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IMBEDDINGS, IMMERSIONS AND INTEGRABILITY OF CHARACTERISTIC CLASSES

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Introduction.

Let $f_r\colon X_r \to \mathrm{BO}(r)$ be a sequence of fibrations with maps $g_r\colon X_r \to X_{r+1}$ such that the usual diagram commutes. For such a situation Lashof defines the concept of an X-structure on manifolds in [8] and proves a Thom isomorphism type theorem for bordism groupes of such manifolds. Many of the usual classes of manifolds mat be described in terms of X-structures, e.g. U, SO, Spin etc., as well as some more esoteric classes of manifolds. For example Hirsch's theorem reduces the study of manifolds which immerse in codimension k to an appropriate X-structure, (Hirsch ([7], Wells [16]).

In this paper we study X-characteristic classes with rational coefficients i.e. the group $H^*(X; \mathrm{Q}) = \varprojlim H^*(X_r; \mathrm{Q})$. In particular we are interested in those rational X-characteristic classes which go to integral cohomology classes by the normal map of all n-manifolds with an X-structure, (by normal map I mean the lifting $M \to X_r$, of the Gauss map $M \to \mathrm{BO}(r)$).

Let I_n^i be the set of all rational X-characteristic classes which are integral cohomology on all n-manifolds with an X-structure, (a cohomology class is called integral if it is the reduction of an integral cohomology class). The set I_n^n has been computed by Hattori and Stong, for certain X-structures, (see [6] and [12], [13]). I_n^{n-1}, and I_n^{n-2} have been computed by the author for U-structures (see [11]). The rest of these sets is unknown.

Let us restrict ourselves to n-manifolds which immerse in codimension k, (k is a positive integer), and they have an X-structure on the normal bundle of the immersion. Let us consider the set $I_{n,k}^i \subseteq H^i(X; \mathrm{Q})$ of those rational X-classes which are integral on all such manifolds. Clearly $I_{n,k}^i \supseteq I_n^i$. Furthermore it is clear that $I_{n,k}^i$ contains the kernel of the map $l_k\colon H^i(X; \mathrm{Q}) \to H^i(X_k; \mathrm{Q})$, defined in the obvious way. The question is what else does it contain. We are going to prove, under certain assumptions on X, that in the range $i \leqq (n+k)/2$, essentially it does not contain anything else, (see Theorem 1 below). The same considerations are applicable to case of imbeddings too, and Theorem 2 below covers some cases.

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In the same spirit we can consider the mod-m X-characteristic classes, $H^i(X; \mathbf{Z}_m) = \varprojlim H^i(X_r; \mathbf{Z}_m)$, ($m$ is a positive integer), and let $I_n^i(m)$ be the subset of $H^i(X; \mathbf{Z}_m)$ which consist of those classes which are zero on all n-dimensional X-manifolds. Also let $I_{n,k}^i(m)$ be the set of the i-dimensional mod-m X-characteristic classes, which are zero on all n-manifolds which immerse in $\mathbf{R}^{n+k}$, with an X-structure on the normal bundle of the immersion. Finally we keep the same notation for the obvious map $l_k: H^*(X; \mathbf{Z}_m) \to H^*(X_k; \mathbf{Z}_m)$.

Next we describe the assumptions that we will need.

From now on m will be a non-negative integer. For all r's X_r contains a finite number of cells in each dimension, and the torsion of its cohomology is prime to m, (if m is 0 this assumption will be interpreted that there is no torsion). We assume that γ_r, the pull-back over X_r of the universal r-linear bundle is orientable. We assume that the map $H^*(X_{r+1}; \mathbf{Z}_p) \to H^*(X_r; \mathbf{Z}_p)$ is an iso up to dimension r, for all primes dividing m, (if m is 0, this will include all primes). Finally we assume that the map $H^*(X_{r+1}; \mathbf{Z}_p) \to H^*(X_r; \mathbf{Z}_p)$ is onto in all dimensions when p is a prime dividing m.

These assumption are satisfied by U, SU-structures for all m's, and by SO, Spin, Pin-structures when m is an odd positive integer.

Under the above assumptions we have.

THEOREM 1. *If* $i \leqq (n+k)/2$, *then* $I_{n,k}^i(m) = I_n^i(m) + (\ker l_k)^i$.

REMARK. If m is zero $I_{n,k}^i(0)$ and $I_n^i(0)$ are respectively $I_{n,k}^i$ and I_n^i.

THEOREM 2. *If a* mod-m *(respectively rational) X-characteristic class, is zero (respectively integral) on all n-manifolds which imbed in* $\mathbf{R}^{n+k}$ *with an X-structure on the normal map of the imbedding, and if this X-characteristic class is of dimension less than k, then it is zero (respectively integral) on all n-dimensional n-manifolds.*

It would be interesting to know what happens in the higher dimensions.

