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ON STABILITY OF C^∞ MAPPINGS OF MANIFOLDS WITH BOUNDARY

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Introduction.

In his paper [1], J. N. Mather proves that a proper, infinitesimally stable mapping of one finite dimensional C^∞ manifold into another is structurally stable. The proof of this theorem requires manifolds with boundary and corners, but the main theorems ([1, § 3]) only deals with manifolds without boundary. The purpose of this paper is to extend Mathers theorem to a theorem which also deals with manifolds with boundary and corners.

The reader is supposed to be familiar with Mathers paper [1]. Concepts like manifold, corner, index, tangent bundle, vector field, infinitesimal and structural stability, etc. will be used without further explanations in the same sense as in this paper.

In section 1 we introduce a new kind of infinitesimal and structural stability. In section 2 we state the main theorem, and in section 4 we prove it using some lemmas proved in section 3. In section 5 we give a few examples.

1. The stratification of a manifold.

Let N be a manifold and TN its tangent bundle. We let $\hat{T}N$ denote the set of tangent vectors $v \in TN$, to which there exists a C^∞ curve $c\colon (-\varepsilon; \varepsilon) \to N$ whose tangent vector in 0 is v. We define

$$\hat{T}N_x = \hat{T}N \cap TN_x$$

and

$$\Gamma^k(\hat{T}N) = \{\xi \in \Gamma^k(TN) \mid \xi(N) \subset \hat{T}N\},$$

$k=0,1,\ldots,\infty$. Let P be another manifold and $f\colon N \to P$ a C^∞ mapping. We shall write

$$\Gamma^k(f^*\hat{T}P) = \{\zeta \in \Gamma^k(f^*TP) \mid \zeta(N) \subset \hat{T}P\},$$

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interpreting $\zeta \in \Gamma^k(f^*TP)$ as a vector field $\zeta\colon N \to TP$ along f.

Note that $\hat{T}N_x = TN_x$ when x is an interior point of N. Especially $\hat{T}N = TN$ when N has no boundary. When N has a boundary and x is a boundary point of N, $\hat{T}N_x$ is the set of tangent vectors $v \in TN_x$ which are not (in local coordinates) pointing out of N or into N "from nowhere". In that case $\hat{T}N$ is not a vector bundle, not even a manifold. However $\Gamma^k(\hat{T}N)$ is a submodule of $\Gamma^k(TN)$ and $\Gamma^k(f^*\hat{T}P)$ a submodule of $\Gamma^k(f^*TP)$, and as $\Gamma^k(TN)$ and $\Gamma^k(f^*TP)$ are finitely generated, it is easy to see that $\Gamma^k(\hat{T}N)$ and $\Gamma^k(f^*\hat{T}P)$ are finitely generated modules too.

DEFINITION. Let N be a manifold of dimension n. By the *stratification of* N we will mean the set S of components of $\{x \in N \mid \text{index}\,(x) = j\}$, $j = 0, 1, \ldots, n$. For $x \in N$ we let $\text{str}\,(x)$ denote the stratum which contains x.

Note that any chart on N has nonvoid intersection with only finitely many strata, and the stratification of N is countable since the topology for N has a countable base. A stratum $M \in S$ of index j is a manifold of dimension $n-j$ without boundary. As TM naturally can be considered as a subset of TN we have

$$\hat{T}N = \bigcup_{M \in S} TM .$$

2. The main result.

Let N and P be manifolds with boundary and corners and $f\colon N \to P$ a C^∞ mapping. The mappings

$$tf\colon \Gamma^\infty(TN) \to \Gamma^\infty(f^*TP) \quad \text{and} \quad \omega f\colon \Gamma^\infty(TP) \to \Gamma^\infty(f^*TP),$$

$$tf(\xi) = Tf \circ \xi, \quad \omega f(\eta) = \eta \circ f,$$

define by restriction the mappings

$$\hat{t}f\colon \Gamma^\infty(\hat{T}N) \to \Gamma^\infty(f^*\hat{T}P)$$

and

$$\hat{\omega}f\colon \Gamma^\infty(\hat{T}P) \to \Gamma^\infty(f^*\hat{T}P) .$$

Since tf is a $C^\infty(N)$ module homomorphism, so is $\hat{t}f$, and since ωf is a module homomorphism over the ring homomorphism $f^*\colon C^\infty(P) \to C^\infty(N)$, so is $\hat{\omega}f$.

