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DETERMINANTS OF A CLASS OF TOEPLITZ MATRICES

TOM HØHOLDT and JØRN JUSTESEN

Abstract.

Let $T_n(f) = (a_{i-j})$, $i, j = 0, 1, \ldots n$ be the finite Toeplitz matrices generated by the Laurent expansion of an arbitrary rational function f. In his paper [2] K. M. Day has calculated $\det T_n(f)$. The present paper contains a new proof of Day's formula (4.1.).

1. Introduction.

Toeplitz matrices occur in a variety of applications of mathematics [4]–[8], and the determinants of Toeplitz matrices of special classes have been studied by several authors [1], [3], [9]. In connection with some problems in digital signal processing [4] it is necessary to calculate the determinants of the Toeplitz matrices generated by the Laurent expansion of a rational function. We have solved the problem by a method somewhat different from that of Day, which gives a new proof of his results. The main ideas in our proofs are: 1) Reduction of the problem to a special case, 2) observing that the determinants are almost the same as those obtained from a Laurent polynomial and finally 3) that the determinants can be calculated by solving a system of linear equations.

2. Preliminaries.

With each formal power series $f(z) = \sum_{\nu=-\infty}^{\infty} a_\nu z^\nu$ we associate the Toeplitz matrices $T_n(f)$, where $t_{ij} = a_{i-j}$, $i, j = 0, 1, \ldots n$. Now let R_1 and R_2 be real numbers such that $0 \leqq R_1 < R_2$. Let $D(z)$ be a complex polynomial of degree k with roots $\delta_1, \delta_2, \ldots, \delta_k$ where $|\delta_i| \leqq R_1$, and $F(z)$ a polynomial of degree h with roots $\varrho_1, \varrho_2, \ldots, \varrho_h$ satisfying $|\varrho_j| \geqq R_2$. Finally let $G(z)$ be a polynomial of degree p, with distinct roots $r_1, r_2, \ldots, r_p$.

Let $\sum_{\nu=-\infty}^{\infty} a_\nu z^\nu$ be the Laurent expansion of $f(z) = G(z)/F(z)D(z)$ in the annulus $\{z \in \mathbf{C} \mid R_1 < |z| < R_2\}$. We want to express $\det T_n(f)$ in terms of the

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roots of $G(z)$, $F(z)$ and $D(z)$. If we assume that $k \geqq 1$, $h \geqq 1$, $p \geqq k+h$ and that $G(z)$ has no factor in common with $F(z)$ or $D(z)$ then the Laurent expansion has both positive and negative values of v such that $a_v \neq 0$.

3. Determination of det $T_n(f)$ in a special case.

Let the notation be as above, and suppose $p \geqq h+k$. Let the polynomials be normed such that

$$D(z) = \prod_{j=1}^{k} (z-\delta_j), \quad F(z) = \prod_{j=1}^{h} (1-\varrho_j^{-1}z), \quad G(z) = \prod_{j=1}^{p} (z-r_j)\,.$$

LEMMA 3.1.

$$\det T_n(f) = \det (A + T_n(z^{-k}G))$$

where A has zeros everywhere except in the $(p-k)\times k$ upper left corner.

PROOF. Let

$$F(z) = \sum_{i=0}^{h} f_i z^i \quad \text{and} \quad z^{-k}D(z) = \sum_{j=0}^{k} d_{-j} z^{-j}.$$

From $\det T_n(F) = 1 = \det T_n(z^{-k}D)$ it follows that

$$\det T_n(f) = \det T_n(F) \det T_n(f) \det T_n(z^{-k}D)$$

and hence

3.1.1. $$\det T_n(f) = \det [T_n(F) T_n(f) T_n(z^{-k}D)]\,.$$

Now let

$$T_n(z^{-k}G) = T_n(F\cdot f\cdot z^{-k}D) = (b_{rs})$$

and $\qquad r,s = 0,1,\ldots,n$

$$T_n(F)T_n(f)T_n(z^{-k}D) = (c_{rs})$$

By calculation we get

$$b_{rs} = \sum_{j=0}^{k} \sum_{i=0}^{h} d_{-j} f_i a_{r-s-i+j}$$

and

$$c_{rs} = \sum_{j=0}^{\min(s,k)} \sum_{i=0}^{\min(r,h)} d_{-j} f_i a_{r-s-i+j}$$

1) If $s \geqq k$ and $r \geqq h$ we conclude that $b_{rs} = c_{rs}$.

