MATH. SCAND. 43 (1978), 185–196

SMOOTHING H-SPACES

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0. Introduction.

It is an old question on the theory of H-spaces, whether a finite H-space is homotopy equivalent to a differentiable manifold. For a long time all known examples were Liegroups. Then Hilton and Roitberg discovered some new H-spaces in 1969 see [8]. These, however, were all differentiable manifolds by their definition. The main attack on the general problem is due to W. Browder, who in a series of papers proved that any 1-connected finite H-space is homotopy equivalent to a topological manifold and in dimensions $\neq 4k+2$ to a differentiable manifold.

The first examples of H-spaces that are not a priori given as differentiable manifolds are the H-spaces obtained by the method of homotopy mixing developed by A. Zabrodsky, see [15], [7]. These H-spaces are not a priori known to be finite, only finitely dominated, which of course, implies finite in the simply connected case since $K_0(\mathbf{Z})=0$. In this paper we consider H-spaces obtained by this method and we prove they are finite and in almost all cases we prove they are homotopy equivalent to differentiable manifolds, the only exception being when the H-space at the prime 2 is a product of $\mathbf{R}P^3$'s, S^7's and $\mathbf{R}P^7$'s. The method is: Given an H-space X of above type, to construct a fibration $S^1 \to X \to Y$ where Y is finitely dominated, and then use the existence of this fibration to prove that finiteness and surgery obstruction vanish.

1. Statement of results.

A CW-complex is quasifinite if the homology of the universal cover is a finitely generated abelian group and all homotopy groups are finitely generated, see [7].

For convenience we remind the reader of the definition of mixing homotopy types, also called Zabrodsky-mixing.

Let $P_1,\ldots,P_n$ be a partition of all primes, $X_1,\ldots,X_n$ quasi-finite CW-complexes that are rationally equivalent, i.e. the localization at 0, $(X_i)_0$ are

Received December 20, 1976; in revised form February 2, 1978.

homotopy equivalent. Having specified homotopy equivalences $(X_i)_0 \simeq (X_{i+1})_0$ we may form the homotopy pullback Z of the diagram

$$\begin{array}{ccccccc} (X_1)_{P_1} & & (X_2)_{P_2} & & & & (X_n)_{P_n} \\ \downarrow & & \downarrow & & & & \downarrow \\ (X_1)_0 & \simeq & (X_2)_0 & \simeq & \dots & \simeq & (X_n)_0 . \end{array}$$

This is the homotopy mixing of $X_1,\dots,X_n$ at $P_1,\dots,P_n$. Clearly Z is a quasifinite CW-complex, and hence dominated by a finite complex (i.e. homotopy equivalent to a retract of a finite complex. This follows from [7]).

We consider quasifinite CW-complexes that are H-spaces at some set of primes of the following type

a) Compact Lie-groups different from $\mathbf{R}P^3$
b) Total spaces of principal, stably trivial S^3-bundles over stably reducible, quasifinite, nilpotent Poincaré duality spaces
c) S^7, $\mathbf{R}P^7$ and $\mathbf{R}P^3$
d) Any stably reducible, nilpotent, quasifinite Poincaré duality space.

Here of course a) and c) are the most well-known examples. Examples of type b) are constructed in [9], and d) notably includes odd-dimensional spheres [1] and examples constructed in [9].

We prove

1.1. Theorem. *Let $P_1,\dots,P_n$ be a partition of all primes and let X_i be homotopy equivalent to products of spaces of the above type. We require that each X_i has a factor of type* a) *or* b). *If Z is an* H-*space obtained by homotopy mixing of $X_1,\dots,X_n$ at $P_1,\dots,P_n$ then Z is homotopy equivalent to a parallelizable differentiable manifold.*

If we choose all X_i the same, but different rational equivalences we may obtain any element in the genus of Z, this follows from [16]. Hence we get

1.2. Corollary. *Let Z be in the genus of a compact Lie group, then Z is homotopy equivalent to a parallelizable manifold.*

Proof. Except for the case $Z = (\mathbf{R}P^3)^k$ this is direct from Theorem 1.1. We defer the proof of this special case to the end.