When U is a neighbourhood of f in $C^\infty(N,P)$ we let $\hat{U}$ denote the set

$$\hat{U} = \{g \in U \mid \forall x \in N(\text{str}\,(f(x)) = \text{str}\,(g(x)))\} .$$

DEFINITION. We will call f *structurally stratum stable* if there exists a neighbourhood U of f in $C^\infty(N,P)$ such that for any $g \in \hat{U}$ there exists C^∞ diffeomorphisms $h_1\colon N \to N$ and $h_2\colon P \to P$ such that $g = h_2 \circ f \circ h_1$. And we will call f *infinitesimally stratum stable* if the mapping

$$\hat{t}f + \hat{\omega}f\colon \Gamma^\infty(\hat{T}N) \oplus \Gamma^\infty(\hat{T}P) \to \Gamma^\infty(f^*\hat{T}P)$$

is onto.

Now we can state the main theorems which are analogous to the theorems 1–3 in § 3 of [1].

THEOREM 2.1. *If f is proper and infinitesimally stratum stable, then f is also structurally stratum stable.*

THEOREM 2.2. *If $f\colon N \to P$ is proper and infinitesimally stratum stable, then there exists a neighbourhood U of f in $C^\infty(N,P)$ and continuous mappings $\hat{H}_1\colon \hat{U} \to \mathrm{Diff}^\infty N$ and $\hat{H}_2\colon \hat{U} \to \mathrm{Diff}^\infty P$ such that $\hat{H}_1(f) = 1_N$, $\hat{H}_2(f) = 1_P$ and*

$$g = \hat{H}_2(g) \circ f \circ \hat{H}_1(g)$$

for any $g \in \hat{U}$.

THEOREM 2.3. *Let M be a closed submanifold of N. Suppose $f \mid M$ is proper and infinitesimally stratum stable. Then there exists a neighbourhood U of $f \mid M$ in $C^\infty(M,P)$ and continuous mappings $\hat{H}_1\colon \hat{U} \to \mathrm{Diff}^\infty N$ and $\hat{H}_2\colon U \to \mathrm{Diff}^\infty P$ such that $\hat{H}_1(f \mid M) = 1_N$, $\hat{H}_2(f \mid M) = 1_P$ and*

$$g = \hat{H}_2(g) \circ f \circ \hat{H}_1(g) \mid M$$

for any $g \in \hat{U}$.

In section 4 we shall prove Theorem 2.3 which clearly implies the theorems 2.1 and 2.2.

3. Preparations for the proof.

PROPOSITION 3.1. *Any manifold P has a family of geodesics (U,γ) with the following property: When $(y_0, y_1) \in U$ and $\mathrm{str}\,(y_0) = \mathrm{str}\,(y_1) = V$ then*

$$\gamma((y_0, y_1) \times I) \subset V.$$

This proposition is an extension of Lemma 3 in § 2 of [1]. It is easily seen that any family of geodesics, constructed as in the proof of this lemma, fulfils the claim of the proposition because they are just "straight lines".

PROPOSITION 3.2. *Let N be a manifold and M a closed submanifold of N and let*

$\pi_N: N \times I \to N$ and $\pi_M: M \times I \to M$ be the projections. There exists a continuous linear extension $E: \Gamma^0(\pi_M^* TM) \to \Gamma^0(\pi_N^* TN)$ such that

$$E(\Gamma^0(\pi_M^* \hat{T}M)) \subset \Gamma^0(\pi_N^* \hat{T}N)\,.$$

PROOF. We can use the construction of E in the proof of the proposition in § 5 of [1]. Note that if $f(x,t)=0$ for all $t>0$, then $E(f)(x,t)=0$ for all $t \in \mathbf{R}$. Now we see that the mapping E from Mathers proposition fulfils the claim of Proposition 3.2 if we put the following modest restriction on the choice of family of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$: φ_α is a diffeomorphism of U_α onto some open subset of a quadrant Q_α for each $\alpha \in A$. We demand that all the quadrants Q_α and Q'_α shall be of the form

$$\{x \in \mathbf{R}^n \mid l_1(x) \geqq 0, \ldots, l_k(x) \geqq 0,\ l_{k+1}(x)=0, \ldots, l_{k'}(x)=0\}\,,$$

where $l_1, \ldots, l_k, l_{k+1}, \ldots, l_{k'}$ are not only linearly independent but also of the form

$$l_i(x_1, \ldots, x_n) = x_j$$

for any $i=1,\ldots,k'$.

