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SIMPLE INJECTIVE MODULES

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Rosenberg and Zelinsky [6] first addressed the question of characterizing for a ring R those simple modules $_RT$ having injective hulls of finite length. In that work they also announced a very special case obtained by Kaplansky: if R is commutative, then every simple R-module is injective iff R is von Neumann regular. A few years later Villamayor (see [4]) characterized those (not necessarily commutative) rings R — now known as V-*rings* — over which every simple left module is injective by the property that every left ideal is an intersection of maximal left ideals. (Also see [2] and [3] for further treatment of V-rings and their bibliography.)

In this note we return to that which is common to both the Rosenberg–Zelinsky and the V-ring studies. That is, we consider the problem of characterizing, for a ring R, those simple modules $_RT$ that are themselves injective. As one application, we prove that if R is a V-ring, then so is the endomorphism ring of every finitely generated projective module P_R.

If M is a left R-module, then for each $X \subseteq M$ and $A \subseteq R$, we set

$$(A:X) = \{r \in R \mid rX \subseteq A\} .$$

1. Characterizations of simple injectives.

Let R be a ring (with identity). Then a left R-module $_RT$ is simple iff

$$T \cong R/M$$

for some maximal left ideal M of R. Indeed, if $_RT$ is simple, then

$$T \cong R/(0:t)$$

for each $0 \neq t \in T$.

DEFINITION. Let $_RI \leq {}_RR$ be a left ideal and let $a \in R$. A left ideal $L \leq {}_RR$ *supports a on I* if

$$L \cap Ra = Ia .$$

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For each left ideal I of R and each $a \in R$ the left ideal Ia is a support for a on I. So since the collection of supports for a on I is clearly inductive, each support for a on I is contained in a maximal support for a on I.

1.1. LEMMA. *Let M be a maximal left ideal of R and let $a \in R$. Then*

$$a \in Ma$$

iff R is a (necessarily unique) maximal support for a on M.

PROOF. Since $Ra \supseteqq Ma$, we have

$$a \in Ma \text{ iff } Ra = Ma \text{ iff } R \cap Ra = Ma .$$

Now our main result is the following characterization of simple injective modules.

1.2. THEOREM. *Let ${}_RM$ be a maximal left ideal of R. Then the simple left R-module R/M is injective iff for each $a \in R$*

$$L + Ra = R$$

for every maximal support L of a on M.

PROOF. ($\Rightarrow$) If $a \in Ma$, then by Lemma 1.1, R is the unique maximal support for a on M. So we may assume that $a \notin Ma$. This means that

$$R/M \cong Ra/Ma$$

so that Ra/Ma is injective. Let L be a maximal support for a on M. Since

$$L \cap Ra = Ma ,$$

we infer that

$$(Ra+L)/Ma \cong Ra/Ma \oplus L/Ma .$$

So by the maximality of the support L, we have $Ra+L=R$.

($\Leftarrow$). Let I be a left ideal of R and let

$$\varphi : I \to R/M$$

be a non zero homomorphism. Let $a \in I$ with

$$\varphi(a) = 1 + M .$$

Then

$$\operatorname{Ker} \varphi = Ma \cap I = Ma .$$

So $a \notin Ma$. Now let ${}_RL \preceq {}_RR$ be a left ideal maximal with respect to

$$Ma \preceq L \quad \text{and} \quad a \notin L .$$

Then since $I(Ma) \cong R/M$ is simple, since $L \cap Ra \prec Ra \preceq I$ and since $Ma \preceq L$, we have

$$L \cap Ra = Ma .$$

It is clear, since $a \notin Ma$, that L is a maximal support for a on M. So by hypothesis,

$$L + Ra = R .$$

Therefore,

$$\psi : l + ra \mapsto \varphi(ra) \qquad (l + ra \in L + Ra)$$

is a well defined homomorphism $\psi : R \to R/M$ extending φ. So by the Injective Test Lemma ([1, p. 205]) R/M is injective.

If R is a V-ring, then every left ideal I of R is fully idempotent (i.e., $I^2 = I$). A local generalization of this fact is the following.