I was motivated to look at these questions by M. Bendersky's results in [1]. The above Theorems hold true for the case of $\mathbf{Z}_p$ coefficients without the assumptions on the torsion of X_r, and without assuming that $H^*(X_{r+1}; \mathbf{Z}_p) \to H^*(X_r; \mathbf{Z}_p)$ is onto. So for $\mathbf{Z}_2$ coefficients O, SO, Spin, Pin-structures are included.

1. Homological Algebra.

In this section we will state a few facts from Homological Algebra, which although I am sure they are well-known, I could not find explicit references.

DEFINITION 1.1. A commutative ring R is called self-injective if it is injective as an R-module.

For details about injectivity and related staff see MacLane's book [9]. Let $\mathbf{Z}_m$ be the ring of integers mod-m, (m is a positive integer).

PROPOSITION 1.2. *The ring* $\mathbf{Z}_m$ *is self-injective.*

PROOF. We will apply Theorem 7.2 of [9]. An ideal of $\mathbf{Z}_m$ is of the form $\mathbf{Z}_a$, where a is a positive integer which divides m. We will compute $\mathrm{Ext}_{\mathbf{Z}_m}(\mathbf{Z}_a, \mathbf{Z}_m)$. For that we consider a free $\mathbf{Z}_m$-resolution of $\mathbf{Z}_a$,

$$\ldots \to \mathbf{Z}_m \xrightarrow{m/a} \mathbf{Z}_m \xrightarrow{a} \mathbf{Z}_m \xrightarrow{1} \mathbf{Z}_a \to 0$$

where its map is the multiplication by the indicated integer. Let $f\colon \mathbf{Z}_m \to \mathbf{Z}_m$ be a map such that $(m/a)f=0$. Let us assume that f is given by multiplication by an integer b, then a must divide b. Consider the map $g\colon \mathbf{Z}_m \to \mathbf{Z}_m$ which is multiplication by b/a, then $f=ag$, which implies by the very definition that $\mathrm{Ext}_{\mathbf{Z}_m}(\mathbf{Z}_a, \mathbf{Z}_m)=0$.

Next we consider the rational numbers $\mathbf{Q}$, and the rational numbers modulo the integers $\mathbf{Q}/\mathbf{Z}$.

PROPOSITION 1.3. *As abelian groups, (namely as* $\mathbf{Z}$*-modules),* $\mathbf{Q}$ *and* $\mathbf{Q}/\mathbf{Z}$ *are injective.*

PROOF. It follows from the obvious fact that both of them are divisible (see Corollary 7.3 of [9]).

PROPOSITION 1.4. *Let* R *be a commutative ring.*

(i) *If* A, X, Y, *are* R*-modules,* Y *is injective, and there is a mono* $A \to X$. *Then the dual map* $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(A,Y)$ *is onto.*

(ii) *If* $A \to B \to C$ *is a sequence of abelian groups, and assume that* C *is a* $\mathbf{Z}_m$*-module, (*m *is positive), then the dual sequence* $\mathrm{Hom}(C,\mathbf{Z}_m) \to \mathrm{Hom}(B,\mathbf{Z}_m) \to \mathrm{Hom}(A,\mathbf{Z}_m)$ *is exact.*

iii) *Let* C *be a chain complex over a self-injective ring* R. *Then the usual map* $H^*(C) \to \mathrm{Hom}(H_*(C),R)$ *is an iso.*

iv) *Let* $A \to B \to C$ *be an exact sequence of* R*-modules and* X *be an injective* R*-module. Then the dual sequence* $\mathrm{Hom}(C,X) \to \mathrm{Hom}(B,X) \to \mathrm{Hom}(A,X)$ *is exact.*

PROOF. Part i) is the definition of injective. The rest is a not difficult exercise.

Caution. Part iii) of the previous Proposition is not included in the usual treatment of the Universal coefficient Theorem if R is not a principal ideal domain. We want to apply the statement for the ring $\mathbf{Z}_m$ which is principal ideal but not a domain always.

2. The case of immersed manifolds.

In this section we will prove Theorem 1.

From now on we adopt the following notational conventions. We will use the symbol $\mathbf{Z}_0$ to denote the integers $\mathbf{Z}$.