PROPOSITION 3.3. *Let U be a manifold (with boundary and corners) and let $\pi_U: U \times I \to U$ be the projection. There exists a neighbourhood $\hat{O}_U$ of 0 in $\Gamma^\infty(\pi_U^* \hat{T}U)$ and a continuous mapping $\hat{\theta}: \hat{O}_U \to C^\infty(U \times I, U)$ such that $\hat{\theta}(\xi)_t$: $x \to \hat{\theta}(\xi)(x,t)$ is a diffeomorphism for each $t \in I$ and $\xi \in \hat{O}_U$,*

$$\hat{\theta}(\xi)_0 = 1_U \quad \text{and} \quad \frac{\partial \hat{\theta}(\xi)_t}{\partial t} \circ \hat{\theta}(\xi)_t^{-1} = \xi_t\,,$$

that is

$$\frac{\partial}{\partial t}\hat{\theta}(\xi)(x,t) = \xi(\hat{\theta}(\xi)(x,t),t)\,,$$

for all $\xi \in \hat{O}_U$ and $(x,t) \in U \times I$.

This proposition is an extension of Lemma 2 in § 7 of [1].

PROOF. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ be a family of charts for U such that $\{U_\alpha\}$ is a locally finite cover. Choose $K_\alpha \subset L_\alpha \subset U_\alpha$ such that K_α and L_α are compact, $\{K_\alpha\}$ covers U, and L_α is a neighbourhood of K_α. Since $\varphi_\alpha(U_\alpha)$ is an open subset of a quadrant $Q_\alpha \subset \mathbf{R}^n$, where n is the dimension of N, there exist an open set $\hat{U}_\alpha \subset \mathbf{R}^n$ such that $\varphi_\alpha(U_\alpha) = Q_\alpha \cap \hat{U}_\alpha$ and a compact set $\hat{L}_\alpha \subset \hat{U}_\alpha$ such that $\varphi_\alpha(K_\alpha) \subset (\hat{L}_\alpha)^\circ$, where $(\hat{L}_\alpha)^\circ$ is the interior (in $\mathbf{R}^n$) of $\hat{L}_\alpha$, and $\varphi_\alpha(L_\alpha) = \hat{L}_\alpha \cap Q_\alpha$. For each

α, let ε_α denote the distance from $\varphi_\alpha(K_\alpha)$ to $\hat{U}_\alpha - \hat{L}_\alpha$. Then $\varepsilon_\alpha > 0$ for each $\alpha \in A$. We define O_U to be the set of all $\xi \in \Gamma^\infty(\pi_U^* TU)$ such that $\|\xi_\alpha(x, t)\| < \varepsilon_\alpha$ for all $x \in \varphi_\alpha(L_\alpha)$ and all $t \in I$. We write $\hat{O}_U = O_U \cap \Gamma^\infty(\pi_U^* \hat{T}U)$. For each $\alpha \in A$, each $x \in K_\alpha$ and each $\xi \in O_U$, there exists a unique curve $[0; \varepsilon] \to \hat{L}_\alpha$, $t \to \gamma_\alpha(\varphi_\alpha(x), t)$ such that $\gamma_\alpha(\varphi_\alpha(x), 0) = \varphi_\alpha(x)$ and

$$\frac{\partial \gamma_\alpha(\varphi_\alpha(x), t)}{\partial t} = \xi_\alpha(\gamma_\alpha(\varphi_\alpha(x), t), t) .$$

If $\xi \in \hat{O}_U$, we can obviously choose $\varepsilon = 1$, since the curve needs more than unit time to get out of $\varphi_\alpha(L_\alpha)$. We can then define

$$\hat{\theta}(\xi)(x, t) = \varphi_\alpha^{-1}\big(\gamma_\alpha(\varphi_\alpha(x), t)\big)$$

for $\xi \in \hat{O}_U$, $(x, t) \in K_\alpha \times I$. That this definition has the properties stated in the proposition is proved in the same way as Lemma 2 in § 7 of [1].

