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REMAINDER ESTIMATES FOR EIGENVALUES AND KERNELS OF PSEUDO-DIFFERENTIAL ELLIPTIC SYSTEMS

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1. Introduction.

Let Σ be an n-dimensional compact C^∞ manifold without boundary, provided with a C^∞ density dx, and let E be a C^∞ complex vector bundle over Σ of dimension q (n and $q \geqq 1$). We assume that E is provided with a smooth Hermitian metric, so that the space of square integrable sections $L^2(E)$, and the Sobolev spaces $H^s(E)$ ($s \in \mathbf{R}$) can be defined; the norms will be denoted $\|u\|_s$ (the L^2-norm denoted $\|u\|_0$, with scalar product (u,v)).

Let P be a classical pseudo-differential operator of order $l \in \mathbf{R}_+$ in E. That P is classical means that P operates on the sections in E in such a way that in each local trivialization $\varkappa\colon E|_X \to U \times \mathbf{C}^q$ (with $U \subset \mathbf{R}^n$), P has the form $P_\varkappa = \operatorname{Op}(p)$,

$$(1.1)\qquad \operatorname{Op}(p)u = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi \quad \text{for } u \in C_0^\infty(U)\,,$$

where $p(x,\xi)$ is a C^∞ $q \times q$-matrix valued function on $U \times \mathbf{R}^n$ satisfying

$$(1.2)\qquad p(x,\xi) \sim \sum_{j=0}^{\infty} p^j(x,\xi)\,,$$

the $p^j(x,\xi)$ being homogeneous in ξ of degree $l-j$ and C^∞ on $U \times (\mathbf{R}^n \setminus \{0\})$. Here (1.2) stands for the property:

$$(1.3)\qquad D_x^\beta D_\xi^\alpha\Big(p(x,\xi)-\sum_{j=0}^{N} p^j(x,\xi)\Big) \text{ is } O(|\xi|^{l-N-1-|\alpha|}), \quad \text{for } |\xi| \to \infty\,,$$

for all multiindices α and β, uniformly for x in compact subsets of U. The principal symbol $p^0(x,\xi)$ can be given an invariant meaning on $T^*(\Sigma)\setminus 0$. (See e.g. Seeley [20] for further explanations.)

We assume that P is *selfadjoint* in $L^2(E)$ (so in particular $p^0(x,\xi)$ is selfadjoint at each (x,ξ)) and, except in Corollary 5.5, that P is *strongly elliptic*, i.e., $p^0(x,\xi)$ is positive definite at each (x,ξ) ($\xi \neq 0$). Note that the functions

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$p^j(x,\xi)$ have locally bounded derivatives in ξ on $\mathbf{R}^n$ up to order $l-j$. Hence for $j \leqq l$, we can apply the formula (1.1) to p^j, defining operators $\mathrm{Op}\,(p^j) = P^j$.

The above hypotheses imply that for $u \in C^\infty(E)$,

$$(1.4) \qquad (Pu,u) \geqq c_0\|u\|_0^2;$$

we may and shall assume that $c_0 > 0$. In the following we consider the maximal realization of P as an operator in $L^2(E)$, which we also denote P; it is a selfadjoint positive operator in $L^2(E)$ with domain $H^l(E)$. Since $l > 0$, the spectrum of P is a sequence of positive real eigenvalues going to ∞. The resolvent

$$(1.5) \qquad Q_\lambda = (P-\lambda I)^{-1}$$

exists for $\lambda \in \mathbf{C} \setminus \mathbf{R}_+$, and it follows easily from (1.4) that for $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}_+}$,

$$(1.6) \qquad \|Q_\lambda u\|_0 \leqq \frac{1}{d(\lambda)}\|u\|_0 \quad \text{for } u \in L^2(E),$$

where $d(\lambda)$ is the distance from λ to $\mathbf{R}_+$.

We shall present a construction of Q_λ (for λ outside a parabolic region around $\mathbf{R}_+$) that can be used to deduce the following estimate for the number $N(t;P)$ of eigenvalues of P less than t:

$$(1.7) \qquad N(t;P) = c_P t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) \quad \text{for } t \to \infty,$$

for any $\varepsilon > 0$, where

$$c_P = \frac{1}{n(2\pi)^n}\int_\Sigma \int_{S_x} \mathrm{tr}\,[p^0(x,\xi)^{-n/l}]\,d\omega\,dx,$$

cf. Theorem 5.4 below. The spectral function of P satisfies the related estimate

$$(1.8) \quad \mathrm{tr}\,e(t;x,x) = \frac{t^{n/l}}{n(2\pi)^n}\int_{S_x} \mathrm{tr}\,[p^0(x,\xi)^{-n/l}]\,d\omega + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) \quad \text{for } t \to \infty,$$

uniformly in $x \in \Sigma$. We furthermore derive from (1.7) that when P is selfadjoint elliptic of order l, but not strongly elliptic, then the numbers $N^\pm(t;P)$ of eigenvalues of P in the intervals $[0,t]$ resp. $[-t,0]$ satisfy

$$(1.9) \qquad N^\pm(t;P) = c_P^\pm t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) \quad \text{for } t \to \infty,$$

where (cf. Corollary 5.5)

$$c_P^\pm = \frac{1}{n(2\pi)^n}\int_\Sigma \int_{S_x} \sum |\lambda_j^\pm(p^0(x,\xi))|^{-n/l}\,d\omega\,dx.$$

The *principal* estimate ((1.7) with the O-term replaced by $o(t^{n/l})$) was shown in Seeley [19]. When P is a *scalar* pseudo-differential operator (i.e., $E=\Sigma\times \mathbf{C}$) or $p^0(x,\xi)$ has *simple eigenvalues*, the remainder in (1.7–8) can be improved to be $O(t^{(n-1)/l})$, see Hörmander [14]. (This possibly extends to the case where the eigenvalues of $p^0(x,\xi)$ have *constant multiplicity*, cf. Duistermaat–Guillemin [7].) For the case where P is a *differential operator*, (1.7–8) follow already from Agmon–Kannai [3] and Hörmander [13]; see also the simplified proof in Nagase [17].

The novelty of the present work is then that it obtains remainder estimates for general pseudo-differential systems. Like Nagase [17] (and earlier Seeley [19], Hörmander [13]) we construct Q_λ as a sum of terms Q_λ^k ($k=0,\dots,N$), with symbols homogeneous in $(\xi,(-\lambda)^{1/l})$, and a remainder term $S_{\lambda,N}$. When P is a differential operator, the Q_λ^k have rational symbols with denominator equal to a power of $\det(p^0(x,\xi)-\lambda I)$; they are C^∞ in $(\xi,\lambda)\in \mathbf{R}^n\times(\mathbf{C}\setminus\overline{\mathbf{R}_+})$ and satisfy convenient estimates with respect to $(1+|\xi|+|\lambda|^{1/l})$ (used in [17]). When P is a pseudo-differential operator, the Q_λ^k have a less simple structure; in particular, the ξ-derivatives of their symbols satisfy convenient estimates in $(1+|\xi|+|\lambda|^{1/l})$ only up to order $l-k$; also $S_{\lambda,N}$ is more complicated. Here we profit from the boundedness theorem of Calderón and Vaillancourt [5] (developed further by Cordes [6] and Kato [15]) which keeps an accurate account of the derivatives needed for each estimate.

It is applied to operators where $|\lambda|^{1/l}$ is built in as an extra variable; from this we deduce Sobolev estimates for our operators in n variables, which imply the appropriate kernel estimates by a well known theorem of Agmon.

A special aspect of our proof is that we have to enlarge the *order* of P (by replacing P by a power $(P)^r$), not just so that it exceeds the dimension n, but actually the larger, the smaller ε in (1.7)–(1.8) is ($rl\sim\varepsilon^{-1}n$). This is not necessary when the same proof is applied to *differential* operators, see Remark 4.9 below. (Other methods of proving L^∞ estimates of the kernels may possibly avoid this phenomenon, but it enters necessarily in our proof of the Sobolev estimates, that are meant to be useful in a generalization to boundary value problems as in [10], [11].)

In Sections 2–4 we construct the approximate resolvent in local coordinates (this is of course of interest also for operators on noncompact manifolds or subsets of $\mathbf{R}^n$). Section 5 proves the main results for operators on Σ. In Section 6, we apply our theorem to obtain an eigenvalue estimate like (1.7) for strongly *Douglis–Nirenberg elliptic* pseudo-differential systems P, with l denoting the lowest order occurring in P; the O-term is under certain circumstances replaced by a weaker estimate, see Theorem 6.3. (This improves a result of Kozevnikov [16], also proved by the author in CIME III, 1973).

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2. Symbols of the resolvent in local coordinates.

In this and the next two sections we consider P in a local trivialization $U \times \mathbf{C}^q$ for E. More precisely, we modify the symbol $p(x,\xi)$ (by multiplying it with a cut-off function) so that we now have (1.2) valid for $x \in \mathbf{R}^n$; p and the p^j having x-support in a fixed compact set K_1, with $p^0(x,\xi)$ being positive definite (for $\xi \neq 0$) for x in another fixed compact set K_2. For simplicity of notation, we again denote $\mathrm{Op}(p) = P$. We also assume in Sections 2 and 4 that l is integer. For integer $N \leqq l-1$, we define

$$\text{(2.1)}\qquad \begin{aligned} P^j &= \mathrm{Op}(p^j) \text{ for } j \leqq N, \qquad P_{\lambda,N} = \sum_{j=0}^{N} P^j - \lambda I\,, \\ T_N &= P - \sum_{j=0}^{N} P^j, \qquad \text{so that} \qquad P - \lambda I = P_{\lambda,N} + T_N\,. \end{aligned}$$

Since $|D_x^\beta p^j(x,\xi)| \leqq c_\beta (1+|\xi|)^{l-j}$ for all β, the P_j are continuous from $H^s(\mathbf{R}^n)$ to $H^{s-l+j}(\mathbf{R}^n)$, and T_N is continuous from $H^s(\mathbf{R}^n)$ to $H^{s-l+N+1}(\mathbf{R}^n)$, for all real s. The following notation will be used throughout: For $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}_+}$, we write

$$\text{(2.2)}\qquad \lambda = -(e^{i\theta}\mu)^l, \text{ where } \mu = |\lambda|^{1/l} \text{ and } \theta = \frac{1}{l}\operatorname{Arg}(-\lambda),\ \theta \in \left]-\frac{\pi}{l}, \frac{\pi}{l}\right[\,.$$

We now construct symbols $q_\lambda^j(x,\xi)$ for $j = 0,1,\dots,N$, so that for $x \in K_2$, $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}_+}$,

$$\sum_{|\alpha| \leqq N} \frac{1}{\alpha!} \partial_\xi^\alpha \Big(\sum_{j=0}^{N} p^j - \lambda I \Big) D_x^\alpha \sum_{k=0}^{N} q_\lambda^k = I + [\text{terms of degree} \leqq -(N+1)];$$

these are determined successively by the formulas

$$\text{(2.3)}\qquad \begin{aligned} &\text{(i) } q_\lambda^0 = (p^0 - \lambda I)^{-1}\,, \\ &\text{(ii) } q_\lambda^k = -q_\lambda^0 \sum_{\substack{|\alpha|+i+j=k \\ j<k}} \frac{1}{\alpha!} \partial_\xi^\alpha p_\lambda^i D_x^\alpha q_\lambda^j\,, \end{aligned}$$

where $p_\lambda^0 = p^0 - \lambda I$, $p_\lambda^i = p^i$ for $i > 0$. We note that q_λ^k is homogeneous of degree $-l-k$ and continuous in $(\xi,\mu) \in \mathbf{R}^n \times \mathbf{R}_+$, for each $k \leqq N$ $(\leqq l-1)$, each $|\theta| < \pi/l$.