1.3. Corollary. *Let M be a maximal left ideal of R with R/M injective. Then for each $a \in R$*

$$aR \subseteq M \Rightarrow a \in Ma .$$

Proof. Let $aR \subseteq M$ and let L be a maximal support for a on M. Then by Theorem 1.2

$$L + Ra = R .$$

Thus,

$$a \in RaR \subseteq RaL + RaRa \subseteq L + Ma \subseteq L ;$$

and so

$$a \in L \cap Ra = Ma .$$

As we shall show later (Corollary 1.7) if R is commutative, then the converse of this last corollary holds. In general, however, the converse is false. Indeed, if R is von Neumann regular, then

$$aR \subseteq M \Rightarrow a \in Ma$$

for all left ideals M of R. But von Neumann regular rings need not be V-rings. (See, e.g., [2], [3], or [1, Exercise 18.4].)

Let R be a ring and let P be a two sided ideal of R. Then each simple left R/P module is a simple R-module. The following result characterizes those injective simple R/P modules that are injective as R-modules.

1.4. THEOREM. *Let P be an ideal of R and let M be a maximal left ideal of R with $P \subseteqq M$. If R/M is R/P injective, then R/M is R injective iff for all $a \in R$*

$$a \in P \Rightarrow a \in Ma .$$

PROOF. ($\Rightarrow$). If $a \in P$, then $aR \subseteqq P \subseteqq M$, so $a \in Ma$ by Corollary 1.3.

($\Leftarrow$). Let $a \in R$ and let L be a maximal support for a on M. Then by Theorem 1.2 it will suffice to show that $L + Ra = R$.

First suppose $P \subseteqq L$. Then

$$L \cap (Ra + P) = (L \cap Ra) + P = Ma + P ,$$

so L/P is a maximal support for $a + P$ on M/P in R/P. Since R/M is R/P injective, it follows from Theorem 1.2 that $L + Ra = L + Ra + P = R$.

On the other hand suppose that $P \nsubseteqq L$. Then $P + L$ does not support a on M. So there is an $x \in R$ with

$$xa \in P + L \quad \text{and} \quad xa \notin Ma .$$

But then $Ra/Ma \cong R/M$ is simple, and for some $r \in R$,

$$a - rxa \in Ma \prec P + L ,$$

so $a \in P + L$. Then $a - y \in P$ for some $y \in L$. So by hypothesis

$$a - y \in M(a - y) \preceq L ,$$

and $a \in L \cap Ra = Ma$. Therefore, by Lemma 1.1, $L = R$.

Recall that if M is a maximal left ideal of R, then $(M : R)$ is a primitive ideal of R with $(M : R) \subseteqq M$.

1.5. COROLLARY. *Let M be a maximal left ideal of R. If $R/(M : R)$ is a V-ring, then the simple R-module R/M is injective iff $a \in Ma$ for all $a \in (M : R)$.*

Let $J = J(R)$ be the Jacobson radical of R. If $J = 0$, then R is *semi-artinian* if every primitive factor ring of R is artinian. Every primitive ring with a polynomial identity is artinian; so for example, if R satisfies a polynomial identity, then R/J is semi-artinian.

1.6. COROLLARY. *Let M be a maximal left ideal of R. If R/J is semi-artinian, then R/M is injective iff $a \in Ma$ for all $a \in (M : R)$.*

PROOF. Since $R/(M:R)$ is primitive, it is artinian and simple, and hence it is a V-ring.

1.7. COROLLARY. *Let R be commutative. Then a simple module $_RT$ is injective iff $a \in (0:T)a$ for all $a \in (0:T)$.*

2. Endomorphism rings of projectives.

Throughout this section let P_R be a finitely generated projective module with endomorphism ring

$$S = \text{End}\,(P_R)\,.$$

If P_R is a generator, then by Morita equivalence (see [1, Chapter 6]) the categories of left R and left S modules are equivalent. In particular, an R-module $_RT$ is injective (projective) iff the S-module

$$P \otimes_R T$$

is injective (projective). From this it is immediate, for example, that if P is a generator and R is a quasi Frobenius, then so is S. On the other hand, when P_R is not a generator, the nature of $P \otimes_R T$ cannot always be readily determined from that of T. However, if $_RT$ is also simple, we can be fairly definitive.