2. The surgery arguments.

2.1. Proposition. *Let $S^1 \to Z \xrightarrow{p} Y$ be a fibration such that*

a) *Y is a stably reducibel finitely dominated Poincaré duality space.*
b) *p induces isomorphism of fundamental groups*
c) *$\pi_1(Y)$ acts trivially on $H^*(S^1; \mathbf{Z})$.*

Then Z is homotopy equivalent to a parallelizable differentiable manifold.

PROOF. First we need to deal with the finiteness obstruction. The formulae of [11] tells us that $\sigma(Z)=\chi(S^1)\cdot\sigma(Y)$ where $\chi(S^1)$ is the Euler-characteristic, hence $\sigma(Z)=0$ and Z is homotopy equivalent to a finite complex. We now let E be the total space of the corresponding D^2-fibration. Then (E, Z) is a Poincaré duality pair and we consider the classifying map $\nu_E\colon E \to BG$. We have the equation

$$\nu_E = p^*(\nu_Y) \oplus p^*(p)^{-1} .$$

Now ν_Y is trivial since Y was assumed to be stably reducible. Also p is an S^1-fibration classified by $G(2)$. But $O(2) \subseteq G(2)$ is a homotopy equivalence so p is fibre homotopy equivalent to an $O(2)$-bindle, actually an S^1-bundle since the fibration was assumed orientable. We thus get a linear reduction ζ of ν_E and the reduction is trivial when restricted to Z, since the pullback of an S^1-bundle to the total space of itself is trivial.

A standard procedure (see e.g. Browder [3, page 38]) sets up a surgery problem (a degree 1 normal map)

$$(M, \partial M) \xrightarrow{\varphi} (E, Z), \qquad \hat{\varphi}\colon \nu_M \to \zeta$$

with $\pi_1(E) \cong \pi_1(Y) \cong \pi_1(Z)$. However E is possibly not of finite homotopy type. We wish to compute the surgery obstruction when restricted to $\partial M \to Z$. If we cross the problem with S^1, $E \times S^1$ is homotopy equivalent to a finite complex [5], and $(E \times S^1, Z \times S^1)$ is a simple Poincaré pair. Hence the obstruction to completing surgery for the relative surgery problem

$$(M \times S^1, \partial M \times S^1) \xrightarrow{\varphi \times 1} (E \times S^1, Z \times S^1)$$

is an element of $L_n^s(\pi_1(E \times S^1), \pi_1(Z \times S^1))$. But $\pi_1(E) = \pi_1(Z)$ so by [14, Ch. IV] the surgery obstruction group is the trivial group, so $\varphi \times 1$ is normally cobordant to a homotopy equivalence. In particular

$$\varphi \times 1 \colon \partial M \times S^1 \to Z \times S^1$$

defines the zero element of $L_{n-1}^s(\pi_1(Z) \oplus \mathbf{Z})$.

Shaneson proves [13] that the map induced by product with S^1

$$L_{n-1}^h(\pi) \to L_n^s(\pi \oplus \mathbf{Z})$$

is a monomorphism, thus our original surgery problem

$$\varphi\colon \partial M \to Z, \qquad \hat{\varphi}\colon \nu_{\partial M} \to \varepsilon$$

must also have vanishing surgery obstruction. Hence Z is homotopy equivalent to a differentiable, parallelizable manifold.

3. 1-tori in H-spaces.