The resolvent will be studied for λ in a region

$$\text{(2.4)}\qquad V_\delta = \{\lambda \in \mathbf{C} \mid |\lambda| \geqq 1,\ \operatorname{Re}\lambda \leqq 0 \text{ or } |\operatorname{Im}\lambda| \geqq |\lambda|^{1-\delta/l}\}\,,$$

where δ will be specified later. Note that when λ runs through V_δ, $e^{i\theta}\mu = (-\lambda)^{1/l}$ (principal branch) runs through a subset V'_δ, (2.5) $$V'_\delta = \{(-\lambda)^{1/l} \mid \lambda \in V_\delta\},$$

of the sector $\{z \in \mathbf{C} \mid |\mathrm{Arg}\, z| < \pi/l\}$ (see fig. 1).

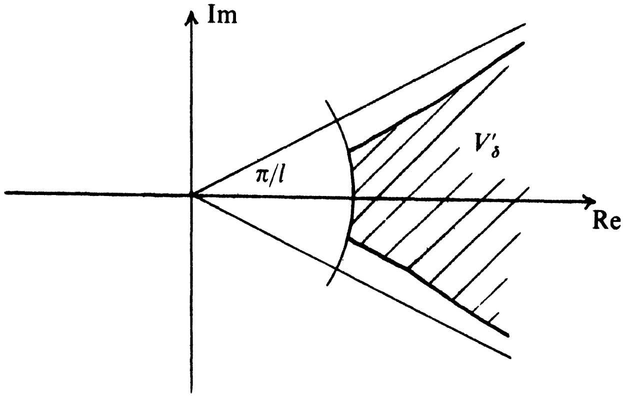

Figure 1.

It is easy to derive from the positivity and homogeneity of $p^0(x,\xi)$ that for $\lambda \in V_\delta$,

$$|p^0(x,\xi)-\lambda I| \geqq |\lambda|^{1-\delta/l} \quad \text{and} \quad |p^0(x,\xi)-\lambda I| \geqq c|\lambda|^{-\delta/l}|\xi|^l$$

(the matrix norm denoted $|\cdot|$), and hence

$$\begin{aligned} |q^0_\lambda(x,\xi)| &\leqq c_1|\lambda|^{\delta/l}(|\lambda|+|\xi|^{-l}) \\ &\leqq c_2\mu^\delta(\mu+|\xi|)^{-l} \qquad \text{for } x \in K_2,\ \xi \in \mathbf{R}^n \text{ and } \lambda \in V_\delta, \end{aligned}$$

cf. (2.2). By successive use of Leibniz' formula:

$$0 = D^\gamma_{x,\xi,\mu}[(p^0-\lambda I)q^0_\lambda] = \sum_{\sigma \leqq \gamma} c_{\sigma,\gamma} D^\sigma_{x,\xi,\mu}(p^0-\lambda I) D^{\gamma-\sigma}_{x,\xi,\mu} q^0_\lambda, \qquad \text{for } \gamma > 0,$$

we then find that for $x \in K_2$, $\xi \in \mathbf{R}^n$ and $\lambda \in V_\delta$,

(2.6) $$|D^\beta_x D^\alpha_\xi D^j_\mu q^0_\lambda(x,\xi)| \leqq c_{\alpha,\beta,j}\mu^{\delta(1+|\alpha|+|\beta|+j)}|\xi|^{l-|\alpha|}(\mu+|\xi|)^{-2l-j}$$

for all multiindices α and β and all integers $j \geqq 0$ (using that λ is polynomial in μ). In particular,

(2.7) $$|D^\beta_x D^\alpha_\xi D^j_\mu q^0_\lambda(x,\xi)| \leqq c_{\alpha,\beta,j}\mu^{\delta(1+|\alpha|+|\beta|+j)}(\mu+|\xi|)^{-l-|\alpha|-j}$$
$$\text{for } (x,\xi,\lambda) \in K_2 \times \mathbf{R}^n \times V_\delta, \text{ when } |\alpha| \leqq l.$$

(The latter estimate is valid for *all* α, when p^0 is *polynomial* in ξ). For any *sector*

$$W = \{|\lambda| \geqq 1 \mid |\mathrm{Arg}(-\lambda)| \leqq \varphi_0 < \pi\}$$

one has the stronger estimates (for $|\alpha| \leqq l$)

$$(2.8) \qquad |D_x^\beta D_\xi^\alpha D_\mu^j q_\lambda^0| \leqq c_{\alpha,\beta,j}(\mu + |\xi|)^{-l-|\alpha|-j} \quad \text{for } (x,\xi,\lambda) \in K_2 \times \mathbf{R}^n \times W.$$

In (2.7), there is a loss of μ^δ for each differentiation in x, ξ and μ; in fact (2.7) implies

$$(2.9) \qquad |D_x^\beta D_\xi^\alpha D_\mu^j q_\lambda^0(x,\xi)| \leqq c_{\alpha,\beta,j}(\mu + |\xi|)^{\delta - l - (|\alpha| + j)\varrho + |\beta|\delta},$$

with $\varrho = 1 - \delta$, which resembles the definition of the classes $S^m_{\varrho,\delta}$ of Hörmander [12] (a cryptical remark to this effect can be found in Eskin [8]). Of course, as function of x and ξ, q_λ^0 satisfies the estimates up to order l required for the class $S^{-l}_{1,0}$ for each λ (cf. also (2.8)), but not uniformly in $\lambda \in V_\delta$. However, the function $a(x,\xi,t,\tau) = q^0_{-(e^{i\theta}\tau)^l}(x,\xi)$ (considered for fixed θ and suitably extended to $\tau \in \mathbf{R}$) satisfies the requirements up to order l for the class $S^{\delta-l}_{\varrho,\delta}(K_2 \times \mathbf{R})$. In order to utilize this, we shall study the connection between certain estimates for operators in $n+1$ variables and families of operators in n variables (generalizing a device found in Agmon [1]).

3. Estimates obtained by addition of a variable.

Specifically for the abovementioned purposes, we introduce the class of symbols $S^m_{\varrho,\delta,k}$ defined as follows:

DEFINITION 3.1. Let $m \in \mathbf{R}$, let ϱ and $\delta \in [0,1]$, and let k be an integer $\geqq 0$. A (possibly matrix valued) function $a(x,\xi,\tau)$ on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ is said to belong to the class $S^m_{\varrho,\delta,k}(\mathbf{R}^{2n+1})$ (or simply $S^m_{\varrho,\delta,k}$) if the following continuous derivatives exist and satisfy the estimates

$$(3.1) \qquad |D_x^\beta D_\xi^\alpha D_\tau^j a(x,\xi,\tau)| \leqq c_{\alpha,\beta,j}(1 + |\xi| + |\tau|)^{m - (|\alpha| + j)\varrho + |\beta|\delta}$$

for $|\alpha| \leqq k$, all β and all j.

When $a \in S^m_{\varrho,\delta,k}$, it defines an operator $\bar{A}$ on $\mathbf{R}^{n+1}$

$$(3.2) \quad (\bar{A}f)(x,t) = \mathrm{Op}_{n+1}(a)f = (2\pi)^{-n-1}\int e^{i(x\cdot\xi + t\tau)}a(x,\xi,\tau)\hat{f}(\xi,\tau)\,d\xi\,d\tau$$

and a family of operators A_τ on $\mathbf{R}^n$ (parametrized by τ)

$$(3.3) \qquad (A_\tau u)(x) = \mathrm{Op}_n(a)u = (2\pi)^{-n}\int e^{ix\cdot\xi}a(x,\xi,\tau)\hat{u}(\xi)\,d\xi.$$

We assume in the following that $\varrho \in]0,1]$ and $\delta \in [0,1[$ are given, with ϱ

$\geqq \delta$. Recall the theorem of Calderón–Vaillancourt [5], improved to the present form by Cordes [6] and Kato [15]:

LEMMA 3.2. *When $p(x,\xi)$ is a function on $\mathbf{R}^d \times \mathbf{R}^d$ such that*

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leqq c_{\alpha,\beta}(1+|\xi|)^{(|\beta|-|\alpha|)\delta}$$

for all $|\alpha| \leqq [d/2]+1$, all $|\beta| \leqq [d/2]+2$ ($|\beta| \leqq [d/2]+1$ if $\delta=0$), then Op (p) *is a bounded operator in $L^2(\mathbf{R}^d)$.*

This implies for our operators

LEMMA 3.3. *Let $k \geqq [(n+1)/2]+1$, and let r be an integer $\geqq 0$. Then if $a \in S_{\varrho,\delta,k}^{-r}$, $\bar{A}$ is continuous from $L^2(\mathbf{R}^{n+1})$ to $H^r(\mathbf{R}^{n+1})$ (with a norm estimated by the constants in* (3.1) *for $|\beta| \leqq [(n+1)/2]+2+r$, $|\alpha|+j \leqq [(n+1)/2]+1$).*

PROOF. It is easy to see from (3.2) that for any multiindex α, any integer $j \geqq 0$,

$$D_x^\alpha D_t^j \bar{A} f = \mathrm{Op}_{n+1}\left(\sum_{\beta \leqq \alpha} \frac{\alpha!}{(\alpha-\beta)!\,\beta!} \xi^{\alpha-\beta} \tau^j D_x^\beta a(x,\xi,\tau)\right) f, \tag{3.4}$$

where $\xi^{\alpha-\beta}\tau^j D_x^\beta a(x,\xi,\tau) \in S_{\varrho,\delta,k}^{-r+|\alpha-\beta|+j+|\beta|\delta} \subset S_{\varrho,\delta,k}^{-r+|\alpha|+j}$. For $|\alpha|+j \leqq r$, we can apply Lemma 3.1 (with $d=n+1$) to each term, showing that $D_x^\alpha D_t^j \bar{A}$is continuous in $L^2(\mathbf{R}^{n+1})$. The last statement is easily checked.

A similar result can be shown for noninteger r, under much heavier assumptions on k.

Concerning A_τ, we first make some primitive observations:

LEMMA 3.4. (i) *If $a \in S_{\varrho,\delta,k}^m$ with $m \leqq -([n/2]+1)\delta$ and $k \geqq [n/2]+1$, then*

$$\|A_\tau u\|_0 \leqq c\|u\|_0 \quad \text{for } u \in L^2(\mathbf{R}^n), \tag{3.5}$$

with c depending only on the constants in (3.1) *for $|\alpha|, |\beta| \leqq [n/2]+1$, $j=0$.*

(ii) *If $a \in S_{\varrho,\delta,0}^m$ and is compactly supported in x, then its Fourier transform in x satisfies*

$$|\hat{a}(\eta,\xi,\tau)| \leqq c_N(1+|\eta|)^{-N}(1+|\xi|+|\tau|)^{m+N\delta} \tag{3.6}$$

for all integers $N \geqq 0$. In particular, if $m \leqq -(n+1)\delta$, A_τ satisfies (3.5) *with a constant that depends only on the size of the support and the constants in* (3.1) *for $|\beta| \leqq n+1$, $\alpha=0$ and $j=0$.*

PROOF. When the assumptions of (i) hold, then

$$|D_x^\beta D_\xi^\alpha a(x,\xi,\tau)| \leqq c_{\alpha,\beta}(1+|\xi|+|\tau|)^{m-|\alpha|\varrho+|\beta|\delta} \leqq c_{\alpha,\beta}$$

for $|\alpha|$ and $|\beta| \leqq [n/2]+1$; then the assertion follows from Lemma 3.1 (with $\delta = 0$).

(ii) Let $a \in S^m_{\varrho,\delta,0}$, vanishing for x outside a compact set K, then

$$\left|\eta^\alpha \int e^{-ix\cdot\eta} a(x,\xi,\tau)\,dx\right| = \left|\int e^{-ix\cdot\eta} D_x^\alpha a(x,\xi,\tau)\,dx\right|$$
$$\leqq c_{0,\alpha,0}(1+|\xi|+|\tau|)^{m+|\alpha|\delta}\int_K 1\,dx$$

for all α; this implies (3.6). Now if $m \leqq -(n+1)\delta$, we have for $u, v \in \mathscr{S}(\mathsf{R}^n)$, using (3.6),

$$|(A_\tau u, v)| = \left|c_1 \int e^{ix\cdot\xi} a(x,\xi,\tau)\hat{u}(\xi)\bar{v}(x)\,d\xi\,dx\right|$$
$$= \left|c_2 \int \hat{a}(\theta-\xi,\xi,\tau)\hat{u}(\xi)\bar{\hat{v}}(\theta)\,d\xi\,d\theta\right|$$
$$\leqq c_3 \int (1+|\theta-\xi|)^{-n-1}(1+|\xi|+|\tau|)^{m+(n+1)\delta}|\hat{u}(\xi)|\,|\hat{v}(\theta)|\,d\xi\,d\theta$$
$$\leqq c_4\|u\|_0\|v\|_0,$$

which implies (3.5).