2) If $s \geqq k$ and $r < h$ we have

$$b_{rs} - c_{rs} = \sum_{i=r+1}^{h} f_i y_i, \quad \text{where } y_i = \sum_{j=0}^{k} d_{-j} a_{r-s-i+j} .$$

It is seen that y_i is the coefficient of $z^{r-s-i+k}$ in the Laurent expansion of $z^k f(z) z^{-k} D(z) = G(z)/F(z)$, in which only nonnegative powers of z occur. Hence y_i equals zero for $i = r+1, r+2, \ldots, h$, and therefore $c_{rs} = b_{rs}$.

3) If $s < k$ and $r \geqq h$ we have

$$b_{rs} - c_{rs} = \sum_{j=s+1}^{k} x_j d_{-j}, \quad \text{where } x_j = \sum_{i=0}^{h} f_i a_{r-s-i+j} .$$

It is seen that x_j is the coefficient of z^{r-s+j} in the Laurent expansion of $F(z) f(z) = G(z)/D(z)$, in which only powers of z with exponent less than $p-k+1$ occur. Hence if $r \geqq p-k$, x_j equals zero for $j = s+1, s+2, \ldots, k$ and therefore $c_{rs} = b_{rs}$. Using the assumption $p-k \geqq h$ the Lemma follows.

LEMMA 3.2. *Let I be a k-subset of $\{1,2,\ldots,p\}$ and $\bar{I} = \{1,2,\ldots,p\} \setminus I$. Let*

$$g_I(z) = \prod_{j \in I} (1 - r_j z^{-1}) \quad \text{and} \quad h_I(z) = \prod_{j \in \bar{I}} (z - r_j)$$

then

3.2.1. $$T_n(h_I) T_n(g_I) = B_I + T_n(z^{-k} G)$$

where B_I has zeros everywhere, except in the $(p-k) \times k$ upper left corner.

PROOF. Note that $h_I(z) g_I(z) = z^{-k} G(z)$. Simple calculation of the elements of $T_n(z^{-k} G)$ and those of $T_n(h_I) T_n(g_I)$ yields the result.

3.3. From the lemmas we see that the matrices

$$T_n(F) T_n(f) T_n(z^{-k} D) \quad \text{and} \quad T_n(h_I) T_n(g_I)$$

differ only in the $(p-k) \times k$ upper left corner. The determinant of the latter is easily calculated:

$$\det T_n(h_I) T_n(g_I) = \prod_{j \in \bar{I}} r_j^{n+1} (-1)^{(p-k)(n+1)} .$$

We now claim that there exist constants x_I, $I \subseteqq \{1,2,\ldots,p\}$, $|I| = k$ such that

3.3.1. $$\det T_n(f) = \sum_I x_I \det T_n(h_I) T_n(g_I)$$

or by 3.1. and 3.2.

3.3.2. $$\det(A+T_n(z^{-k}G)) = \sum_I x_I \det(B_I+T_n(z^{-k}G))\,.$$

Expressing the determinants of a sum of matrices as a sum of products of minors and complementary minors 3.3.2. is implied by

3.3.3. $$A_l = \sum_I x_I B_{I,l}; \qquad l=1,2,\ldots,\binom{p}{k} = q$$

where A_l is the lth minor of the $(p-k)\times k$ upper left submatrix A' of A, and $B_{I,l}$ the corresponding minor of the submatrix B_I' of B_I.