Let $K(m)$ be the Eilenberg–MacLane space $K(\mathbf{Z}_m, n-i)$, where m is any positive integer or zero. We put $K(m)_+ = K(m) \cup \{\text{point}\}$, and let c be the fundamental class of $K(m)$. If G is an abelian group, we put $G^*(m) = \operatorname{Hom}(G, \mathbf{Z}_m)$ when m is positive, and $G^*(0) = \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$. We select Thom classes $U_r(m) \in H^r(T\gamma_r; \mathbf{Z}_m)$, ($\gamma_r$ is the pull-back over X_r of the universal r-linear bundle) such that if $Tg_r\colon T(\gamma_r + \varepsilon) \to T\gamma_{r+1}$, is the Thomification of the map $g_r\colon X_{r+1} \to X_r$, we have $Tg_r^*(U_{r+1}) = SU_r$, (the letter S denotes the suspension). N will be a natural number which will be taken vary big with respect to n, k. We put

$$A' = S^{N-k}T\gamma_k \wedge K(m)_+, \qquad A = S^{N-k}T\gamma_k \wedge K(m)\,,$$

$$B' = T\gamma_N \wedge K(m)_+, \qquad B = T\gamma_N \wedge K(m)\,.$$

From now on, usually, we will drop m from the symbols $K(m)$, $G^*(m)$, etc. and all cohomology and homology groups will have $\mathbf{Z}_m$-coefficients. Every Proposition below refers to one non-negative integer m.

LEMMA 2.1. *The maps $\pi_j^*(A) \to \pi_j^*(A')$ and $\pi_j^*(B) \to \pi_j^*(B')$, induced by the obvious projections $A' \to A$ and $B' \to B$, respectively, are monomorphisms in the stable range.*

PROOFS. We will prove the second case only, the other one being similar. We observe that the spaces B, B' are highly connected. Let us consider the obvious projections $B' \to B$; and $B' \to T\gamma_N$. They induce a map $B' \to B \times T\gamma_N$ which is an isomorphism in $\mathbf{Z}_p$ cohomology in the stable range, (this follows easily from the Künneth formula and Thom isomorphism) for all primes p, so it induces an iso among the homotopy groups in the same range. Furthermore the projection $B \times T\gamma_N \to B$, induces an onto map among homotopy groups, so the composite projection $B' \to B$ induces an onto map among homotopy groups, so the dual map $\pi_j^*(B) \to \pi_j^*(B')$ is mono in the stable range.

LEMMA 2.2. *Let s be a positive integer and Y a simply connected complex* *having a finite number of cells in each dimension, such that* $H^j(Y;\ \mathbf{Z}_p)=0$ *for* $j \leqq s$ *whenever* p *is a prime number dividing* m, *(in the case where* m *is* 0, *that will include all primes). Furthermore we assume that all torsion elements of* $H^*(Y;\ \mathbf{Z})$, *have order prime to* m, *(in the case where* m *is* 0 *this means that there is no torsion). Under those condition, there is a map from* $Y \wedge K(\mathbf{Z}_m, n-i)$ *to a product of* $K(\mathbf{Z}_{m'})$*'s which is an iso in* mod-p *cohomology for all prime numbers, up to dimension* $s+1+2(n-i)$.

PROOF. Let S be a minimal set generating $H^*(Y;\ \mathbf{Z}_m)$ as a $\mathbf{Z}_m$-module. Every element x of S determines a map f_x: $Y \wedge K(\mathbf{Z}_m;\ n-i) \to K(\mathbf{Z}_m, n-i+\dim x)$ which pulls-back the fundamental class to $x \wedge c$, (recall c is the fundamental class of $K(\mathbf{Z}_m, n-i)$. Next we consider the product of all those maps, this provides the required iso.

Consider the cofibration $S^{N-k}T\gamma_k \to T\gamma_N \to L$, where L is the cofibre of the map $S^{N-k}T\gamma_k \to T\gamma_N$, which is the Thomification of the obvious map $X_k \to X_N$. By smashing it with K we get the cofibration $A \to B \to L \wedge K$, which is a fibration too since we are in the stable range.

LEMMA 2.3. *The homotopy groups of* $L \wedge K$ *in dimensions non-greater than* $N+k+2(n-i)$, *are free* $\mathbf{Z}_m$*-modules.*

PROOF. The map $X_k \to X_N$ gives an iso in mod-p cohomology up to dimension k, for all primes dividing m. By the Thom isomorphism, $S^{N-k}T\gamma_k \to T\gamma_N$ gives an iso up to dimension $N+k$, for the same primes. So by the long exact cohomology sequence of the cofibration, $H^j(L;\ \mathbf{Z}_p)=0$ for $j \leqq N+k$. By the same long exact sequence and because of the assumptions on X_r's we deduce that all the torsion of L is prime to m. So we can apply the previous Lemma and the result follows free of charge.

We introduce the map $F: H^i(X_k;\ \mathbf{Z}_m) \to \pi^*_{N+n}(A)$ when m is positive and F: $H^i(X_k;\ \mathbf{Q}) \to \pi^*_{N+n}(A)$ when m is zero, defined by the following formula,

$$F(x)([a]) = a_*(xS^{N-k}U_k \wedge c)([S^{N+n}]) .$$

In exactly the analogous way we introduce the map G: $H^i(X_N;\ \mathbf{Z}_m) \to \pi^*_{N+n}(B)$ for m positive and G: $H^i(X_N;\ \mathbf{Q}) \to \pi^*_{N+n}(B)$ for m being zero, defined by the formula

$$G(x)([a]) = a_*(xU_N \wedge c)([S^{N+n}]) .$$

PROPOSITION 2.4. $I^i_{n,k}(m)$ *and* $I^i_n(m)$ *are respectively the kernel of* Fl_k *and* G.