We start with a definition:

3.1. DEFINITION. Let Z be a quasifinite nilpotent space. We say that Z admits a 1-torus if there exists a fibration $S^1 \to Z \xrightarrow{p} Y$ such that

a) Y is a quasifinite, stably reducible nilpotent complex
b) p induces isomorphism of fundamental groups
c) $\pi_1(Y)$ acts trivially on $H^*(S^1; \mathbf{Z})$
d) There is a (rational) space B so that the rational type of the fibration $S^1 \to Z \xrightarrow{p} Y$ is given by

$$\begin{tikzcd} S_0^1 \arrow[r] \arrow[d, "\simeq"] & Z_0 \arrow[r] \arrow[d, "\simeq"] & Y_0 \arrow[d, "\simeq"] \\ K(\mathbf{Q},1) \arrow[r] & K(\mathbf{Q},3) \times B \arrow[r, "h \times 1"] & S_0^2 \times B \end{tikzcd}$$

where h is the localization of the hopf map $S^3 \to S^2$.

REMARKS. We note that the nilpotency condition in a) will automatically be satisfied in case Z is nilpotent since the long exact sequence in homotopy is a sequence of $\pi_1(Z)$, hence of $\pi_1(Y)$-modules where the action on $\pi_1(S^1)$ is trivial.

We also note there are some obvious H-spaces that do not admit a 1-torus due to condition d) notably T^k and S^7. However T^k is uninteresting in the context of mixing homotopy types since it is a $K(\mathbf{Z}^k, 1)$ so it always splits off as a factor. Further S^7 (and $\mathbf{R}P^7$) are the only known H-spaces at the prime 2 whose rational type does not contain a $K(\mathbf{Q}, 3)$ factor, so there is nothing to mix with. We shall see (Proposition 3.5) that all compact nonabelian Lie groups do admit a 1-torus in this sense except $(\mathbf{R}P^3)^k$ (because of condition b)). This fact does restrict the application of our method to smooth H-spaces which at the prime 2 are e.g. $(\mathbf{R}P^3)^k \times (S^7)^l \times (\mathbf{R}P^7)^m$. In spite of this we are able to say something in this case too (see section 4 final remarks).

The main reason for condition d) in definition 3.1 is that it ensures that the existence of 1-tori is a generic property for H-spaces:

3.2. PROPOSITION. Let Z be an H-space which admits a 1-torus. Then any $\bar{Z} \in G(Z)$ also admits a 1-torus.

PROOF. The proof of 3.2 is based on results from [16], but extended to the non-simply connected case as in [10]. According to [10] theorem 4.1 if $\bar{Z}$ is in the genus of Z and Z has finite fundamental group, then if

$$\bar{Z} \to \prod K(\mathbf{Z}, \text{odd}) \qquad Z \to \prod K(\mathbf{Z}, \text{odd})$$

define a basis for $QH^*(\bar{Z})$/torsion and $QH^*(Z)$/torsion respectively then there is a commutative diagram

$$\begin{array}{ccc} \bar{Z} & \xrightarrow{\bar{f}} & Z \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\ \prod K(\mathbf{Z}, \text{odd}) & \xrightarrow{f} & \prod K(\mathbf{Z}, \text{odd}) \end{array}$$

Moreover f is described by a diagonal matrix and if P denotes the set of primes occurring in the fibre of p in dim$<$dim$(\mathbf{Z})+1$ then f and $\bar{f}$ are P-equivalences. Clearly p and $\bar{p}$ will be $\bar{P}$-equivalences in dimensions $<$dim (Z) $+1$, where $\bar{P}$ is complement of P. The restriction that $\pi_1(Z)$ is finite is not serious here since free summands split off as product with S^1. It is now clear that $\bar{f}$: $\bar{Z} \to Z$ is a P-equivalence and

$$\bar{Z} \xrightarrow{\bar{p}} \prod K(\mathbf{Z}, \text{odd}) \xleftarrow{p} Z$$

is a $\bar{P}$-equivalence in dimensions $<\dim Z+1$.