These observations are helpful in the deduction of a much stronger result, Proposition 3.7 below.

For $\mu \in \mathsf{R}$ and $s \in \mathsf{R}$ we denote by $H^{s,\mu}(\mathsf{R}^n)$ the Sobolev space $H^s(\mathsf{R}^n)$ provided with the norm

$$\|u\|_{s,\mu} = \left(\int_{\mathsf{R}^n} (1+|\xi|^2+\mu^2)^s|\hat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}. \tag{3.7}$$

It is easily seen that for each $s \geqq 0$, this norm is equivalent with the norm $(\|u\|_s^2 + |\mu|^{2s}\|u\|_0^2)^{\frac{1}{2}}$, uniformly in μ. $H^{s,\mu}(\mathsf{R}^n)$ and $H^{-s,\mu}(\mathsf{R}^n)$ are anti-duals of each other (with respect to an extension of (u,v)); and when $s > s' > s''$, $H^{s',\mu}(\mathsf{R}^n)$ is an interpolated space between $H^{s,\mu}(\mathsf{R}^n)$ and $H^{s'',\mu}(\mathsf{R}^n)$ in an obvious way. Let $\zeta(t)$ denote a function on R with the properties: $\zeta \in C_0^\infty(\mathsf{R})$, $\zeta = 1$ for $|t| \leqq \frac{1}{2}$, $\zeta = 0$ for $|t| \geqq 1$ and $0 \leqq \zeta(t) \leqq 1$ for all t. One easily shows (or one may consult Agmon [1, pp. 272–273]):

LEMMA 3.5. *Let r be an integer $\geqq 0$. There exist three positive constants c_1, c_2 and c_3 (depending on ζ and r) so that*

$$(3.8)\qquad c_1\|u\|_{H^{r,\mu}(\mathbf{R}^n)} \leqq \|u(x)\zeta(t)e^{it\mu}\|_{H^r(\mathbf{R}^{n+1})} \leqq c_2\|u\|_{H^{r,\mu}(\mathbf{R}^n)}\,,$$

for all $|\mu|\geqq c_3$.

We shall now prove

PROPOSITION 3.6. *Let* $a(x,\xi,\mu)\in S^m_{\varrho,\delta,k}$ *with* $k\geqq[(n+1)/2]+1$, *and assume that a vanishes for x outside a compact set. Let* $\varphi\in\mathscr{S}(\mathbf{R})$. *When* $m\leqq\varrho$, *there exist constants* c_1 *and* c_2, *depending on* φ *and on a certain number of the estimates* (3.1), *so that*

$$(3.9)\qquad \|\tilde{A}(u(x)\varphi(t)e^{it\mu})-(A_\mu u)(x)\varphi(t)e^{it\mu}\|_{L^2(\mathbf{R}^{n+1})} \leqq c_1\|u\|_{L^2(\mathbf{R}^n)}$$

for all $u\in L^2(\mathbf{R}^n)$, *all* $|\mu|\geqq c_2$.

PROOF. For $u\in\mathscr{S}(\mathbf{R}^n)$,

$$(3.10)\qquad e^{-it\mu}\tilde{A}(u(x)\varphi(t)e^{it\mu}) = (2\pi)^{-n-1}\int_{\mathbf{R}^{n+2}} e^{i(x\cdot\xi+(\tau-\mu)(t-s))}a(x,\xi,\tau)\hat{u}(\xi)\varphi(s)\,ds\,d\xi\,d\tau\,.$$

Now for any $N\geqq 0$,

$$a(x,\xi,\tau) = \sum_{j=0}^{N}\frac{1}{j!}\partial_\tau^j a(x,\xi,\mu)(\tau-\mu)^j + a_N(x,\xi,\tau-\mu,\mu)\,,$$

where

$$a_N(x,\xi,\sigma,\mu) = \frac{1}{N!}\sigma^{N+1}\int_0^1 (1-h)^N\partial_\tau^{N+1}a(x,\xi,\mu+h\sigma)\,dh$$

satisfying, for all α,

$$|D_x^\alpha a_N(x,\xi,\sigma,\mu)| \leqq c_2(1+|\xi|+|\mu|)^{m-(N+1)\varrho+|\alpha|\delta}(1+|\sigma|)^{N+1+|m-(N+1)\varrho+|\alpha|\delta|}\,,$$

and hence for all $M\geqq 0$ (cf. the proof of Lemma 3.4(ii)),

$$(3.11)\qquad |\hat{a}_N(\eta,\xi,\sigma,\mu)| \leqq c_M(1+|\eta|)^{-M}(1+|\xi|+|\mu|)^{m-(N+1)\varrho+M\delta}(1+|\sigma|)^{N+1+|m-(N+1)\varrho+M\delta|}\,.$$

(We constantly use the estimate $(1+|a+b|)^r\leqq(1+|a|)^r(1+|b|)^{|r|}$.) Moreover, we have that

$$(2\pi)^{-n-1}\int e^{i(x\cdot\xi+(\tau-\mu)(t-s))}\partial_\tau^j a(x,\xi,\mu)(\tau-\mu)^j\hat{u}(\xi)\varphi(s)\,ds\,d\xi\,d\tau = \mathrm{Op}_n(\partial_\tau^j a(x,\xi,\mu))u(x)D_t^j\varphi(t) \equiv (A_\mu^{(j)}u)(x)D_t^j\varphi(t)\,.$$

Inserting this in (3.10) and multiplying by $e^{it\mu}$, we find

$$\bar{A}(u(x)\varphi(t)e^{it\mu}) = \sum_{j=0}^{N} \frac{1}{j!}(A_\mu^{(j)}u)(x)D_t^j\varphi(t)e^{it\mu} + R_N(\mu,u)(x,t) ,$$

where

$$R_N(\mu,\mu)(x,t) = (2\pi)^{-n-1}\int e^{i(x\cdot\xi+t\tau)}a_N(x,\xi,\tau-\mu,\mu)\hat{u}(\xi)\hat{\varphi}(\tau-\mu)\,d\xi\,d\tau .$$

The last term will be estimated first: For any $v \in \mathscr{S}(\mathbf{R}^{n+1})$,

$$\begin{aligned} &|(R_N(\mu,u)(x,t),v(x,t))_{L^2(\mathbf{R}^{n+1})}| \\ &= c_1\left|\int e^{i(x\cdot\xi+t\tau)}a_N(x,\xi,\tau-\mu,\mu)\hat{u}(\xi)\hat{\varphi}(\tau-\mu)\bar{v}(x,t)\,d\xi\,d\tau\,dx\,dt\right| \\ &= c_2\left|\int \hat{a}_N(\xi-\eta,\xi,\tau-\mu,\mu)\hat{u}(\xi)\hat{\varphi}(\tau-\mu)\bar{\hat{v}}(\eta,\tau)\,d\xi\,d\tau\,d\eta\right| \\ &\leqq c_3\int (1+|\xi-\eta|)^{-(n+1)}(1+|\xi|+|\mu|)^{N'}(1+|\tau-\mu|)^{N+1+|N'|} \\ &\qquad |\hat{u}(\xi)\hat{\varphi}(\tau-\mu)\hat{v}(\eta,\tau)|\,d\xi\,d\eta\,d\tau \end{aligned}$$

by (3.11) with $M=n+1$; here

$$N' = m-(N+1)\varrho+(n+1)\delta .$$

When $N'\leqq 0$, it now follows by a standard application of the Schwarz inequality, using that $\hat{\varphi} \in \mathscr{S}(\mathbf{R})$, that

$$|(R_N(\mu,u)(x,t),v(x,t))| \leqq c\|u\|_{L^2(\mathbf{R}^n)}\|v\|_{L^2(\mathbf{R}^{n+1})} ,$$

and hence

$$\|R_N(\mu,u)(x,t)\|_0 \leqq c\|u\|_0 , \tag{3.12}$$

where c does not depend on u and μ.

This terminates the proof for the case where $m\leqq -(n+1)\delta+\varrho$, for then (3.12) holds with $N=0$, and $R_0(\mu,u)(x,t)$ is simply equal to

$$\bar{A}(u(x)\varphi(t)e^{it\mu})-(A_\mu u)(x)\varphi(t)e^{it\mu}.$$

When m is larger, we proceed by induction: Assume that (3.9) has been proved for all $m\leqq m_0$ $(m_0\leqq\varrho)$, and let $m\leqq m_0+\varrho$ $(m\leqq\varrho)$. Then

$$\bar{A}(u\varphi e^{it\mu})-(A_\mu u)\varphi e^{it\mu} = \sum_{j=1}^{N} (A_\mu^{(j)}u)(D_t^j\varphi)e^{it\mu}+R_N(\mu,u)(x,t) ,$$

where we choose N so large that $N' \leqq 0$. Then $R_N(\mu, u)$ satisfies (3.12), and on the other hand, we can apply the induction hypothesis to each operator $A_\mu^{(j)}$, with φ replaced by $D_t^j\varphi$, which shows that for $j \geqq 1$,

$$\begin{aligned} \|A_\mu^{(j)} u D_t^j \varphi e^{it\mu}\|_0 &\leqq \|A_\mu^{(j)} u D_t^j \varphi e^{it\mu} - \overline{A^{(j)}}(u D_t^j \varphi e^{it\mu})\|_0 + \\ &\quad + \|\overline{A^{(j)}}(u D_t^j \varphi e^{it\mu})\|_0 \\ &\leqq c_1 \|u\|_0 + c_2 \|u D_t^j \varphi e^{it\mu}\|_0 \leqq c_3 \|u\|_0 , \end{aligned}$$

using that $\overline{A^{(j)}}$ is of order $m - j\varrho \leqq 0$. Altogether, we find (3.9) for a. Any $m \leqq \varrho$ is reached by a finite number of induction steps.

We can finally show

PROPOSITION 3.7. *Let r and k be nonnegative integers with $k \geqq [(n+1)/2]+1$, and let $a \in S_{\varrho,\delta,k}^{-r}$, with compact x-support. For any $s \leqq r$ there is a constant c_s so that*

$$(3.13) \qquad \|A_\mu u\|_{r-s,\mu} \leqq c_s \|u\|_{-s,\mu} \quad \textit{for all } u \in H^{-s}(\mathbf{R}^n), \textit{ all } \mu \in \mathbf{R} ,$$

and hence also

$$(3.14) \qquad \|A_\mu^* v\|_{s,\mu} \leqq c_s \|v\|_{s-r,\mu} \quad \textit{for all } v \in H^{s-r}(\mathbf{R}^n), \textit{ all } \mu \in \mathbf{R} .$$

PROOF. Case 1: $r=0$, $s=0$. Here we have that for $|\mu| \geqq c_1$ (c_1 being a suitable constant),

$$\begin{aligned} \|A_\mu u\|_0 &\leqq c_2 \|(A_\mu u)(x)\zeta(t) e^{it\mu}\|_0 && \text{(by Lemma 3.5)} \\ &\leqq c_2 \|\bar{A}(u(x)\zeta(t)e^{it\mu})\|_0 + c_3\|u\|_0 && \text{(by Proposition 3.6)} \\ &\leqq c_4 \|u\|_0 && \text{(by Lemmas 3.3 and 3.5)} . \end{aligned}$$

For $|\mu| \leqq c_1$, we obtain the estimate, uniformly in μ, by applying Lemma 3.2 (with $d=n$) directly to a.