The next lemmas show that the homogeneous system corresponding to 3.3.3. has the zero solution only and hence the solution of 3.3.3. is unique. Moreover we shall derive an explicit expression for the x_I's satisfying 3.3.3. Baxter and Schmidt [1] have expressed $\det T_n(z^{-k}G)$ as a function of the r_j's and have given a simple derivation. From lemma 3.1. it follows that $\det T_n(f)$ satisfies the same linear recurrence equation as $\det T_n(z^{-k}G)$. Thus the claim follows directly from this observation and the result of [1]. We now procede to solve 3.3.3.

LEMMA 3.4. *Suppose $p-k=k$. With the same notation as above we have*:

$$x_I = (-1)^k \det(A'-B_{\bar I}') \prod_{\substack{i\in\bar I\\ j\in I}} \left|(r_i-r_j)^{-1}\right.$$

PROOF. In 3.3.3.

$$A_l = \sum_I x_I B_{I,l}$$

we enumerate the minors such that A_l and A_{q-l} are complementary. Let J be a k-subset of $\{1,2,\ldots,p\}$. Multiplying each of the equations in 3.3.3. by $B_{J,q-l}$ $(-1)^{\alpha+\beta}$, where α equals the sum of row and column indices used in forming A_l, and where β is the number of rows in $B_{J,q-l}$ and adding all the equations leads to

$$\sum_{l=1}^{q} A_l B_{J,q-l}(-1)^\alpha = \sum_{l=1}^{q}\sum_I x_I B_{I,l} B_{J,q-l}(-1)^\alpha$$

which is

3.4.1. $$\det(A'-B_J') = \sum_I x_I \det(B_I'-B_J')$$

Since $p-k=k$ we have

$$B'_I - B'_J = T_{k-1}(h_I)T_{k-1}(g_I) - T_{k-1}(h_J)T_{k-1}(g_J)$$

Let Z be the $k \times k$-matrix with elements $z_{ij} = \delta_{i-1,j}$. Then

$$T_{k-1}(h_I) = \prod_{j \in \bar{I}} (Z - r_j E) \quad \text{and} \quad T_{k-1}(g_I) = \prod_{j \in I} (E - r_j Z^{\mathrm{T}}),$$

where E is the $k \times k$ unit matrix and $^{\mathrm{T}}$ denotes transpose. Therefore:

$$B'_I - B'_J = \prod_{j \in \bar{I}} (z - r_j E) \prod_{j \in I} (Z - r_j Z^{\mathrm{T}}) - \prod_{m \in \bar{J}} (Z - r_m E) \prod_{m \in J} (E - r_m Z^{\mathrm{T}}),$$

which equals

$$\prod_{j \in \bar{I} \cap \bar{J}} (Z - r_j E) \Bigg[\prod_{l \in \bar{I} \setminus \bar{J}} (Z - r_l E) \prod_{l \in I \setminus J} (E - r_l Z^{\mathrm{T}}) - \\ - \prod_{m \in \bar{J} \setminus \bar{I}} (Z - r_m E) \prod_{m \in J \setminus I} (E - r_m Z^{\mathrm{T}}) \Bigg] \prod_{j \in J \cap I} (E - r_j Z^{\mathrm{T}}).$$

Noting that $\bar{I} \setminus \bar{J} = J \setminus I$ and $I \setminus J = \bar{J} \setminus \bar{I}$, it is seen that if $J \cap I \neq \emptyset$, the matrix in square brackets contains a row of zeros, and therefore $\det(B'_I - B'_J)$ equals zero in this case.

If $I \cap J = \emptyset$, that is $J = \bar{I}$, we consider $\det(B'_I - B'_J)$ as a polynomial in the r_i's. Determination of the degree and one of the coefficients of this polynomial yields

$$\det(B'_I - B'_{\bar{I}}) = (-1)^k \prod_{\substack{j \in I \\ i \in \bar{I}}} (r_i - r_j)$$

3.4.1. then reduces to

$$\det(A' - B'_{\bar{I}}) = (-1)^k x_I \prod_{\substack{i \in \bar{I} \\ j \in I}} (r_i - r_j)$$

from which the lemma follows.