Therefore $\bar{Z}$ is obtained as the homotopy pullback of

$$\begin{array}{ccccc} & & & & Z_P \\ & & & & \downarrow \\ Z_{\bar{P}} & \to & Z_0 & \xrightarrow{f_0} & Z_0 \end{array}$$

where f_0 is the localization of f at 0, p_0 is thought of as an identification. We want to prove that the fibration $S^1 \to Z \to Y$ can be split up in P- and $\bar{P}$-primary parts and recombined to produce a fibration $S^1 \to \bar{Z} \to \bar{Y}$. To do this it suffices to find a homotopy equivalence k_0: $Y_0 \to Y_0$ so that the diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{f_0} & Z_0 \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow{k_0} & Y_0 \end{array}$$

is homotopy commutative. This done we construct $\bar{Y}$ as the homotopy pullback of the diagram

$$\begin{array}{ccccc} & & & & Y_P \\ & & & & \downarrow \\ Y_{\bar{P}} & \to & Y_0 & \xrightarrow{k_0} & Y_0 \end{array}$$

and since $\bar{Y}$ is in the genus of Y, $\bar{Y}$ is a quasifinite, stably reducible, nilpotent complex [7]. By naturality of homotopy pullback (and localization) we then get a map $\bar{Z} \xrightarrow{\bar{p}} \bar{Y}$ with fibre in the genus of S^1 hence equivalent to S^1. Finally $\bar{p}$ induces isomorphism of π_1 since it does so at every prime, and the fibration is orientable since that is only a 2-primary question. To construct k_0 as above we first have to change f_0 slightly. If we change f_0 by multiplying the diagonal matrix representing f_0 by d^{-1}, where $d \notin P$ that does not change $\bar{Z}$. This is so because the map $\lambda\colon Z_P \to Z_P$, which is d'th power in any bracketing is a homotopy equivalence (since $d \notin P$) the inverse of which induces multiplication by d^{-1} on $H^*(Z_0; \mathbf{Q})$. We thus get a homotopy pullback diagram

$$\begin{tikzcd}
\bar{Z} \arrow[rr] \arrow[d] & & Z_P \arrow[r, "(\lambda)^{-1}"] \arrow[d] & Z_P \arrow[d] \\
Z_{\bar{P}} \arrow[r] & Z_0 \arrow[r, "f_0"] & Z_0 \arrow[r] & Z_0
\end{tikzcd}$$

proving the claim. Since all entries in f_0's matrix are in $\bar{P}$ we may thus arrange to have one entry equal to 1. We may now construct k_0 as follows: By (d) in definition 3.1 the rational type of the fibration $S^1 \to Z \to Y$ is

$$K(\mathbf{Q},1) \to K(\mathbf{Q}^n,3) \times \prod K(\mathbf{Q},\text{odd}) \to S_0^2 \times K(\mathbf{Q}^{n-1},3) \times \prod K(\mathbf{Q},\text{odd})$$

where odd $\neq 3$. It is clear how to define k_0 on the $\prod K(\mathbf{Q},\text{odd})$ factor. To see how to define k_0 on $S_0^2 \times \prod K(\mathbf{Q}^{n-1},3)$ we have to choose the basis in which f_0 is described carefully. Consider the diagram with short exact row

$$\begin{tikzcd}
 & & \mathbf{Z}^n \arrow[d] & & \\
0 \arrow[r] & \mathbf{Q}^{n-1} \arrow[r, "\varphi"] & \mathbf{Q}^n \arrow[r] & \mathbf{Q} \arrow[r] & 0
\end{tikzcd}$$

where the vertical map is the induced map $QH^3(Z,\mathbf{Z})/\text{torsion} \to QH^3(Z;\mathbf{Q})$, and