Case 2: $r>0$, $s=0$. It is seen from (3.3) that for all multiindices α,

$$(3.15) \qquad D_x^\alpha \mathrm{Op}_n(a) = \mathrm{Op}_n\Big(\sum_{\beta \leqq \alpha} \frac{\alpha!}{(\alpha-\beta)!\,\beta!} \xi^{\alpha-\beta} D_x^\beta a\Big) = \mathrm{Op}_n(a_\alpha) ,$$

where $a_\alpha(x, \xi, \mu) \in S_{\varrho,\delta,k}^{-r+|\alpha|}$. Furthermore, $\mu^r a(x,\xi,\mu) \in S_{\varrho,\delta,k}^0$. Then altogether,

$$\begin{aligned} \|A_\mu u\|_{r,\mu} &\leqq c_5\Big(\|\mu^r A_\mu u\|_0 + \sum_{|\alpha| \leqq r} \|D_x^\alpha A_\mu u\|_0\Big) \\ &< c_6 \|u\|_0 \end{aligned}$$

for all $\mu \in \mathbf{R}$, by application of Case 1 to a_α (for $|\alpha| \leqq r$) and to $\mu^r a$.

Case 3: $r \geqq 0, s \leqq r$. Let $u \in H^{-s}(\mathrm{R}^n)$ and let $w = \mathrm{Op}_n((1+|\xi|^2+\mu^2)^{-s/2})u$. Then $\|u\|_{-s,\mu} = \|w\|_0$, and

$$\|A_\mu u\|_{r-s,\mu} = \|A_\mu \mathrm{Op}_n((1+|\xi|^2+\mu^2)^{s/2})w\|_{r-s,\mu}.$$

When s is integer, (3.13) then follows by applying the preceding cases to

$$A_\mu \mathrm{Op}_n((1+|\xi|^2+\mu^2)^{s/2}) = \mathrm{Op}_n(a(x,\xi,\mu)(1+|\xi|^2+\mu^2)^{s/2})$$

whose symbol is in $S^{-r+s}_{\varrho,\delta,k}$. Next, when s is not an integer, the result is obtained by interpolation. (3.14) is an immediate consequence, by the duality of $H^{s,\mu}(\mathrm{R}^n)$ and $H^{-s,\mu}(\mathrm{R}^n)$.

When A_μ is a family of operators parametrized by μ (running through R or an interval I of R), satisfying the estimates (3.13)–(3.14) for all $s \leqq r$, we say for brevity that A_μ is *of μ-order* $-r$ (for $\mu \in I$).

The conclusion of Proposition 3.7 will be needed for families of operators A_μ that are à priori only given for μ on a halfline $\{\mu \geqq \mu_0\}$. We therefore include a lemma showing how to extend the symbols $a(x,\xi,\mu)$ of such a family to all values of $\mu \in \mathrm{R}$, with a control over the estimates (3.1) that is *independent of μ_0 and a*.

To be more precise, we introduce, for any $\tau_0 \in \mathrm{R}$, the class $S^m_{\varrho,\delta,k}(\mathrm{R}^{2n} \times [\tau_0, \infty[)$ of functions $a(x,\xi,\mu)$ on $\mathrm{R}^n \times \mathrm{R}^n \times [\tau_0,\infty[$ for which the seminorms

(3.16) $$|||a(x,\xi,\tau)|||_{\alpha,\beta,j} = \sup_{x,\xi,\tau} (1+|\xi|+|\tau|)^{-m+(|\alpha|+j)\varrho-|\beta|\delta} |D^\beta_x D^\alpha_\xi D^j_\tau a(x,\xi,\tau)|$$

are finite for $|\alpha| \leqq k$, all β and all j (the mentioned derivatives being continuous). The spaces $S^m_{\varrho,\delta,k}(\mathrm{R}^{2n+1})$ and $S^m_{\varrho,\delta,k}(\mathrm{R}^{2n} \times [\tau_0,\infty[)$ are provided with the topologies defined by the seminorms (3.16), where (x,ξ,τ) runs through R^{2n+1} respectively $\mathrm{R}^{2n} \times [\tau_0,\infty[$.

LEMMA 3.8. *Let $m \in \mathrm{R}$, ϱ and $\delta \in [0,1]$ and k integer $\geqq 0$ be given. For each $\tau_0 \geqq 0$ there exists a linear extension operator*

$$E_{\tau_0}: S^m_{\varrho,\delta,k}(\mathrm{R}^{2n} \times [\tau_0,\infty[) \to S^m_{\varrho,\delta,k}(\mathrm{R}^{2n+1})$$

(sending functions $a(x,\xi,\tau)$ on $\mathrm{R}^{2n} \times [\tau_0,\infty[$ into functions $(E_{\tau_0}a)(x,\xi,\tau)$ on R^{2n+1} that coincide with $a(x,\xi,\tau)$ for $\tau \geqq \tau_0$), such that

(3.17) $$|||E_{\tau_0}a|||_{\alpha,\beta,j} \leqq C(\alpha,\beta,j) \sup_{\alpha' \leqq \alpha, \beta' \leqq \beta, j' \leqq j} |||a|||_{\alpha',\beta',j'},$$

for all $|\alpha| \leqq k$, all β and j, with constants $C(\alpha,\beta,j)$ independent of a and of τ_0. *(In particular, the operators E_{τ_0} are continuous, uniformly in $\tau_0 \geqq 0$.)*

PROOF. For each τ_0 we introduce the auxiliary function

$$(3.18) \qquad \varphi(\tau_0;\, \xi, \tau) = \zeta\left(\frac{\tau}{(1+|\xi|^2+\tau_0^2)^{\frac{1}{2}}}\right)$$

(where $\zeta(\tau)$ was defined before Lemma 3.5), and observe that it satisfies the estimates

$$(3.19) \qquad |D_\xi^\alpha D_\tau^j \varphi(\tau_0;\, \xi, \tau)| \leqq C(\alpha, j)(1+|\xi|+|\tau|)^{-|\alpha|-j}$$

for all α and j, with constants $C(\alpha, j)$ *independent of* τ_0. To see this, we note that the derivatives of φ vanish for $|\tau| \notin [\frac{1}{2}(1+|\xi|^2+\tau_0^2)^{\frac{1}{2}},\ (1+|\xi|^2+\tau_0^2)^{\frac{1}{2}}]$; and on the other hand, when $\tau \in [\frac{1}{2}(1+|\xi|^2+\tau_0^2)^{\frac{1}{2}},\ (1+|\xi|^2+\tau_0^2)^{\frac{1}{2}}]$, then

$$\frac{1}{(1+|\xi|^2+\tau_0^2)^{\frac{1}{2}}} \leqq \frac{1}{\frac{1}{2}(1+|\xi|^2)^{\frac{1}{2}}+\frac{1}{2}|\tau|} \leqq \frac{3}{1+|\xi|+|\tau|}\,.$$

Using this we find, denoting $\max_\tau |\zeta^{(j)}(\tau)| = c_j$,

$$|D_\tau \varphi| = |\zeta'((1+|\xi|^2+\tau_0^2)^{-\frac{1}{2}}\tau)(1+|\xi|^2+\tau_0^2)^{-\frac{1}{2}}| \leqq 3c_1(1+|\xi|+|\tau|)^{-1}\,,$$

$$|D_{\xi_i}\varphi| = |\zeta'((1+|\xi|^2+\tau_0^2)^{-\frac{1}{2}}\tau)\tau\xi_i(1+|\xi|^2+\tau_0^2)^{-3/2}| \leqq 3c_1\,(1+|\xi|+|\tau|)^{-1}\,,$$

and so on, showing (3.19).

Now let $\tau_0 \geqq 0$ be given, and let $a \in S^m_{\varrho,\delta,k}(\mathsf{R}^{2n} \times [\tau_0, \infty[)$. Then it easily follows by use of (3.19) that the product

$$a_1(x, \xi, \tau) = \varphi(\tau_0;\, \xi, \tau) a(x, \xi, \tau)$$

belongs to $S^m_{\varrho,\delta,k}(\mathsf{R}^{2n} \times [\tau_0, \infty[)$, and that

$$(3.20) \qquad |||a_1|||_{\alpha,\beta,j} \leqq C_1(\alpha, \beta, j) \sup_{\alpha' \leqq \alpha,\, \beta' \leqq \beta,\, j' \leqq j} |||a|||_{\alpha',\beta',j'}\,,$$

for all $|\alpha| \leqq k$, all β and j, with constants $C_1(\alpha, \beta, j)$ independent of τ_0 and a.

We shall use the extension method of Seeley [21] to extend a to values of τ less than τ_0. Recall from [21] that there exist sequences $\{g_k\}$, $\{h_k\}$ (for $k = 0, 1, 2, \ldots$) such that:

(i) $h_k \geqq 1$ for all k;
(ii) $\sum_{k=0}^\infty |g_k|\,|h_k|^n < \infty$ for $n = 0, 1, 2, \ldots$;
(iii) $\sum_{k=0}^\infty g_k(-h_k)^n = 1$ for $n = 0, 1, 2, \ldots$; and
(iv) $h_k \to \infty$ for $k \to \infty$.

(One may e.g. take $h_k = 2^k$, or $h_k = k+1$.) When $\tau_0 = 0$, we define the extended symbol by

$$a'(x, \xi, \tau) = \sum_{k=0}^\infty g_k \varphi(0;\, \xi, -h_k\tau) a(x, \xi, -h_k\tau)$$

$$= \sum_{k=0}^\infty g_k a_1(x, \xi, -h_k\tau), \qquad \text{for } \tau \leqq 0\,,$$

it is a kind of reflection in the line $\tau=0$. For general τ_0 we take the analogous reflection in the line $\tau=\tau_0$, given by the formula

$$a'(x,\xi,\tau) = \sum_{k=0}^{\infty} g_k a_1(x,\xi,h_k(\tau_0-\tau)+\tau_0), \quad \text{for } \tau \leqq \tau_0 . \tag{3.21}$$

As shown in [21], this series and its termwise derived series converge uniformly on compact sets, defining a function a' for $\tau \leqq \tau_0$ having as many continuous derivatives as a; these derivatives match the derivatives of a at $\tau=\tau_0$. We define $E_{\tau_0}a$ as the function equal to a' for $\tau \leqq \tau_0$ and equal to a for $\tau \geqq \tau_0$.

For the estimations of the seminorms we observe that by the definition of $\varphi(\tau_0;\xi,\tau)$, the function $a_1(x,\xi,h_k(\tau_0-\tau)+\tau_0)$ (defined for $\tau \leqq \tau_0$) and its derivatives can be $\neq 0$ only when

$$h_k(\tau_0-\tau)+\tau_0 \leqq (1+|\xi|^2+\tau_0^2)^{\frac{1}{2}} . \tag{3.22}$$

Since $(1+|\xi|^2+\tau_0^2)^{\frac{1}{2}} \leqq 1+|\xi|+\tau_0$, (3.22) implies

$$h_k(\tau_0-\tau) \leqq 1+|\xi|$$

and hence

$$\tau_0 \leqq h_k^{-1}(1+|\xi|)+\tau \leqq h_k^{-1}(1+|\xi|)+|\tau| ,$$

so that altogether (3.22) gives

$$\begin{aligned} h_k(\tau_0-\tau)+\tau_0 &\leqq (1+h_k)\tau_0+h_k|\tau| \\ &\leqq c(1+|\xi|)+(2h_k+1)|\tau| , \end{aligned} \tag{3.23}$$

where $c=\max_k(1+h_k^{-1})$. On the other hand, we have when $\tau \in [0,\tau_0]$,

$$h_k(\tau_0-\tau)+\tau_0 \geqq \tau_0 \geqq |\tau| ,$$

and when $\tau \leqq 0$ (so that $|\tau| \leqq \tau_0-\tau$),

$$h_k(\tau_0-\tau)+\tau_0 \geqq h_k(\tau_0-\tau) \geqq c'|\tau| ,$$

where $c'=\min_k h_k$; so altogether

$$h_k(\tau_0-\tau)+\tau_0 \geqq c''|\tau|, \quad \text{when } \tau \leqq \tau_0 , \tag{3.24}$$

with $c''=\min(1,c')$.