What remains is to derive an expression for $\det(A' - B'_{\bar{I}})$. We do this in the next lemma.

LEMMA 3.5. *Using the same notation as before*

$$\det(A' - B'_{\bar{I}}) = (-1)^k \prod_{\substack{s \in \{1,\dots,k\} \\ t \in \{1,\dots,h\}}} (\varrho_t - \delta_s)^{-1} \prod_{\substack{i \in \bar{I} \\ s \in \{1,\dots,k\}}} (r_i - \delta_s) \prod_{\substack{j \in I \\ t \in \{1,\dots,h\}}} (\varrho_t - r_j).$$

PROOF. It is enough to prove the lemma in the case where the roots of $F(z)D(z)$ is distinct and different from zero. The general validity of lemma 3.5. then follows by continuity. We have

$$A' - B'_{\bar{I}} = \prod_{t=1}^{h} (E - \varrho_t^{-1} Z) T_{k-1}(f) \prod_{s=1}^{k} (E - \delta_s Z^{\mathrm{T}}) - \prod_{j \in I} (Z - r_j E) \prod_{j \in \bar{I}} (E - r_j Z^{\mathrm{T}}) .$$

If $r_{j_1} = \delta_{s_1}$ for some j_1 in $\bar{I}$ we have

$$A' - B'_{\bar{I}} = \Bigg[\prod_{t=1}^{h} (E - \varrho_t^{-1} Z) T_{k-1}(\hat{f}) \prod_{\substack{s=1 \\ s \neq s_1}}^{k} (E - \delta_s Z^{\mathrm{T}}) - \prod_{j \in I} (Z - r_j E) \prod_{\substack{j \in \bar{I} \\ j \neq j_1}} (E - r_j Z^{\mathrm{T}}) \Bigg] (E - r_{j_1} Z^{\mathrm{T}})$$

where $\hat{f}(z) = G_1(z)/(F(z) D_1(z))$ the degree of $G_1(z)$ is $p-1$ and the degree of $D_1(z)$ is $k-1$.

By direct calculation as in 3.1. and 3.2. it is seen that the matrix in square brackets has zeros in its last column, so in this case $\det(A' - B'_{\bar{I}})$ is equal to zero.

Similarly, if $r_{i_1} = \varrho_{t_1}$ for some i_1 in I, $\det(A' - B'_{\bar{I}}) = 0$.

Since $\det(A' - B'_{\bar{I}})$ is a polynomial in the r_i's we have

$$\det(A' - B'_{\bar{I}}) = a \prod_{\substack{i \in \bar{I} \\ s \in \{1, \ldots, k\}}} (r_i - \delta_s) \prod_{\substack{j \in I \\ t \in \{1, \ldots, h\}}} (\varrho_t - r_j)$$

where a is independent of the r_i's. If we set all r_i's equal to zero the above expression is

$$\det T_{k-1}(\tilde{f}) = (-1)^k a \prod_{s=1}^{k} \delta_s^k \prod_{t=1}^{h} \varrho_t^k$$

where $\tilde{f}(z) = z^{2k}/F(z) D(z)$.