$$\varphi = p^*\colon QH^3(Y;\mathbf{Q}) \to QH^3(Z;\mathbf{Q})\,.$$

We wish to find a basis $\{l_1,\ldots,l_n\}$ of $\mathbf{Z}^n$ over $\mathbf{Z}$ so that the image of l_i belongs to $\mathbf{Q}^{n-1} i > 1$. Clearly $\mathbf{Q}^{n-1} \cap \mathbf{Z}^n$ is a free abelian group of rank $n-1$. We can assume that the codimension 1 vectorspace $\mathbf{Q}^{n-1}$ is defined by some equation $p_1x_1 + \ldots + p_nx_n = 0$ with the p_i integral and relatively prime. Then $\sum p_ix_i$ defines an isomorphism of $\mathbf{Z}^n/\mathbf{Q}^{n-1} \cap \mathbf{Z}^n$ with $\mathbf{Z}$ so we may pick a basis $\{l_2,\ldots,l_n\}$ in $\mathbf{Z}^n \cap \mathbf{Q}^{n-1}$ and choose l_1 to be some element with $\sum p_ix_i = 1$. If we assume (as we may) that the matrix of f_0 is diagonal in this basis with the entry corresponding to l_1 equal to 1, we can choose k_0 to be the identity on S_0^2 and the obvious map on $K(\mathbf{Q}^{n-1},3)$. This ends the proof.

We proceed to construct 1-tori in specific H-spaces.

3.3. LEMMA. *Let G be a compact connected Lie group, S^1 a subgroup such that S^1 determines a torsion element of $\pi_1(G)$. There is then a decomposition $G_0 = K(\mathbf{Q},3)\times\prod K(\mathbf{Q},\text{odd})$ such that the rational type of the fibration $S^1 \to G \to G/S^1$ is*

$$\begin{tikzcd}
S^1_0 \arrow[r, "\simeq"] \arrow[d] & S^1_0 \arrow[d] \\
G_0 \arrow[r, "\simeq"] \arrow[d] & K(\mathbf{Q},3)\times\prod K(\mathbf{Q},\text{odd}) \arrow[d, "h\times 1"] \\
(G/S^1)_0 \arrow[r, "\simeq"] & S^2_0\times\prod K(\mathbf{Q},\text{odd})
\end{tikzcd}$$

where $h\colon K(\mathbf{Q},3)\to(S^2)_0$ is the rationalization of the Hopf map $S^3\to S^2$.

PROOF. Let T be a maximal torus of G containing S^1. The fibration $S^1\to G\to G/S^1$ is a pullback of a fibration over G/T, and G/T is simply connected. Hence the fibration is orientable. We now apply the Serre spectral sequence with rational coefficents to compute $H^*(G/S^1;\mathbf{Q})$. The 1-dimensional generator of $H_1(S^1,\mathbf{Z})$ represents torsion of $H_1(G;\mathbf{Z})$, hence the map $H^1(G;\mathbf{Q})\to H^1(S^1,\mathbf{Q})$ is zero so the generator of $H^1(S^1;\mathbf{Q})$ transgresses to a 2-dimensional element $y\in H^2(G/S^1;\mathbf{Q})$. We need to prove $y^2=0$. Embed $G\subset \mathrm{SU}(N)$ as a subgroup for some big N, and consider

$$\begin{tikzcd}
S^1 \arrow[r, equal] \arrow[d] & S^1 \arrow[r] \arrow[d] & T \arrow[d] \\
G \arrow[r] \arrow[d] & \mathrm{SU}(N) \arrow[r, equal] \arrow[d] & \mathrm{SU}(N) \arrow[d] \\
G/S^1 \arrow[r] & \mathrm{SU}(N)/S^1 \arrow[r] & \mathrm{SU}(N)/T
\end{tikzcd}$$

where T is a maximal torus for $\mathrm{SU}(N)$ extending S^1. In [2] it is proved for an arbitrary compact Lie group H with maximal torus T and Weyl group W that