Now for $\tau \leqq \tau_0$,

$$|D_x^\beta D_\xi^\alpha D_\tau^j a'(x,\xi,\tau)| \leqq \sum_{k=0}^{\infty} |g_k|\,|h_k|^j |D_x^\beta D_\xi^\alpha D_\sigma^j a_1(x,\xi,\sigma)|_{\sigma=h_k(\tau_0-\tau)+\tau_0} ,$$

where

$$|D_x^\beta D_\xi^\alpha D_\sigma^j a_1(x,\xi,\sigma)|_{\sigma=h_k(\tau_0-\tau)+\tau_0} \leqq |||a_1|||_{\alpha,\beta,j}(1+|\xi|+|h_k(\tau_0-\tau)+\tau_0|)^N ,$$

with $N = m - (|\alpha| + j)\varrho + |\beta|\delta$. When $N > 0$ we use that (3.23) holds on the support of the symbol. Hence

$$|D_x^\beta D_\xi^\alpha D_\tau^j a'(x,\xi,\tau)| \leqq \sum_{k=0}^{\infty} |g_k|\,|h_k|^{j+N} C(N) |||a_1|||_{\alpha,\beta,j} (1+|\xi|+|\tau|)^N$$
$$\leqq C(N,j) |||a_1|||_{\alpha,\beta,j} (1+|\xi|+|\tau|)^N ,$$

by the property (ii) of the sequences $\{g_k\}$, $\{h_k\}$. When $N \leqq 0$, we simply use (3.24), showing that

$$|D_x^\beta D_\beta^\alpha D_\tau^j a'(x,\xi,\tau)| \leqq \sum_{k=0}^{\infty} |g_k|\,|h_k|^j |||a_1|||_{\alpha,\beta,j} (1+|\xi|+c''|\tau|)^N$$
$$\leqq C'(N,j) |||a_1|||_{\alpha,\beta,j} (1+|\xi|+|\tau|)^N ,$$

by the property (ii). In view of (3.20), this altogether shows that the extended symbol $E_{\tau_0} a$ satisfies (3.17), so that E_{τ_0} has the asserted properties.

ADDED IN PROOF. In (3.18), ζ should be replaced by a function $\zeta_1(t)$ that is 1 for $|t| \leqq 1$, 0 for $|t| \geqq 3/2$, with subsequent changes in constants.

4. Local remainder estimates.

Consider the (matrix-formed) symbol $q_\lambda^0(x,\xi)$ defined in Section 2, and recall the convention (2.2): $\lambda = -e^{i\theta l}\mu^l$ where $\mu = |\lambda|^{1/l}$ and $|\theta| < \pi/l$. We shall replace $q^0_{-e^{i\theta l}\mu^l}(x,\xi)$ by a closely related symbol defined for all $x \in \mathbf{R}^n$ and all $\mu \in \mathbf{R}$, by the following definitions: Let K_3 be a compact subset of $\mathring{K}_2$ and let $\eta(x) \in C_0^\infty(\mathbf{R}^n)$ with $\eta(x) = 1$ on K_3 and $\operatorname{supp}\eta \subset K_2$. For each $\theta \in \,]-\pi/l, \pi/l[$, let

$$\mu_\theta = \inf\{\mu \mid e^{i\theta}\mu \in V'_\delta\};$$

clearly $\mu_\theta \geqq 1$, and $\mu_\theta \to \infty$ when θ approaches $-\pi/l$ or π/l. Then set (cf. Lemma 3.8)

$$\tilde{q}_\theta^0(x,\xi,\mu) = E_{\mu_\theta}(\eta(x) q^0_{-e^{i\theta l}\mu^l}(x,\xi)) \tag{4.1}$$

(extended by 0 for $x \notin K_2$). Clearly $\tilde{q}_\theta^0(x,\xi,\mu) = q^0_{-e^{i\theta l}\mu^l}(x,\xi)$ for $x \in K_3$ and $\mu \geqq \mu_\theta$; and because of the uniform estimates in Lemma 3.8 it follows from (2.7) that we have estimates like (3.1):

$$|D_x^\beta D_\xi^\alpha D_\mu^j \tilde{q}_\theta^0(x,\xi,\mu)| \leqq c'_{\alpha,\beta,j} (1+|\xi|+|\mu|)^{-l+\delta-(|\alpha|+j)(1-\delta)+|\beta|\delta} \quad \text{on } \mathbf{R}^{2n+1}, \tag{4.2}$$

for all $|\alpha| \leqq l-1$, all β and j, with constants $c'_{\alpha,\beta,j}$ independent of θ. We express this briefly by saying that

$$\tilde{q}_\theta^0(x,\xi,\mu) \in S^{-l+\delta}_{1-\delta,\delta,l-1}, \quad \textit{uniformly in } \theta . \tag{4.3}$$

By the way, the symbols

$$p^0_\lambda(x,\xi) = p^0(x,\xi)-\lambda I, \quad p^j_\lambda(x,\xi) = p^j(x,\xi) \quad \text{for } j>0\,,$$

can be viewed as functions of $(x,\xi,\mu) \in \mathbf{R}^{2n+1}$ when we replace λ by $-e^{i\theta l}\mu^l$ for each θ; and it is easy to check that for each $j \leqq l-1$,

$$p^j_{-e^{i\theta l}\mu^l}(x,\xi) \in S^{l-j}_{1,0,l-j-1}, \quad \text{uniformly in } \theta\,. \tag{4.4}$$

(We use that l is integer, and $(1+|\xi|)^s \leqq (1+|\xi|+|\mu|)^s$ for $s \geqq 0$.) When $j>0$, we may omit the index λ or $-e^{i\theta l}\mu^l$. Let us finally define the functions $\tilde q^k_\theta(x,\xi,\mu)$ on $\mathbf{R}^{2n+1}$ by successive application of (2.3) (ii)

$$\tilde q^k_\theta = -\tilde q^0_\theta \sum_{\substack{|\alpha|+j+j'=k\\ j'<k}} \frac{1}{\alpha!}\partial^\alpha_\xi p^j_{-e^{i\theta l}\mu^l} D^\alpha_x \tilde q^{j'}_\theta\,; \tag{4.5}$$

then $\tilde q^k_\theta(x,\xi,\mu) = q^k_\lambda(x,\xi)$ for $\lambda = -e^{i\theta l}\mu^l \in V_\delta$ and $x \in K_3$.

LEMMA 4.1. *For each* $k \leqq l-1$,

$$\tilde q^k_\theta \in S^{-l+\delta-k(1-2\delta)}_{1-\delta,\delta,l-k-1}, \quad \textit{uniformly in } \theta\,. \tag{4.6}$$

PROOF. The statement was proved above for $k=0$. For general k, we proceed by induction on k, using the elementary observation: when $a_i \in S^{m_i}_{\varrho_i,\delta_i,k_i}$ for $i=1,2$, then

$$a_1a_2 \in S^{m_1+m_2}_{\min\varrho_i,\max\delta_i,\min k_i}\,.$$

Assume that (4.6) has been proved for all $k \leqq k_0$ (where $k_0 \leqq l-2$). Then by (4.5), $\tilde q^{k_0+1}_\theta \in S^N_{1-\delta,\delta,M}$, where

$$\begin{aligned} N &= \max\{-l+\delta+l-j-|\alpha|-l+\delta-j'(1-2\delta)+|\alpha|\delta \mid |\alpha|+j+j' \\ &\qquad\qquad = k_0+1, j' \leqq k_0\} \\ &= \max\{-l+2\delta-j-j'(1-2\delta)-|\alpha|(1-\delta) \mid |\alpha|+j+j'=k_0+1, j' \leqq k_0\} \\ &= -l+\delta-(k_0+1)(1-2\delta) \end{aligned}$$

(the maximum is obtained for $j'=k_0$, $|\alpha|=1$ and $j=0$), and

$$\begin{aligned} M &= \min\{l-j-|\alpha|-1, l-j'-1 \mid |\alpha|+j+j'=k_0+1,\ j' \leqq k_0\} \\ &= l-(k_0+1)-1 \end{aligned}$$

(the minimum is obtained for $|\alpha|+j=k_0+1$, $j'=0$). This shows the statement for $k=k_0+1$.

For $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}_+}$ we define (cf. (3.3))

(4.7) $$Q_\lambda^k = \mathrm{Op}_n(\tilde{q}_\theta^k(x,\xi,\mu))$$

where $\lambda = -e^{i\theta l}\mu^l$ as usual (so $\mu > 0, |\theta| < \pi/l$); we note that $Q_\lambda^k = \mathrm{Op}(\eta(x)q_\lambda^k(x,\xi))$ when $\lambda \in V_\delta$. (The operators $\mathrm{Op}_n(\tilde{q}_\theta^k(x,\xi,\mu))$ are also defined for $\mu \leqq 0$, but since they are not needed here, we do not introduce a special notation.) Then Proposition 3.7 gives immediately:

LEMMA 4.2. *When* $-l+\delta-k(1-2\delta)\leqq 0$ *and* $l-k-1\geqq[(n+1)/2]+1$, *then* Q_λ^k *is of* μ*-order* $-[l-\delta+k(1-2\delta)]$ (*for* $\mu>0$), *uniformly in* θ.

It is important to observe here that the μ-order *does not improve with increasing k, unless* $\delta<\frac{1}{2}$. Since such a property is needed, we assume from now on:

(4.8) $$\delta = \tfrac{1}{2}-\varepsilon \quad \text{for a given } \varepsilon \in \left]0,\tfrac{1}{2}\right];$$

one is interested in *small values of* ε. With this notation,

(4.9) $$\tilde{q}_\theta^k \in S^{-l+\delta-2k\varepsilon}_{1-\delta,\delta,l-k-1} = S^{-l+\frac{1}{2}-(2k+1)\varepsilon}_{\frac{1}{2}+\varepsilon,\frac{1}{2}-\varepsilon,l-k-1}\,.$$

Define now, for $\lambda \in \mathbf{C}\setminus\overline{\mathbf{R}_+}$

(4.10) $$Q_{\lambda,N} = \sum_{k=0}^{N} Q_\lambda^k\,.$$

Lemma 4.2 applies to $Q_{\lambda,N}$ when $l-N-1\geqq[(n+1)/2]+1$ or, for simplicity, when $N\leqq l-n/2-3$. More restrictions on N will occur below, where we investigate how well $(P-\lambda I)Q_{\lambda,N}$ approximates the identity operator. Since the symbols $\tilde{q}_\theta^k$ are defined in such a way that

$$\sum_{i\leqq N}\ \sum_{|\alpha|+j+k=i} \frac{1}{\alpha!}\partial_\xi^\alpha p_\lambda^j D_x^\alpha \tilde{q}_\theta^k = I \quad \text{for } x\in K_3,\ \lambda\in V_\delta\,,$$

we have that

(4.11) $$(P-\lambda I)Q_{\lambda,N} = \mathrm{Op}(f_\lambda(x,\xi)) + R'_{\lambda,N} + R''_{\lambda,N} + R'''_{\lambda,N}\,,$$

where

(4.12) $$f_\lambda(x,\xi) = I \quad \text{for } x\in K_3 \text{ and } \lambda\in V_\delta;$$

(4.13) $$R'_{\lambda,N} = T_N Q_{\lambda,N};$$

(4.14) $$R''_{\lambda,N} = \sum_{\substack{|\alpha|,j,k\leqq N\\ |\alpha|+j+k\geqq N+1}} \mathrm{Op}_n\left(\frac{1}{\alpha!}\partial_\xi^\alpha p^j(x,\xi) D_x^\alpha \tilde{q}_\theta^k(x,\xi,\mu)\right);$$

(4.15) $$R'''_{\lambda,N} = \sum_{j,k\leqq N}\left[P^j Q_\lambda^k - \mathrm{Op}_n\left(\sum_{|\alpha|\leqq N}\frac{1}{\alpha!}\partial_\xi^\alpha p^j D_x^\alpha \tilde{q}_\theta^k\right)\right].$$

The terms $R'_{\lambda,N}$ and $R'''_{\lambda,N}$ can be avoided when P is a differential operator, by taking $N \geqq l$, as in [17]. The three remainder terms (4.13)–(4.15) will now be estimated separately.