2.1. THEOREM. *If $_RT$ is a simple injective (projective) R-module, then $P \otimes_R T$ is either zero or a simple injective (projective) S-module.*

PROOF. Since the result is true when P_R is a generator, we may assume that $P = eR$ for some idempotent $e \in R$ and that $S \cong eRe$. In particular,

$$P \otimes_R T \cong eT\,.$$

Suppose then that $_RT$ is simple and that $eT \neq 0$. Then for each $0 \neq et \in eT$,

$$eRe(et) = e(Ret) = eT\,,$$

and so eT is eRe-simple.

Suppose now that T is projective. If $eT \neq 0$, then as R-modules, since T is simple,

$$Re \cong T \oplus V$$

for some $_RV$. Thus, as eRe modules,

$$eRe \cong eT \oplus eV\,,$$

and eT is eRe projective.

Finally, for the interesting case, assume that ${}_RT$ is injective. Suppose $eT \neq 0$. Say $0 \neq et \in T$. Set

$$M = (0:et) .$$

Then M is a maximal left ideal of R with $e \notin M$ and

$$Me \subseteq M \quad \text{and} \quad T \cong R/M .$$

So as an eRe module

$$eT \cong eRe/eMe .$$

Let $a \in eRe$ and let $L \preceq eRe$ be a maximal support for a on $eMea$. We claim that $RL \cap Ra \subseteq Ma$. For let $x_1, \ldots, x_n \in L$ and suppose

$$r_1x_1 + \ldots + r_nx_n = sa \in RL \cap Ra .$$

If $se \in M$, then $sa = sea \in Ma$. Otherwise, if $se \notin M$, then since ${}_RM$ is maximal, $1 - yse \in M$ for some $y \in R$, so since $a = eae$,

$$a - eysea \in eMea .$$

But then since $L = eLe$,

$$\begin{aligned} ey(r_1x_1 + \ldots + r_nx_n) &= eyr_1ex_1 + \ldots + eyr_nex_n \\ &= eysea \in L \cap eRea , \end{aligned}$$

and so since L supports a on eMe,

$$eysea \in eMea .$$

Thus, $a \in eMea$ whence $sa \in Ma$ as claimed.

Now since $RLe = RL$ and $Rae = Ra \supseteq Ma$, we have

$$[RL + Ma + R(1-e)] \cap Ra = Ma .$$

Thus there is a left ideal K of R maximal with respect to

$$RL + Ma + R(1-e) \preceq K \quad \text{and} \quad K \cap Ra = Ma .$$

So K is a maximal support for a on M. But by hypothesis R/M is injective, so by Theorem 1.2,

$$K + Ra = R \quad \text{and} \quad eKe + eRea = eRe .$$

We claim next that $eKe = L$. Certainly $L \subseteq eKe$. But since $R(1-e) \subseteq K$, we have

$$K = Ke + R(1-e) ,$$

so $eKe \subseteq K$. Thus

$$L \cap eRea \subseteqq eKe \cap eRea \subseteqq e(K \cap Ra) \subseteqq eMa = eMea\ ,$$

so since L is a maximal support, $L = eKe$. But then

$$L + eRea = eKe + eRea = eRe$$

and thus, by Theorem 1.2, eT is injective.

A ring R is a GV-*ring* (see [5] for the basic theory of GV-rings) in case each simple left R-module is either projective or injective.

2.2. COROLLARY. *If R is a* V-*ring* (GV-*ring*), *then S is a* V-*ring* (GV-*ring*).

PROOF. It will suffice to prove that every simple S-module is isomorphic to

$$P \otimes_R T$$

for some simple R-module ${}_RT$. Again we may assume $P = eR$ and $S = eRe$.

Let L be a maximal left ideal of eRe. Then there is a maximal left ideal M of R with

$$RL + R(1-e) \preceq M \quad \text{and} \quad e \notin M\ .$$

Since $eMe = L$, we have

$$e(R/M) = e(Re/Me) \cong eRe/eMe = eRe/L.$$

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