$$H^*(H/T;\mathbf{Q}) \cong H^*(BT;\mathbf{Q})//H^*(BT;\mathbf{Q})^W$$

that is the cohomology of H/T with rational coefficients is a polynomial algebra on rank (H) 2-dimensional generators and relations generated by elements invariant under the Weyl group. In case of $\mathrm{SU}(N)$ the maximal torus is diagonal matrices with determinant 1 and the Weyl group is Σ^n acting as permutations of the diagonal entries. The rational cohomology of $\mathrm{SU}(N)/T$ is thus a polynomial algebra on 2-dimensional generators $x_1,\ldots,x_N$; $\sum x_i=0$, with relations generated by symmetric polynomials. Let x denote the 2-dimensional generator of $H^*(\mathrm{SU}(N)/S^1;\mathbf{Q})$. Then x_i maps to some rational multiple $a_i x$ of x and not all a_i are 0. Since the Weyl group acts by permutation $x_1^2 + \ldots + x_N^2$ is a relation. Hence $\sum a_i^2 x^2 = 0$ in $H^4(\mathrm{SU}\,(N)/S^1; \mathbf{Q})$ so $x^2 = 0$ and hence $y^2 = 0$.

We may now solve the spectral sequence for $S^1 \to G \to G/S^1$: $H^*(G/S^1, \mathbf{Q})$ is an exterior algebra on a 2-dimensional generator y, and odd-dimensional generators. By obstruction theory there is a map

$$(G/S^1)_0 \to (S^2)_0 \times \prod K(\mathbf{Q}, \mathrm{odd})$$

inducing isomorphism on rational cohomology, hence a homotopy equivalence, (the universal cover is obtained by omitting the $K(\mathbf{Q}, 1)$'s). The rational type of the classifying map $G/S^1 \to BS^1$ is given by y. Hence it factors as

$$(G/S^1)_0 \simeq S_0^2 \times \prod K(\mathbf{Q}, \mathrm{odd}) \to S_0^2 \to (BS^1)_0 .$$

and our statement follows by taking homotopy fibres of this map.

3.4. LEMMA. *Let G be a compact Lie group, S^1 a subgroup, then G/S^1 is stably parallelizable. In particular G/S^1 is stably reducible.*

PROOF. It is well-known that G/T, T a maximal torus, is stably parallelizable. Consider the sequence of fibrations

$$G/S^1 \to G/S^1 \times S^1 \to \ldots \to G/T$$

Since G/T is parallelizable (see e.g. [2]), and the pullback of an orientable S^1-bundle over itself is trivial, we conclude that G/S^1 is stably parallelizable.

Lemma 3.3 and 3.4 together establish

3.5. PROPOSITION. *Let G be a non-abelian compact Lie group $\neq (\mathbf{R}P^3)^k$. Then G admits a 1-torus.*

PROOF. We use classification of compact Lie groups (as given in e.g. [6, p. 346]) to find a subgroup $S^1 \subset T \subset G$ of the maximal torus that determines the 0-element in $\pi_1(G)$. By lemma 3.3 and 3.4 we see that $S^1 \to G \to G/S^1$ satisfies the conditions of definition 3.1. We next consider spaces of type b) in our main theorem.

3.6. LEMMA. *Let $S^3 \to A \to B$ be a stably trivial principal S^3-bundle. The free action of S^1 on S^3 induces a free action of S^1 on A. The fibration*

$$S^1 \to A \xrightarrow{p} A/S^1$$

is orientable, p induces isomorphism of fundamental groups and the rational type of the fibration is

$$S^1_0 \to B_0 \times S^3_0 \xrightarrow{1 \times h} B_0 \times S^2_0 ,$$

h the Hopf map. If further B is a stably reducible, quasifinite Poincaré space then so is A/S^1, so A in that case admits a 1-torus.