It is well known, at least for $F(z) = D(z^{-1})$, that

3.5.1. $$\det T_{k-1}(\tilde{f}) = \prod_{s=1}^{k} \delta_s^k \prod_{\substack{s \in \{1,2,\ldots,k\} \\ t \in \{1,2,\ldots,h\}}} (1 - \delta_s \varrho_t^{-1})^{-1} .$$

This determinant may be calculated directly, nothing that the relevant terms of the Laurent series $\tilde{f}(z) = \sum_{v=-\infty}^{\infty} \tilde{a}_v z^v$ are

$$\tilde{a}_v = \sum_{j=1}^{k} b_j \delta_j^{2k-1-v}, \quad v < 2k$$

where

$$b_j^{-1} = \prod_{t=1}^{h} (1 - \delta_s \varrho_t^{-1}) \prod_{\substack{s=1 \\ s \neq j}}^{k} (\delta_j - \delta_s) .$$

Thus $T_{k-1}(\tilde{f})=MN$ where $m_{ij}=b_j\delta_j^{2k-1-i}$, $n_{ij}=\delta_i^j$, $i,j=0,1,\ldots,k-1$ and

$$\det T_{k-1}(\tilde{f}) = \prod_{j=1}^{h} b_j\delta_j^k \prod_{i\neq j} (\delta_j-\delta_i)$$

which proves 3.5.1. We conclude that

$$a = (-1)^k \prod_{\substack{t\in\{1,\ldots,h\}\\ s\in\{1,\ldots,k\}}} (\varrho_t-\delta_j)^{-1}$$

which finally proves the lemma.

Summarizing the previous results, we have so far proven that if $p-k=k$, and $f(z)=G(z)/F(z)D(z)$ then

3.6. $$\det T_n(f) = \sum_I \prod_{\substack{i\in \bar{I}\\ s\in\{1,\ldots,k\}}} (r_i-\delta_s) \prod_{\substack{j\in I\\ t\in\{1,\ldots,h\}}} (\varrho_t-r_j) \prod_{\substack{s\in\{1,\ldots,k\}\\ t\in\{1,\ldots,h\}}} (\varrho_t-\delta_s)^{-1} \prod_{\substack{i\in \bar{I}\\ j\in I}} (r_i-r_j)^{-1} \prod_{i\in\bar{I}} r_i^{n+1}$$

where the summation runs over all k-subsets of $\{1,\ldots,p\}$.

In the next section we will extend this result to the case where we only assume that $p-k\geqq h$, that is to Day's result.

4. The theorem of K. M. Day.

THEOREM 4.1. *Let R_1 and R_2 be real numbers such that $0\leqq R_1<R_2$. Let $D(z)$ be a complex polynomial of degree k with roots $\delta_1,\delta_2,\ldots,\delta_k$ satisfying $|\delta_i|\leqq R_1$, and $F(z)$ a polynomial of degree h with roots $\varrho_1,\varrho_2,\ldots,\varrho_h$ satisfying $|\varrho_j|\geqq R_2$. Let $G(z)$ be a polynomial of degree p with distinct roots $r_1,\ldots,r_p$. Let the polynomials be normed such that*

$$D(z) = \prod_{j=1}^{k} (z-\delta_j), \quad F(z) = \prod_{j=1}^{h} (1-\varrho_j^{-1}z), \quad G(z) = \prod_{j=1}^{p} (z-r_j)\,.$$

Let $\sum_{\nu=-\infty}^{\infty} a_\nu z^\nu$ be the Laurent expansion of $f(z)=G(z)/F(z)D(z)$ in the annulus $\{z\in \mathbf{C} \mid R_1<|z|<R_2\}$. Let $T_n(f)=(a_{i-j})$, $i,j=0,1,\ldots,n$. Then *if $p=k+m$, $m\geqq h$,*

4.1.1. $$\det T_n(f) = (-1)^{m(n+1)}\sum_I \prod_{\substack{j\in I\\ s\in\{1,\ldots,k\}}} (r_i-\delta_s) \prod_{\substack{j\in \bar{I}\\ t\in\{1,\ldots,h\}}} (\varrho_t-r_j) \prod_{\substack{t\in\{1,\ldots,h\}\\ s\in\{1,\ldots,k\}}} (\varrho_t-\delta_s)^{-1} \prod_{\substack{i\in I\\ j\in \bar{I}}} (r_i-r_j)^{-1} \prod_{i\in I} r_i^{n+1}$$

where the summation runs over all m-subsets of $\{1,2,\ldots k+m\}$ *and* $\bar{I} = \{1,2,\ldots,k+m\} \setminus I$.