LEMMA 4.3. *When $N \leqq l-n/2-3$, $R'_{\lambda,N}$ is of μ-order $-N$ (for $\mu>0$), uniformly in θ.*

PROOF. As remarked in the beginning of Section 2, T_N is (for $N+1 \leqq l$) continuous from $H^s(\mathbf{R}^n)$ to $H^{s-l+N+1}(\mathbf{R}^n)$ for all real s; the same holds for the adjoint T_N^*. Then for integer $s \geqq l-N-1$ ($\geqq 0$) we have, by elementary inequalities,

$$\begin{aligned}\|T_N u\|_{s-l+N+1,\mu} &\leqq c_s(\|T_N u\|_{s-l+N+1} + \mu^{s-l+N+1}\|T_N u\|_0)\\ &\leqq c_s'(\|u\|_s + \mu^{s-l+N+1}\|u\|_{l-N-1}) \leqq c_s''\|u\|_{s,\mu},\end{aligned}$$

and similarly

$$\|T_N^* v\|_{s-l+N+1,\mu} \leqq c_s'''\|v\|_{s,\mu},$$

for all $u, v \in \mathscr{S}$, uniformly in μ. The last statement gives by duality

$$\|T_N u\|_{t-l+N+1,\mu} \leqq c_t\|u\|_{t,\mu} \tag{4.16}$$

for integer $t \leqq 0$. Then it follows by interpolation that (4.16) holds for all $t \in \mathbf{R}$, uniformly in μ. (In particular, T_N is of μ-order $l-N-1$.) Since $Q_{\lambda,N}$ is of μ-order $-[l-\delta] = -l+1$, uniformly in θ, it follows that the composed operator $R'_{\lambda,N} = T_N Q_{\lambda,N}$ is of μ-order $-N$, uniformly in θ.

LEMMA 4.4. *Let $N \leqq \frac{1}{2}(l-1)$. For each $|\alpha|, j, k \leqq N$,*

$$\partial_\xi^\alpha p^j(x,\xi) D_x^\alpha \tilde{q}_\theta^k(x,\xi,\mu) \in S_{1-\delta,\delta,l-\max(|\alpha|+j,k)-1}^{\delta-j-2k\varepsilon-|\alpha|(1-\delta)}. \tag{4.17}$$

Moreover,

$$\sum_{\substack{|\alpha|,j,k \leqq N\\ |\alpha|+j+k \geqq N+1}} \frac{1}{\alpha!}\partial_\xi^\alpha p^j D_x^\alpha \tilde{q}_\theta^k \in S_{1-\delta,\delta,l-2N-1}^{-2(N+1)\varepsilon}; \tag{4.18}$$

so that the corresponding operator family $R''_{\lambda,N}$ is of μ-order $-[2(N+1)\varepsilon]$ (for $\mu>0$), uniformly in θ, when $N \leqq \frac{1}{2}(l-n/2-3)$.

PROOF. (4.17) follows immediately from (4.4) and (4.9). Then the sum in (4.18) is of order r, where

$$\begin{aligned} r &= \max\{\delta - j - 2k\varepsilon - |\alpha|(1-\delta) \mid |\alpha|, j, k \leqq N, |\alpha|+j+k \geqq N+1\}\\ &= \delta - 2N\varepsilon - (1-\delta) = -2(N+1)\varepsilon,\end{aligned}$$

since $1-\delta=\frac{1}{2}+\varepsilon\geqq 2\varepsilon$. By Proposition 3.7, (4.18) defines a family of operators of μ-order $-[2(N+1)\varepsilon]$, when

$$l-2N-1 \geqq \frac{n}{2}+2, \text{ that is, } N \leqq \frac{1}{2}\left(l-\frac{n}{2}-3\right).$$

LEMMA 4.5. *Let $N\leqq\frac{1}{2}(l-2)$. Then*

(4.19)
$$R'''_{\lambda,N} = \sum_{j,k\leqq N}\left(P^jQ^k_\lambda-\mathrm{Op}_n\left(\sum_{|\alpha|\leqq N}\frac{1}{\alpha!}\partial^\alpha_\xi p^j D^\alpha_x\tilde{q}^k_\theta\right)\right) = \mathrm{Op}_n\left(r_\theta(x,\xi,\mu)\right),$$

where

$$r_\theta(x,\xi,\mu)\in S^{-N-1+(n+l+2)\delta}_{1-\delta,\delta,l-2N-2}. \tag{4.20}$$

Hence $R'''_{\lambda,N}$ is of μ-order $-[N+1-(n+l+2)\delta]$ (for $\mu>0$) uniformly in θ, when

$$(n+l+2)\delta-1 \leqq N \leqq \frac{l}{2}-\frac{n}{4}-2; \tag{4.21}$$

there exist integers N satisfying (4.21), *when*

$$l \geqq \frac{1}{\varepsilon}\left(\frac{3}{4}n+3\right)-n-2. \tag{4.22}$$

PROOF. For each j and k,

$$\begin{aligned}(P^jQ^k_\lambda u)(x) &= (2\pi)^{-2n}\int e^{i(x\cdot\xi-z\cdot\xi+z\cdot\eta)}p^j(x,\xi)\tilde{q}^k_\theta(z,\eta,\mu)\hat{u}(\eta)\,d\eta\,dz\,d\xi\\ &= (2\pi)^{-n}\int e^{ix\cdot\eta}s_{j,k,\theta}(x,\eta,\mu)\hat{u}(\eta)d\eta,\end{aligned} \tag{4.23}$$

where

$$s_{j,k,\theta}(x,\eta,\mu) = (2\pi)^{-n}\int e^{i(x-z)\cdot(\xi-\eta)}p^j(x,\xi)\tilde{q}^k_\theta(z,\eta,\mu)\,dz\,d\xi \tag{4.24}$$

(the integral is seen to converge by applying (3.6) to $\tilde{q}^k_\theta$). Inserting

$$\begin{aligned}p^j(x,\xi) &= \sum_{|\alpha|\leqq N}\frac{1}{\alpha!}\partial^\alpha_\xi p^j(x,\eta)(\xi-\eta)^\alpha\\ &\quad+\sum_{|\alpha|=N+1}\frac{N+1}{\alpha!}(\xi-\eta)^\alpha\int_0^1(1-h)^N\partial^\alpha_\xi p^j(x,\eta+h(\xi-\eta))\,dh\end{aligned} \tag{4.25}$$

one finds (using the Fourier transform and a substitution $\xi-\eta=\zeta$)

$$(4.26)\qquad s_{j,k,\theta}(x,\eta,\mu) = \sum_{|\alpha|\leqq N} \frac{1}{\alpha!}\partial_\xi^\alpha p^j(x,\eta) D_x^\alpha \tilde{q}_\theta^k(x,\eta,\mu) + r_{j,k,\theta}(x,\eta,\mu)\,,$$

where

$$(4.27)\qquad r_{j,k,\theta}(x,\eta,\mu) = (2\pi)^{-n} \sum_{|\alpha|=N+1} \frac{N+1}{\alpha!} \int e^{ix\cdot\zeta}\zeta^\alpha \int_0^1 (1-h)^N \partial_\xi^\alpha p^j(x,\eta+h\zeta)\,dh\hat{\tilde{q}}_\theta^k(\zeta,\eta,\mu)\,d\zeta\,.$$

By (4.9) and (3.6),

$$\begin{aligned}|r_{j,k,\theta}(x,\eta,\mu)| &\leqq c_M \int |\zeta|^{N+1}(1+|\eta|+|\zeta|)^{l-j-N-1}(1+|\zeta|)^{-M}\cdot \\ &\qquad (1+|\eta|+|\mu|)^{-l+\delta-2k\varepsilon+M\delta}\,d\zeta \\ &\leqq c'_M(1+|\eta|+|\mu|)^{-j-2k\varepsilon-N-1+(M+1)\delta}\end{aligned}$$

for $M>n+l-j$. In particular, for $M=n+l-j+1$,

$$\begin{aligned}|r_{j,k,\theta}(x,\eta,\mu)| &\leqq c(1+|\eta|+|\mu|)^{-j-2k\varepsilon-N-1+(n+l-j+2)\delta} \\ &= c(1+|\eta|+|\mu|)^{-j(1+\delta)-2k\varepsilon-N-1+(n+l+2)\delta}\,.\end{aligned}$$

The derivatives are estimated similarly, using that p^j vanishes for x outside K_1, and we find altogether

$$(4.28)\qquad r_{j,k,\theta}(x,\eta,\mu) \in S_{1-\delta,\delta,l-j-N-2}^{-j(1+\delta)-2k\varepsilon-N-1+(n+l+2)\delta}\,.$$

Then $r_\theta(x,\eta,\mu)=\sum_{j,k\leqq N} r_{j,k,\theta}(x,\eta,\mu)$ satisfies

$$r_\theta(x,\eta,\mu) \in S_{1-\delta,\delta,l-2N-2}^{-N-1+(n+l+2)\delta}\,,$$

so that, when $l-2N-2\geqq n/2+2$ and $N\geqq(n+l+2)\delta-1$, $R'''_{N,\lambda}=\operatorname{Op}(r_\theta(x,\eta,\mu))$ is of μ-order $-[N+1-(n+l+2)\delta]$, by Proposition 3.7. The set of integers N satisfying these requirements is nonempty, when

$$\frac{l}{2}-\frac{n}{4}-2 \geqq (n+l+2)(\tfrac{1}{2}-\varepsilon)\,,$$

i.e., when $l\geqq 1/\varepsilon(3n/4+3)-n-2$.

REMARK 4.6. Our estimate of the remainder in (4.19) is not nearly as strong as the estimates in Hörmander [12]; this comes from the fact that we do not dispose of higher ξ-derivatives (in fact, an application of [12, Theorem 2.6] would require more than $l+n+1$ derivatives in ξ, where our symbols have only up to l well-behaved derivatives).

A common feature of Lemmas 4.4 and 4.5 is that the smaller ε is, the larger l has to be (inverse proportionally to ε), in order for N to exist so that $R''_{\lambda,N}$ respectively $R'''_{\lambda,N}$ has a given negative order. Let us find conditions on l and N for which the remainders are of μ-order $-r$.

THEOREM 4.7. *Let $\varepsilon > 0$ be given, and let r integer $\geqq 0$. Then*

$$(P - \lambda I)Q_{\lambda,N} = \text{Op}\,(f_\lambda(x,\xi)) + R_{\lambda,N}\,, \tag{4.29}$$

where $f_\lambda(x,\xi) = I$ for $x \in K_3$ and $\lambda \in V_\delta$, and $R_{\lambda,N}$ is of μ-order $-r$ (for $\mu > 0$), uniformly in θ, when $N = [l/2 - n/4 - 2]$ with $l \geqq 1/\varepsilon(\frac{3}{4}n + 3 + r) - n - 2$, or simply

$$l \geqq \varepsilon^{-1}(n + 3 + r)\,. \tag{4.30}$$

PROOF. By (4.11)–(4.15)

$$R_{\lambda,N} = R'_{\lambda,N} + R''_{\lambda,N} + R'''_{\lambda,N}\,,$$

where the terms are estimated in Lemmas 4.3–4.5. These lemmas require

$$N \leqq \min\left\{l - \frac{n}{2} - 3, \frac{1}{2}\left(l - \frac{n}{2} - 3\right), \frac{1}{2}\left(l - \frac{n}{2} - 4\right)\right\} = \frac{l}{2} - \frac{n}{4} - 2\,, \tag{4.31}$$

so we take $N = [l/2 - n/4 - 2]$ in the following. Then $R_{\lambda,N}$ is of μ-order $-r$, if (cf. Lemmas 4.3–4.5)

$$N \geqq \max\left\{r, \frac{r}{2\varepsilon} - 1, (n + l + 2)\left(\frac{1}{2} - \varepsilon\right) + r - 1\right\}. \tag{4.32}$$

A computation shows that this holds, when

$$\begin{aligned} l &\geqq \max\left\{2r + \frac{n}{2} + 4, \frac{r}{\varepsilon} + \frac{n}{2} + 4, \frac{1}{\varepsilon}\left(\frac{3}{4}n + 3 + r\right) - n - 2\right\} \\ &= \frac{1}{\varepsilon}\left(\frac{3}{4}n + 3 + r\right) - n - 2; \end{aligned}$$

the latter expression is $\leqq \varepsilon^{-1}(n + 3 + r)$.