PROOF. The diagram of fibrations

$$\begin{array}{ccccc} S^2 & \leftarrow & S^3 & \leftarrow & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ A/S^1 & \leftarrow & A & \leftarrow & S^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ B & \leftarrow & B & \leftarrow & * \end{array}$$

proves $\pi_1(p)$ is an isomorphism and as p has S^1 as structure group it is orientable. From [12] we have that A/S^1 is a Poincaré complex since both S^2 and B are. The normal bundle $\nu(A/S^1)$ is the Whitney sum of the pullback of ν_B and the inverse of the tangent bundle along the fibres of π. We have assumed ν_B is trivial and must check that the other summand is also trivial. Now $S^2 \to A/S^1 \to B$ is the projective bundle of $S^3 \to A \to B$, and is classified by a map into $B\mathrm{SU}(2) = BS^3$. Taking tangent bundles along the fibres (of SU(2)-bundles) induce a mapping $B\mathrm{SU}(2) \to B\mathrm{SO}(2)$, but $B\mathrm{SO}(2) = K(\mathbf{Z}, 2)$ so any such map is homotopically trivial. This proves that $\nu(A/S^1)$ is trivial. We now need to prove that $(A/S^1)_0$ is $B_0 \times S^2_0$. First we observe that $S^3_0 \to A_0 \to B_0$ is trivial since the first rational Pontryagin class must be 0 by stably triviality, hence $A_0 \cong B_0 \times S^3_0$. In the Serre spectral sequence of $S^1 \to A \to A/S^1$ the 1-dimensional element $x \in H^1(S^1; \mathbf{Q})$ transgresses to a non-zero element y by naturality, comparing with the spectral sequence of $S^1 \to S^3 \to S^2$. In the latter spectral sequence $y \otimes x$ survives so it must in the former since some element must hit $y \otimes x$ and there are no extension problems. Therefore $y^2 = 0$ in $H^4(A/S^1; \mathbf{Q})$. By obstruction theory we produce a map $(A/S^1)_0 \to B_0 \times S^2_0$ inducing isomorphism of rational cohomology and since the spaces involved are nilpotent it is a homotopy equivalence [4]. This completes the proof.

We are now in a position to prove our main theorem. Since we assume X_i has a factor of type (a) or (b) it follows from lemma 3.6 and proposition 3.5 that each X_i admits a 1-torus. The proof now follows from proposition 2.1 once we have the following

3.7. LEMMA. *Assume X is obtained by homotopy mixing of X_i at P_i and each X_i admits a 1-torus $S^1 \to X_i \to Y_i$. Then X admits a 1-torus.*

PROOF. All the Y_i must have rational type $S_0^2 \times \prod K(\mathbf{Q}, \mathrm{odd})$ so they are rationally equivalent and we may form $\bar{Y}$ as the homotopy mixing of Y_i at p_i. The rational equivalences of $(Y_i)_0$ are covered by rational equivalences of $(X_i)_0$. So forming the homotopy pullback of the localized S^1-fibrations creates an S^1-fibration $S^1 \to \bar{Z} \to \bar{Y}$ as in the proof of proposition 3.2. $\bar{Y}$ is clearly quasifinite, nilpotent and we see that $\bar{Y}$ is stably reducible by mixing the stable reductions. $\bar{Z}$ is in the genus of Z and since $\bar{Z}$ admits a 1-torus so does Z by proposition 3.2. This ends the proof.

The corollary follows immediately except for $Z = (\mathbf{R}P^3)^k$. It follows from [16] however, that if $\bar{Z} \in G(Z)$ then $\bar{Z} = X^k$ where $X \in G(\mathbf{R}P^3)$, and X is obtained as a homotopy pullback

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{R}P^3_{\bar{P}} \\ \downarrow & & \downarrow \\ \mathbf{R}P^3_P \to K(\mathbf{Q},3) & \simeq & K(\mathbf{Q},3) \end{array}$$

the map $K(\mathbf{Q},3) \simeq K(\mathbf{Q},3)$ being multiplication by p/q. If we write $p/q = p_1/q_1 \cdot p_2/q_2$ where p_1, q_1 are products of primes in P, p_2, q_2 primes in $\bar{P}$, we may realize the rational maps p_1/q_1 and q_2/q_2 as homotopy equivalences

$$\begin{array}{ccc} \mathbf{R}P^3_P & \xrightarrow{q_2/p_2} & \mathbf{R}P^3_P \\ \downarrow & & \downarrow \\ \mathbf{R}P^3_{\bar{P}} & \xrightarrow{p_1/q_1} & \mathbf{R}P^3_{\bar{P}} \end{array}$$

and

and it thus follows that $X = \mathbf{R}P^3$.