4. Proof of the main result.

Now we will prove Theorem 2.3. As the procedure of this proof is just the same as that in § 7 of [1], we will restrict ourselves to point out the differences from this proof.

The assumption is that

$$\hat{\mu} = \big(\hat{\omega}(f \mid M), \hat{t}(f \mid M), \Gamma^\infty(\hat{T}P), \Gamma^\infty(\hat{T}M), \Gamma^\infty((f \mid M)^* \hat{T}P)\big)$$

is surjective. From Mather we have an isomorphism of $\Gamma^\infty(TP) \otimes C^\infty(P \times I)$ with $\Gamma^\infty(\pi_P^* TP)$. By restriction we get a bijection of $\Gamma^\infty(\hat{T}P) \otimes C^\infty(P \times I)$ with $\Gamma^\infty(\pi_P^* \hat{T}P)$. As $\hat{T}P$ is the union of the vector bundles TQ over all strata Q of P, it is easy to see that this bijection is also an isomorphism. It follows that $\iota(\hat{\mu})$ is isomorphic to the mixed homomorphism

$$\iota(\hat{\mu})' = \big(\alpha, \beta, \Gamma^\infty(\pi_P^* \hat{T}P), \Gamma^\infty(\pi_M^* \hat{T}M), \Gamma^\infty(((f \mid M) \circ \pi_M)^* \hat{T}P)\big)$$

where

$$\alpha = \omega'(f \circ \pi_M) \mid \Gamma^\infty(\pi_P^* \hat{T}P) \quad \text{and} \quad \beta = t'(f \circ \pi_M) \mid \Gamma^\infty(\pi_M^* \hat{T}M) .$$

Instead of the parameter space X we shall use

$$\hat{X} = \{g \in X \mid \forall z \in M\big(g(z \times I) \subset \operatorname{str}(f(z))\big)\} ,$$

but the base point is still $x_0 = f \circ \pi_M$. We let $\hat{f} \in C^{\hat{X}}(M \times I, P)$ denote the germ of the inclusion of $\hat{X}$ into $C^\infty(M \times I, P)$. And we let $\Gamma^{\hat{X}}(\pi_P^* \hat{T}P)$ denote the set of germs u at x_0 of mappings $\hat{X} \to C^\infty(M \times I, \hat{T}P)$ having the following property: for suitable representatives $\tilde{\pi}_P$ of the germ π_P and $\tilde{u}$ of u, we have $\tilde{\pi}_P(x) =$

$\hat{\pi}(P)\circ\hat{u}(x)$ for all x near x_0. Here $\hat{\pi}(P)$ denotes the projection $\hat{T}P \to P$. We thus get a mixed homomorphism $\hat{v}$ over $C^{\hat{X}}(\hat{\bar{f}}, \pi_2)$,

$$\hat{v} = (\omega\hat{\bar{f}}, t\hat{\bar{f}}, \Gamma^{\hat{X}}(\pi_P^* \hat{T}P), \Gamma^{\hat{X}}(\pi_M^* \hat{T}M), \Gamma^{\hat{X}}(\hat{\bar{f}}^* \hat{T}P))$$

and an isomorphism of ev $\hat{v}$ with $\iota(\hat{\mu})'$. Hence kern ev $\hat{v}$ is surjective.

By restriction of $\tilde{\Delta}: X \to C^\infty(M \times I, TP)$ we get a continuous mapping

$$\hat{\tilde{\Delta}}: \hat{X} \to C^\infty(M \times I, \hat{T}P)\,.$$

Let $\hat{\Delta}$ denote the germ of $\hat{\tilde{\Delta}}$ at $x_0 \in \hat{X}$. Since $\hat{\Delta} \in$ kern ev $\Gamma^{\hat{X}}(\hat{\bar{f}}^* \hat{T}P)$ there exists $\xi_0 \in$ kern ev $\Gamma^{\hat{X}}(\pi_M^* \hat{T}M)$ and $\eta \in$ kern ev $\Gamma^{\hat{X}}(\pi_P^* \hat{T}P)$ such that

$$\hat{\Delta} = t\hat{\bar{f}}(\xi_0) + \omega\hat{\bar{f}}(\eta)\,.$$

We may choose representatives $\tilde{\xi}_0$ of ξ_0 and $\tilde{\eta}$ of η such that