Recall that the present calculations are concerned with a localized situation. In order to pass to the global statements in the next section, we need to show that operators of the form $\psi Q_\lambda^k \varphi$ with $\psi\varphi = 0$ are of relatively low order. An easy variant of the proof of Lemma 4.5 gives

LEMMA 4.8. *Let $l \geqq n/2 + 4$, let $N = [l/2 - n/4 - 2]$, and let $k \leqq N$. If ψ and $\varphi \in C_0^\infty(\mathbf{R}^n)$ with*

$$\psi(x) = 0 \quad \textit{for } x \in \operatorname{supp}\varphi\,, \tag{4.33}$$

then $\psi Q_\lambda^k \varphi$ is of μ-order $-[\frac{5}{4}l - n/8 - 1]$.

PROOF. One finds, like in (4.23)–(4.27)

$$(\psi Q_\lambda^k \varphi u)(x) = (2\pi)^{-n} \int e^{ix\cdot\eta} s_\theta^k(x,\eta,\mu)\hat{u}(\eta)\,d\eta ,$$

where (for $M+1 \leqq l-k$)

$$s_\theta^k(x,\eta,\mu) = \psi(x) \sum_{|\alpha|\leqq M} \frac{1}{\alpha!}\partial_\xi^\alpha \tilde{q}_\theta^k(x,\xi,\mu) D_x^\alpha \varphi(x) + r_{\theta,M}^k(x,\eta,\mu);$$

here the sum over $|\alpha|\leqq M$ is 0 because of (4.33), and

$$r_{\theta,M}^k(x,\eta,\mu) = (2\pi)^{-n}\psi(x) \sum_{|\alpha|=M+1} \frac{M+1}{\alpha!} \int e^{ix\cdot\zeta}\zeta^\alpha \int_0^1 (1-h)^M \partial_\xi^\alpha \tilde{q}_\theta^k(x,\eta+h\zeta)\,dh\,\hat{\varphi}(\zeta)\,d\zeta .$$

It follows from (4.9) that

$$r_{\theta,M}^k(x,\eta,\mu) \in S_{1-\delta,\delta,l-k-M-2}^{-l+\delta-2k\varepsilon-(M+1)(1-\delta)} \subset S_{1-\delta,\delta,l-N-M-2}^{-l-M/2} .$$

Taking $M=[l-N-2-n/2-2]=N$ or $N+1$, we find by Proposition 3.7, that $\psi Q_\lambda^k \varphi = \mathrm{Op}_n(r_{\theta,M}^k)$ is of μ-order $-[\frac{5}{4}l-n/8-1]$.

REMARK 4.9. When P is a *differential* operator, the symbol $\tilde{q}_\theta^0(x,\xi,\mu)$ is in $S_{1-\delta,\delta,j}^{-l+\delta}$ for *all* $j\geqq 0$, and similarly the symbols $\tilde{q}_\theta^k(x,\xi,\mu)$ are defined as elements of $S_{1-\delta,\delta,j}^{-l+\delta-2k\varepsilon}$ for all $k\geqq 0$, all $j\geqq 0$. Then no upper bound on N (as in (4.31)) is imposed, so that for P of any order l (integer >0), we obtain a remainder $R_{\lambda,N}$ of μ-order $-r$ by taking N satisfying (4.32), which can here be replaced by

$$N \geqq \max\left\{\frac{r}{2\varepsilon}-1, \frac{n}{2}+\frac{l}{2}+r\right\}.$$

In the differential operator case one may in fact conveniently take $N\geqq l$, whereby $R'_{\lambda,N}$ and $R'''_{\lambda,N}$ will be zero, so that the remainder $R_{\lambda,N}$ equals $R''_{\lambda,N}$, which is of μ-order $-[2(N+1)\varepsilon]$; and our Sobolev estimates are valid without limitations on l. The operators Q_λ^k are of μ-order $-[l-\delta+2k\varepsilon]$ for all $k\geqq 0$. (Hence in the development (5.6) below, one can take N arbitrarily large, obtaining, for $N\geqq l$, that $S'_{\lambda,N}$ is of μ-order $-[l-\delta+2(N+1)\varepsilon]$.)

5. Global constructions.

Recall that P was originally given as an operator in a complex q-dimensional vector bundle E over a compact n-dimensional manifold Σ, and Sections 2–4 refer to an operator defined from P in a local chart. We shall now define an approximate resolvent of P on Σ. Let $\varkappa_i$: $E|_{X_i} \to \Omega_i \times \mathbf{C}^q$ ($\Omega_i \subset \mathbf{R}^n$) be a family of charts so that $\bigcup_{i\leqq i_0} E|_{X_i} = E$; let $\{\varphi_i\}_{i\leqq i_0}$ be a partition of unity subordinate to the X_i (that is, $\varphi_i \in C_0^\infty(X_i)$, $\sum_{i \leqq i_0} \varphi_i = 1$ on Σ) and let $\{\psi_i\}_{i \leqq i_0}$, $\{\sigma_i\}_{i \leqq i_0}$ and $\{\varrho_i\}_{i \leqq i_0}$ be three other families of C^∞ functions on Σ with ψ_i, σ_i and $\varrho_i \in C_0^\infty(X_i)$, $\psi_i = 1$ on $\operatorname{supp} \varphi_i$, $\sigma_i = 1$ on $\operatorname{supp} \psi_i$ and $\varrho_i = 1$ on an open set ω_i containing $\operatorname{supp} \sigma_i$, for each i. We use the same notations $(\varphi_i, \psi_i, \sigma_i, \varrho_i, \omega_i)$ for the functions and sets carried over to Ω_i. For each i, P defines an operator $P_{\varkappa_i}$ on the q-tuples of functions on Ω_i, by the formula

$$\varkappa_i^* \mathrm{P}_{\varkappa_i} u = P(\varkappa_i^* u) \quad \text{for } u \in C_0^\infty(\Omega_i, \mathbf{C}^q) \tag{5.1}$$

($\varkappa_i^*$ denoting the pull-back of sections in $\Omega_i \times \mathbf{C}^q$ to $E|_{X_i}$, defined from $\varkappa_i$); $P_{\varkappa_i}$ is of the form (1.1).

For each i, we apply definition (5.1) to $\varrho_i P \varrho_i$, which gives a $q \times q$-matrix formed pseudo-differential operator $(\varrho_i P \varrho_i)_{\varkappa_i}$ on $\mathbf{R}^n$ with a symbol

$$p_i(x, \xi) \sim \sum_{j=0}^{\infty} p_i^j(x, \xi),$$

satisfying the hypotheses of Section 2 with $K_1 = \operatorname{supp} \varrho_i$ and $K_2 = \overline{\omega_i}$. Letting $K_3 = \operatorname{supp} \sigma_i$, we then construct (for some $N \leqq l-1$)

$$Q_{\lambda, N, i} = Q_{\lambda, i}^0 + \ldots + Q_{\lambda, i}^N \tag{5.2}$$

from the symbols p_i^j, as described in Sections 2–4. Finally, set

$$Q_{\lambda, N} = \sum_{i \leqq i_0} (\psi_i Q_{\lambda, N, i} \varphi_i)_{\varkappa_i^{-1}} \tag{5.3}$$

(where each $(\psi_i Q_{\lambda, N, i} \varphi_i)_{\varkappa_i^{-1}}$ is "extended by 0" outside X_i).

Defining $H^{s,\mu}(E)$ (for each $s \in \mathbf{R}$) as the Sobolev space of sections $H^s(E)$ provided with a norm $\|u\|_{s,\mu}$ obtained from the norms in $H^{s,\mu}(\mathbf{R}^n)^q$ by use of local charts, we say that a family of operators A_μ on the sections of E is of μ-order $-r$ when the estimates (3.13)–(3.14) hold for $u \in H^{-s}(E)$ respectively $v \in H^{s-r}(E)$, $s \leqq r$, uniformly in μ ($\mu \in \mathbf{R}$ or a subset of $\mathbf{R}$).

Proposition 5.1. *Let $\varepsilon \in \,]0, 1/4]$ be given; let l be an integer $\geqq \varepsilon^{-1}(n+5)$ and let $N = [l/2 - n/4 - 2]$. Then*

$$(P - \lambda I) Q_{\lambda, N} = I - S_{\lambda, N}, \tag{5.4}$$

where $S_{\lambda, N}$ is of μ-order -2, uniformly for $\lambda \in V_\delta$ ($\lambda = -(e^{i\theta}\mu)^l$).

PROOF. Using that $\varrho_i\psi_i=\psi_i$, we have

$$\begin{aligned}(P-\lambda I)Q_{\lambda,N} &= \sum_{i\leqq i_0}(P-\lambda I)(\psi_i Q_{\lambda,N,i}\varphi_i)_{\varkappa_i^{-1}}\\ &= \sum_{i\leqq i_0}[\sigma_i(P-\lambda I)\varrho_i(\psi_i Q_{\lambda,N,i}\varphi_i)_{\varkappa_i^{-1}}+\\ &\quad +(1-\sigma_i)(P-\lambda I)\varrho_i(\psi_i Q_{\lambda,N,i}\varphi_i)_{\varkappa_i^{-1}}]\,.\end{aligned}$$

In the second term, $(1-\sigma_i)(P-\lambda I)\psi_i=(1-\sigma_i)P\psi_i$ is of order $-\infty$ (since $(1-\sigma_i)\psi_i=0$) and hence continuous in $H^{s,\mu}(E)$ for each s, uniformly in μ; then by Lemma 4.2, the second term is of μ-order $-l+1$ for each i.

For the first term we have, since $\sigma_i\varrho_i=\sigma_i$,

$$\begin{aligned}&(\sigma_i(P-\lambda I)\varrho_i)_{\varkappa_i}\psi_i Q_{\lambda,N,i}\varphi_i\\ &= \sigma_i(\varrho_i(P-\lambda I)\varrho_i)_{\varkappa_i}Q_{\lambda,N,i}\varphi_i+\sigma_i(P-\lambda I)_{\varkappa_i}\varrho_i(\psi_i-1)Q_{\lambda,N,i}\varphi_i\,.\end{aligned}$$

By Lemma 4.8, the second term here is of μ-order

$$l-\left[\frac{5}{4}l-\frac{n}{8}-1\right] = -\left[\frac{l}{4}-\frac{n}{8}-1\right] \leqq -\left[n+5-\frac{n}{8}-1\right] \leqq -\left[\frac{7}{8}n+4\right],$$

since $\varepsilon\leqq\frac{1}{4}$. (We use that P satisfies (4.16) with $N+1$ replaced by 0, for all t.) To the first term we can apply Theorem 4.7, which gives

$$\begin{aligned}\sigma_i(\varrho_i(P-\lambda I)\varrho_i)_{\varkappa_i}Q_{\lambda,N,i}\varphi_i &= \sigma_i\,\mathrm{Op}\,(f_{\lambda,i}(x,\xi))\varphi_i+\sigma_i R_{\lambda,N,i}\varphi_i\\ &= \varphi_i+\sigma_i R_{\lambda,N,i}\varphi_i\end{aligned}$$

(since $f_{\lambda,i}(x,\xi)=I$ for $x\in\operatorname{supp}\sigma_i$), with $\sigma_i R_{\lambda,N,i}\varphi_i$ of μ-order -2. Altogether,

$$\begin{aligned}(P-\lambda I)Q_{\lambda,N} &= \sum_{i\leqq i_0}\varphi_i+[\text{terms of }\mu\text{-order }\leqq -2]\\ &= I-S_{\lambda,N}\end{aligned}$$

as asserted.

In the next theorem, we shall use the following immediate consequence of Agmon's theorem [2, Theorem 3.1]:

LEMMA 5.2. *When T_μ and T_μ^* are bounded linear operators from $L^2(\mathbf{R}^n)$ to $H^{r,\mu}(\mathbf{R}^n)$ for some $r>n$, uniformly for μ in an interval I, then T_μ is an integral operator on $\mathbf{R}^n$ with a continuous kernel $K(T_\mu)(x,y)$ satisfying*

$$|K(T_\mu)(x,y)| \leqq c|\mu|^{-r+n} \quad \textit{for all } \mu\in I\,, \tag{5.5}$$

for some constant c. This holds in particular if T_μ is of μ-order $-r$ (for $\mu\in I$).