4. Final remarks.

In our main theorem we excluded the case $Z_2 = ((\mathbf{R}P^3)^k \times (S^7)^l \times (\mathbf{R}P^7)^m)$. The main technique of the paper does, however, work in some of these cases, but then depends on actual surgery group computations rather than just the $\pi - \pi$ theorem of Wall. If for instance Z is the homotopy pullback of

$$\begin{array}{ccc} Z & \longrightarrow & \mathrm{Sp}(2)_P \\ \downarrow & & \downarrow \\ (\mathbf{R}P^3 \times \mathbf{R}P^7)_2 & \to & K(\mathbf{Q},3) \times K(\mathbf{Q},7) \end{array}$$

P all odd primes.

We may then construct a diagram

$$\begin{tikzcd}
 & \mathrm{Sp}(2)_P \arrow[d] \arrow[dr] & \\
(\mathbf{R}P^3 \times \mathbf{R}P^7)_2 \arrow[r] \arrow[dr] & K(\mathbf{Q},3) \times K(\mathbf{Q},7) \arrow[dr] & (\mathrm{Sp}(2)/S^1)_P \arrow[d] \\
 & (\mathbf{R}P^3 \times \mathbf{R}P^7)_2 \arrow[r] & S_0^2 \times K(\mathbf{Q},7)
\end{tikzcd}$$

giving by homotopy pullback a fibration $S^1 \to Z \to Y$. Here $\pi_1(Z) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $\pi_1(Y) = \mathbf{Z}_2$. Also Y is a quasifinite stably reducible Poincaré complex and the fibration is orientable. Since $K_0(\mathbf{Z}(\mathbf{Z}_2))$ is 0, Y is a finite complex. The stable reduction of Y establishes a surgery problem $M \to Y$ with obstruction in $L_1(\mathbf{Z}_2)$. This group is 0 so Y is homotopy equivalent to a differentiable manifold. Since S^1-fibrations are equivalent to 0(2)-bundles Z is homotopy equivalent to a differentiable manifold. Since Sp(2) is the only Lie-group of type (3, 7) this method can be extended to prove that if Z is mixed of $(\mathbf{R}P^3)^k \times (S^7)^l \times (\mathbf{R}P^7)^m$ and compact Lie-groups, then there is a differentiable parallelizable manifold in the genus of Z and hence since Mislin [10] has proved that finiteness is a generic property for H-space Z is the homotopy type of a finite complex.

Added in Proof. The surgery argument in Section 2 is in error: $(E \times S^1, Z \times S^1)$ is a Poincaré pair, but only a simple Poincaré pair if the finiteness obstruction of E is 0. It follows from a result of Varadarajan (J. Pure Appl. Algebra, 12 (1978), 137–146) that the finiteness obstruction of E lies in the deficiency subgroup $D(\mathbf{Z}\pi)$, so what the argument does prove is that the surgery obstruction is 0 in the L group based on projectives in $D(\mathbf{Z}\pi)$. This is certainly sufficient in case $D(\mathbf{Z}\pi) = 0$, e.g. for the trivial group, $\mathbf{Z}/4\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z} + \mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$/We should thus add the assumption to Theorem 1.1 that $D(\mathbf{Z}\pi) = 0$. As far as Corollary 1.2 is concerned, using that $G(X + Y) = X \times G(Y)$ and the above remarks about D groups, we still get that corollary except for SU $(n+1)/\mathbf{Z}/(n+1)\mathbf{Z}$.

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