PROOF. Suppose that $k > p-k \geqq h$. Choose nonzero element $r_{p+1}, r_{p+2}, \ldots, r_{2k}$ distinct and different from the roots of $G(z)$, such that $|r_i| \geqq R_2$, $i = p+1,\ldots,2k$. Then

$$f(z) = (-1)^{2k-p} \prod_{l=p+1}^{2k} (z-r_l)G(z) \Big/ F(z)D(z) \prod_{l=p+1}^{2k} (1-r_l^{-1}z)(r_{p+1} \cdot r_{p+2} \cdots r_{2k})$$

Hence by the formula (3.6.) already proven:

$$\det T_n(f) = b \sum_I \prod_{\substack{i \in \bar{I} \\ s \in \{1,\ldots,k\}}} (r_i - \delta_s) \prod_{\substack{j \in I \\ t \in \{1,\ldots,v\}}} (\varrho_t - r_j)$$

$$\prod_{\substack{s \in \{1,\ldots,k\} \\ t \in \{1,\ldots,v\}}} (\varrho_t - \delta_s)^{-1} \prod_{\substack{i \in \bar{I} \\ j \in I}} (r_i - r_j)^{-1} \prod_{i \in \bar{I}} r_i^{n+1}$$

where $v = h + 2k - p$ and $b = (-1)^{k(n+1)}(-1)^{(2k-p)(n+1)}(r_{p+1} \cdots r_{2k})^{-(n+1)}$ and the summation now runs over all k-subsets I of $\{1,2,\ldots,p,p+1,\ldots,2k\}$ and where $\varrho_{h+1} = r_{p+1}, \ldots, \varrho_{h+2k-p} = r_{2k}$. It is seen that if $I \cap \{p+1,\ldots,2k\} \neq \emptyset$, then the product equals zero, and if $I \cap \{p+1,\ldots,2k\} = \emptyset$ cancellation of equal terms leads to

4.1.2. $$\det T_n(f) = (-1)^{(p-k)(n+1)} \sum_I \prod_{\substack{i \in \bar{I} \\ s \in \{1,\ldots,h\}}} (r_i - \delta_s) \prod_{\substack{j \in I \\ t \in \{1,\ldots,h\}}} (\varrho_t - r_j)$$

$$\prod_{\substack{s \in \{1,\ldots,k\} \\ t \in \{1,\ldots,h\}}} (\varrho_t - \delta_s)^{-1} \prod_{\substack{i \in \bar{I} \\ j \in I}} (r_i - r_j)^{-1} \prod_{i \in \bar{I}} r_i^{n+1}$$

where the summation now runs over all k-subsets I of $\{1,2,\ldots,p\}$. If $p > 2k$, we choose nonzero distinct elements $r_{p+1}, \ldots, r_{p+(p-2k)}$ different from the roots of $G(z)$. Then

$$f(z) = G(z) \prod_{l=1}^{p-2k} (z - r_{p+l}) \Big/ F(z)D(z) \prod_{l=1}^{p-2k} (z - r_{p+l})$$

$$= G_1(z)/F(z)D_1(z)$$

where $D_1(z)$ now has degree $p-k$ and $G_1(z)$ has degree $2(p-k)$. Applying the result 3.6. once more yields 4.1.2. Inserting $p-k=m$, noting that

$$\binom{p}{k} = \binom{k+m}{k} = \binom{k+m}{m}$$

and changing indices finally results in 4.1.1.

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DEPARTMENT OF MATHEMATICS
AND
INSTITUTE OF CIRCUIT THEORY AND TELECOMMUNICATION

TECHNICAL UNIVERSITY OF DENMARK
DK-2800 LYNGBY, DENMARK