$$\tilde{\xi}_0: \hat{Y} \to \Gamma^\infty(\pi_M^* \hat{T}M) \quad \text{and} \quad \tilde{\eta}: \hat{Y} \to \Gamma^\infty(\pi_P^* \hat{T}P)$$

are continuous, and such that

$$\tilde{\Delta}(y) = t'y(\tilde{\xi}_0(y)) + \omega' y(\tilde{\eta}(y)), \qquad \text{for all } y \in \hat{Y},$$

where $\hat{Y}$ is a suitable neighbourhood of x_0 in $\hat{X}$. By Proposition 3.2 we get a continuous mapping $\tilde{\xi}: \hat{Y} \to \Gamma^\infty(\pi_N^* \hat{T}N)$ such that $\tilde{\xi}_0(y)$ is the restriction of $\tilde{\xi}(y)$ for all $y \in \hat{Y}$.

By Proposition 3.1 we get a family of geodesics (V, γ). Let U be a neighbourhood of $f\,|\,M$ in $C^\infty(M, P)$ such that $(f, g)(M) \subset V$ for all $g \in U$. We define $\hat{\gamma}_*: \hat{U} \to C^\infty(M \times I, P)$ by $\hat{\gamma}_*(g)(z, t) = \gamma(f(z), g(z), t)$. Note that

$$\hat{\gamma}_*(g)(z \times I) \subset \operatorname{str}(f(z))$$

for all $g \in \hat{U}$ and $z \in M$.

Since $\hat{Y}$ is a neighbourhood of x_0 in $\hat{X}$ and consequently in

$$\{h \in C^\infty(M \times I, P) \mid \forall z \in M(h(z \times I) \subset \operatorname{str}(f(z)))\}\,,$$

we may choose U and $\hat{Y}$ so that $\hat{\gamma}_*(\hat{U}) \subset \hat{Y}$, and $-\tilde{\xi}(\hat{\gamma}_*(\hat{U})) \subset \hat{O}_N$ and $\tilde{\eta}(\hat{\gamma}_*(\hat{U})) \subset \hat{O}_P$ (cf. Proposition 3.3).

The rest of the proof is not different from Mather's proof.

5. A few examples.

It is obvious that if a mapping $f \in C^\infty(N, P)$ is structurally stable, f is structurally stratum stable too. It is also obvious that a structurally stratum stable mapping $f \in C^\infty(N, P)$ is not generally structurally stable, e.g. the identity of a closed interval is not structurally stable.

An example of a mapping which is infinitesimally stable but not infinitesimally stratum stable is the following: Let $N=[0;\ 1]\cup[2;\ 3]$ and $P=\mathbf{R}$ and let $f\colon N\to P$ be the mapping

$$f(x) = \begin{cases} x, & x\in[0;\ 1] \\ x-2, & x\in[2;\ 3]\ . \end{cases}$$

Since $tf(\Gamma^\infty(TN))=\Gamma^\infty(f^*TP)$, f is infinitesimally stable, but

$$\hat{t}f(\xi)(0) = \hat{t}f(\xi)(2) = 0$$

(i.e. the vector part of the tangent vector is 0) for every $\xi\in\Gamma^\infty(\hat{T}N)$ and

$$\hat{\omega}f(\eta)(0) = \hat{\omega}f(\eta)(2) = \eta(0)$$

for every $\eta\in\Gamma^\infty(\hat{T}P)$. Thus $\zeta(0)=\zeta(2)$ for every $\zeta\in\hat{t}f(\Gamma^\infty(\hat{T}N)) +\hat{\omega}f(\Gamma^\infty(\hat{T}P))$, and f is not infinitesimally stratum stable.

Finally let $N=[0;\ \infty)\times[-1;\ 1]$ and

$$P = \{(y_1,y_2)\in\mathbf{R}^2 \mid y_1\geqq 0,\ -y_1\leqq y_2\leqq y_1\}$$

and let $f\colon N\to P$ be the mapping $f(x_1,x_2)=(x_1,x_1x_2)$ ("collapse of $0\times[-1;\ 1]$"). It is not difficult to check that f is infinitesimally stratum stable but not infinitesimally stable.

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