PROOF. It follows easily from (2) and (3), because of the Lemma below.

LEMMA 2.5. *Let M be an orientable compact closed smooth manifold of dimension n.*

i) *Let m be positive and x an element of $H^i(M; Z_m)$ which has zero cup product with all elements of $H^{n-i}(M; Z_m)$. Then x is zero.*

ii) *Let x be an element of $H^i(M; Q)$ which has integral cup product with all elements of $H^{n-i}(M; Z)$. Then x is integral.*

PROOF. i) Via the cup product, x induces the zero homomorphism in the map

$$H^i(M; Z_m) \to \mathrm{Hom}\,(H^{n-i}(M; Z_m), Z_m) = \mathrm{Hom}\,(H_i(M; Z_m), Z_m)\,.$$

But this is an iso because of Proposition 1.4. And the result follows.

ii) Let us consider the commutative diagram

$$\begin{array}{ccc} H^i(M; Q) & \to & \mathrm{Hom}\,(H_i(M; Z), Q) \\ \downarrow & & \downarrow \\ H^i(M; Q/Z) & \to & \mathrm{Hom}\,(M; Z), Q/Z) \end{array}$$

where both the horizontal map are iso since Q and Q/Z are injective. By assumption x goes to zero under the top map, and the result follows.

And now we are ready to prove:

PROOF OF THEOREM 1. Consider the following commutative diagram

$$\begin{array}{ccccc} \pi^*_{N+n}(L \wedge K) & \xleftarrow{H^*} & H^{N+n}(L \wedge K) & & \\ \downarrow & & \downarrow & & \\ \pi^*_{N+n}(B) & \xleftarrow{H^*} & H^{N+n}(B) & \xleftarrow{U_N \wedge c} & H^i(X_N) \\ \downarrow & & \downarrow & & \downarrow l_k \\ \pi^*_{N+n}(A) & \xleftarrow{H^*} & H^{N+n}(A) & \xleftarrow{S^{N-k}U_k \wedge c} & H^i(X_k) \end{array}$$

Which is defined as follows. If m is positive the cohomology groups have Z_m coefficients and if m is zero the coefficient group is Q. The H^*'s are the dual of the Hurewicz homomorphism. The two horizontal maps on the right are defined multiplying with the indicated element. The two vertical sequences come from the homotopy and cohomology sequences of the fibration. They are both exact by Proposition 1.2, 1.3, 1.4. The top H^* map is onto by Lemma 2.3. Finally the composite of the two horizontal sequences are G and F. And the result follows by chasing the diagram.

3. The case of imbedded manifolds.

The proof of Theorem 2, follows the same line as Theorem 1.
Analogously with A', A, B', B we define

$$C' = T\gamma_k \wedge K_+, \quad C = T\gamma_k \wedge K, \quad D' = \Omega^{N-k}T\gamma_N \wedge K_+,$$
$$D = \Omega^{N-k}T\gamma_N \wedge K .$$

Analogously with Lemma 2.1, we have.

LEMMA 3.1. *The maps* $\pi_j^*(C) \to \pi_j^*(C')$ *and* $\pi_j^*(D) \to \pi_j^*(D')$, *induced by the obvious projections* $C' \to C$ *and* $D' \to D$ *respectively, are mono in dimensions less than* $2k+(n-i)$.

PROOF. Like Lemma 2.1.

PROOF OF THEOREM 2. Analogously with F, G we define maps $R\colon H^i(X_N) \to \pi_{n+k}^*(D)$ and $P\colon H^i(X_k) \to \pi_{n+k}^*(C)$, defined by the formulas

$$R(x)([a]) = a_*(\Omega^{N-k}(xU_N) \wedge c)([S^{n+k}]) ,$$

and

$$P(x)([a]) = a_*(xU_k \wedge c)([S^{n+k}]) .$$

Again, if i is less than $k, I_n^i(m)$ is the kernel of R, and the kernel of P is the set of those X-characteristic classes in $\mathbf{Z}_m$-cohomology (respectively $\mathbf{Q}$-cohomology) which are zero (respectively integral) on all n-manifolds which imbed in R^{n+k} with an X-structure on the normal bundle of the imbedding. On the other hand the obvious map $C \to D$, induces iso in cohomology up to dimension $2k+n-i-1$, so we get a monomorphism in the map $\pi^*(D) \to \pi^*(C)$ in this range of dimensions, and the Theorem follows.

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