. Since the regularity is questionable only in a neighbourhood of H, we may assume that p and q are close enough to H in order that $\Gamma_p \cap \tilde{H}$ and $\Gamma_q \cap \tilde{H}$ are non-empty. They consist of one point each, $\{p'\}$ and $\{q'\}$ with parameter values $s(p)$ and $s(q)$ on the L-curves.

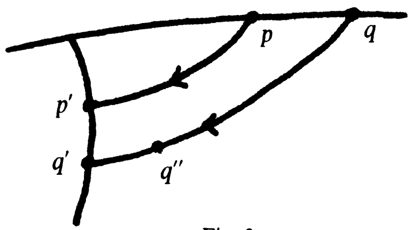

Fig. 3.

We assume that $|s(p)| \leqq |s(q)|$ and denote by q'' the point $x_q(s(p))$. Let $\lambda_0 = \lambda$ and assume that $\lambda_1, \ldots, \lambda_k$ have been found such that $u \in C^{1+\lambda_k}(\bar{\Omega})$ and $\lambda_k \leqq \lambda + \varepsilon$. Again we use the representation (4.2) to get new estimates. This time we consider two cases. In fact, if

$$|s(p)| \leqq C_1 \cdot |p-q|^{1/2(m+1)}$$

(where C_1 has to be specified later on) then we consider the inequality

$$|Du(p) - Du(q)| \leqq C_2 \cdot |\operatorname{grad} u(p') - \operatorname{grad} u(q'')| + \\ + C_3 \cdot |p-q|^{\lambda_k} \cdot |s(p)|^{1-(\lambda_k - \lambda)(m+1)} .$$

The last term can be estimated by $C \cdot |p-q)^{\lambda''}$ where $\lambda'' = (\lambda_k + \lambda + (m+1)^{-1})/2$.

The first term splits up into

$$|\operatorname{grad} u(p') - \operatorname{grad} u(q')| + |\operatorname{grad} u(q') - \operatorname{grad} u(q'')|$$

where the first difference is estimated by $C \cdot |p'-q'|^{\lambda'}$ according to the proposition and the second difference is represented by an integral of grad $(\partial u/\partial L)$ along a curve of length $|s(p) - s(q)| \leqq C \cdot |p-q|$. Hence if $\lambda'' \leqq \lambda + \varepsilon$ it follows that we can take $\lambda_{k+1} = \lambda''$, i.e. the exponents λ_k are given by

$$\lambda_k = \lambda + (1 - 2^{-k})(m+1)^{-1} .$$

However, we must also consider the case when $|s(p)| > C_1 \cdot |p-q|^{1/2(m+1)}$. In this case it is enough to reproduce the proof in section 4 since the optimal value of t, used to get (4.4) is $t = C_0 \cdot |p-q|^{1/2(m+1)}$. Hence we can choose $C_1 = 2C_0$ and the proof is complete.

ACKNOWLEDGMENTS. This work was supported by the Swedish Natural Science Research Council.

REFERENCES

1. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions* I, Comm. Pure Appl. Math. 12 (1959), 623–727.
2. Ju. V. Egorov, *Sub-elliptic pseudo-differential operators*, Soviet Math. Dokl. 10 (1969), 1056–1059.
3. Ju. V. Egorov and V. A. Kondrat'ev, *The oblique derivative problem*, Mat. Sb. 78 (1969), 139–169.
4. L. Hörmander, *Pseudo-differential operators and non-elliptic boundary problems*, Ann. of Math. 83 (1966), 129–209.
5. V. G. Maz'ja, *On a degenerating problem with directional derivative*, Mat. Sb. 87 (1972), 129–169.
6. A. Melin and J. Sjöstrand, *Fourier integral operators with complex phase functions and parametrix for an interior boundary value problem*, Comm. Partial Differential Equations 1 (1976), 313–400.
7. K. Taira, *Sur le problème de la dérivée oblique*, C.R. Acad. Sci. Paris Sér. A 284 (1977), 1511–1513.
8. B. Winzell, *Solutions of second order elliptic partial differential equations with prescribed directional derivative on the boundary*, Linköping Studies in Science and Technology, Dissertations No 003, 1975.
9. B. Winzell, *The oblique derivative problem* I, Math. Ann. 229 (1977), 267–278.

DEPARTMENT OF MATHEMATICS
LINKÖPING UNIVERSITY
S-58183 LINKÖPING
SWEDEN