Concerning operators on E we remark that when E is trivial, $E=\Sigma\times\mathbf{C}^q$, then an operator T on the sections of E for which T and T^* are continuous from $L^2(E)$ to $H^r(E)$ with $r>n$, has a well-defined kernel $K(T)(x,y)$ that is a $q\times q$-matrix valued continuous function on $\Sigma\times\Sigma$; Lemma 5.2 extends immediately to such operators. For general E, T has a kernel in every local chart, and $K(T)(x,x)$ has a meaning on Σ (as a continuous section in Hom (E,E)).

THEOREM 5.3. *Let $\varepsilon>0$ be given with $\varepsilon\leqq 1/4$, let l be an integer $>\varepsilon^{-1}(n+5)$ and let $N=[l/2-n/4-2]$. There exists $\lambda_0>0$ so that for $\lambda\in V_\delta$ with $|\lambda|\geqq\lambda_0$ (cf. (2.4), $\delta=\frac{1}{2}-\varepsilon$), the resolvent $Q_\lambda=(P-\lambda I)^{-1}$ is of the form (cf. (5.4))*

$$(5.6)\qquad Q_\lambda = Q_\lambda^0+\ldots+Q_\lambda^N+S'_{\lambda,N},\quad \textit{with}$$

$$Q_\lambda^0+\ldots+Q_\lambda^N = Q_{\lambda,N},\quad \textit{and}\quad S'_{\lambda,N} = Q_{\lambda,N}\sum_{r=1}^{\infty}(S_{\lambda,N})^r;$$

here the Q_λ^k are pseudo-differential operators of order $-l-k$ and of μ-order $-[l-\frac{1}{2}+(2k+1)\varepsilon]$, and $S'_{\lambda,N}$ is a pseudo-differential operator of order $-l-N-1$ and of μ-order $-l-1$ (uniformly in Arg λ, *with $\mu=|\lambda|^{1/l}$). The kernels of the operators satisfy, in each local chart $U\times\mathbf{C}^q$,*

$$(5.7)\quad |x-y|^j|K(Q_\lambda^k)(x,y)| \leqq c_{j,k}|\lambda|^{-1+l^{-1}(n+\frac{1}{2}-(2k+1)\varepsilon-j(\frac{1}{2}+\varepsilon))}\quad \textit{for } j\leqq l-k\,,$$

uniformly on compact subsets of U,

$$(5.8)\qquad K(Q_\lambda^k)(x,x) = c_k(x)(-\lambda)^{-1+(n-k)/l}\,,$$

for certain C^∞ $q\times q$-matrix valued functions $c_k(x)$; and

$$(5.9)\qquad |K(S'_{\lambda,N})(x,y)| \leqq c|\lambda|^{-1+(n-1)/l}\,.$$

In particular, $c_0(x)$ is defined on Σ by

$$(5.10)\qquad c_0(x) = (2\pi)^{-n}\int_{T_x^*}(p^0(x,\xi)+I)^{-1}\,d\xi\,.$$

PROOF. Since $S_{\lambda,N}$ is of μ-order -2, there exists λ_0 so that the operator norm in $L^2(E)$ of $S_{\lambda,N}$ is $\leqq\frac{1}{2}$ for $|\lambda|\geqq\lambda_0$. Then the series of iterates $\sum_{r=0}^\infty(S_{\lambda,N})^r$ converges uniformly, and

$$(P-\lambda I)Q_{\lambda,N}\sum_{r=0}^{\infty}(S_{\lambda,N})^r = \sum_{r=0}^{\infty}(S_{\lambda,N})^r-\sum_{r=1}^{\infty}(S_{\lambda,N})^r = I$$

so that

$$(P-\lambda I)^{-1} = Q_{\lambda,N}\sum_{r=0}^{\infty}(S_{\lambda,N})^r = Q_{\lambda,N}+Q_{\lambda,N}\sum_{r=1}^{\infty}(S_{\lambda,N})^r\,.$$

Defining $S'_{\lambda,N}=Q_\lambda-Q_{\lambda,N}$ and setting

$$Q^k_\lambda = \sum_{i\leqq i_0} (\psi_i Q^k_{\lambda,i}\varphi_i)_{\kappa_i^{-1}} \quad \text{for } k\leqq N \tag{5.11}$$

(cf. (5.2)), we find (5.6). The statements on the orders follow from Lemma 4.2 and Proposition 5.1 (when we use the elementary fact that if A^1_μ and A^2_μ are of μ-orders $-r_1$ respectively $-r_2$ (r_1 and $r_2\geqq 0$), then $A^1_\mu A^2_\mu$ is of μ-order $-r_1-r_2$). For the kernels we have in local coordinates, by (4.9),

$$\begin{aligned}|(x-y)^\alpha K(Q^k_{\lambda,i})(x,y)| &= (2\pi)^{-n}\left|\int e^{i(x-y)\cdot\xi}D^\alpha_\xi q^k_{\lambda,i}(x,\xi)\,d\xi\right|\\ &\leqq c\int (1+|\xi|+\mu)^{-l+\frac12-(2k+1)\varepsilon-|\alpha|(\frac12+\varepsilon)}\,d\xi\\ &\leqq c_1(1+\mu)^{n-l+\frac12-(2k+1)\varepsilon-|\alpha|(\frac12+\varepsilon)}\end{aligned}$$

for $|\alpha|\leqq l-k$, this implies (5.7). For $x=y$, we have in particular:

$$K(Q^k_{\lambda,i})(x,x) = (2\pi)^{-n}\int q^k_{\lambda,i}(x,\xi)\,d\xi\ ,$$

where $q^k_{\lambda,i}(x,\xi)$ is homogeneous in $(\xi,(-\lambda)^{1/l})$ of degree $-l-k$, and analytic in $\lambda\in\mathbf{C}\setminus\overline{\mathbf{R}_+}$. Using the homogeneity for $\lambda\in\mathbf{R}_-$ and continuing analytically, we find that

$$K(Q^k_\lambda)(x,x) = (-\lambda)^{-1+(n-k)/l}(2\pi)^{-n}\int q^k_{-1,i}(x,\eta)\,d\eta\ .$$

By (5.11), this leads to (5.8). In particular, the formula

$$K(\psi_i Q^0_{\lambda,i}\varphi_i)(x,x) = (-\lambda)^{-1+n/l}(2\pi)^{-n}\varphi_i(x)\int (p^0_i(x,\eta)+I)^{-1}\,d\eta\ ,$$

in local coordinates, carries over to Σ where it gives (5.10) after a summation over i ($d\xi$ denoting the Lebesgue measure in T^*_x induced by dx).

Finally, Lemma 5.2 applied to $S'_{\lambda,N}$ gives (5.9).

Further applications of Theorem 4.7 show that for $r>0$, $S'_{\lambda,N}$ is of μ-order $-l-r-1$ when $l\geqq\varepsilon^{-1}(n+r+5)$.

It is now an easy matter to deduce estimates of the spectral function of P and the eigenvalue distribution, by methods like those used in Agmon–Kannai [3] and Beals [4]. The spectral function is the kernel $e(t;\,x,y)$ (which is in fact globally defined) of the projector $\mathscr{E}_t$ in the spectral resolution $\{\mathscr{E}_t\}_{t\in\mathbf{R}}$ of P; we note that since $\mathscr{E}_t u=\sum_{\lambda_j\leqq t}(u,u_j)u_j$, (5.12) $$\operatorname{tr} e(t; x, x) = \sum_{\lambda_j \leqq t} \langle u_j(x), u_j(x) \rangle \quad \text{(scalar product in } E_x) ,$$

defined from the normalized eigenfunctions u_j belonging to the eigenvalues $\lambda_j \leqq t$; and e is C^∞ in x and y for each t. We shall also remove the various hypotheses on l.

THEOREM 5.4. *Let $\varepsilon > 0$ be given, and let P be as defined in Section 1, of order $l > 0$. The spectral function of P satisfies*

(5.13) $$\operatorname{tr} e(t; x, x) = c_P(x) t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}), \quad \text{for } t \to \infty ,$$

uniformly in x, where

(5.14) $$c_P(x) = \frac{1}{n(2\pi)^n} \int_{\xi \in S_x} \operatorname{tr}[p^0(x, \xi)^{-n/l}]\, d\omega ,$$

and the number of eigenvalues less than t satisfies

(5.15) $$N(t; P) = c_P t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) \quad \text{for } t \to \infty ,$$

where

(5.16) $$c_P = \int_\Sigma c_P(x)\, dx .$$

PROOF. Choose a number $r \in \mathbf{R}_+$ for which $l' = rl$ is an integer $\geqq \varepsilon^{-1}(n+5)$, and let $(P)^r$ be the rth power of P defined by the calculus of Seeley [19]; then Theorem 5.3 applies to $(P)^r$. Since

$$((P)^r - \lambda I)^{-1} = \int_0^\infty (t^r - \lambda)^{-1}\, d\mathscr{E}_t ,$$

$$\operatorname{tr} K(((P)^r - \lambda I)^{-1})(x, x) = \int_0^\infty (t^r - \lambda)^{-1} d \operatorname{tr} e(t; x, x) .$$

Here $\operatorname{tr} e(t; x, x)$ is a nondecreasing function of $t \in \mathbf{R}$, so we may apply a tauberian theorem of Malliavin, cf. Pleijel [18], Beals [4]: The estimate

(5.17) $$\int_0^\infty (t^r - \lambda)^{-1}\, d\sigma(t) = c_0(-\lambda)^\alpha + O(|\lambda|^\beta)$$

for $|\lambda| \to \infty$ with $\operatorname{Re} \lambda \geqq 0$ and $|\operatorname{Im} \lambda| = |\lambda|^\gamma$, where $-1 < \beta < \alpha < 0 < \gamma < 1$, and $\sigma(t)$ is nondecreasing in t; implies

(5.18) $$\sigma(t) = c_0 \frac{\sin \pi(\alpha+1)}{\pi(\alpha+1)} t^{(\alpha+1)r} + O(t^{(\alpha+\gamma)r}) + O(t^{(\beta+1)r})$$

as $t \to \infty$. We have by Theorem 5.3 that (5.17) is valid with $\sigma(t) = \operatorname{tr} e(x; t, t)$, $\alpha = -1 + n/rl$ and $\beta = -1 + (n-1)/rl$ (cf. (5.8)–(5.9)), $\gamma = 1 - (\frac{1}{2} - \varepsilon)/rl$ and

$$c_0 = (2\pi)^{-n} \int_{T_x^*} \operatorname{tr}\left[(p^0(x,\xi)^r + I)^{-1}\right] d\xi .$$

Then, by (5.18)

$$\begin{aligned} \operatorname{tr} e(t; x, x) &= c_0 \frac{\sin(\pi n/rl)}{\pi n/rl} t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) + O(t^{(n-1)/l}) , \\ &= c_P(x) t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) , \end{aligned}$$

where $c_P(x) = c_0(\sin \pi n/rl) rl/\pi n$. Choosing a norm $|\xi|$ in the fibres of $T^*(\Sigma)$ and a measure $d\omega$ on the unit sphere S_x in each fibre so that

$$\int_{T_x^*} f(\xi)\, d\xi = \int_0^\infty \int_{S_x} f(\xi)|\xi|^{n-1}\, d\omega\, d|\xi| ,$$

we find (5.14) by using the homogeneity of $p^0(x,\xi)$ together with a diagonalization. ((5.14) can be given an invariant meaning, cf. Hörmander [14, p. 216].) Finally, (5.15) and (5.16) follow from the fact that $N(t; P) = \int_\Sigma \operatorname{tr} e(t; x, x)\, dx$, cf. (5.12).

One advantage of having a result for pseudo-differential operators is that it permits manipulations with differential operators, like taking fractional powers, etc. We can for instance easily obtain

COROLLARY 5.5. *Let $\varepsilon > 0$ be given, and let P be an invertible selfadjoint classical pseudo-differential operator in E of order $l > 0$ (not necessarily strongly elliptic). Then the numbers $N^+(t; P)$ and $N^-(t; P)$ of eigenvalues of P in the interval $[0, t]$ respectively $[-t, 0]$ satisfy*

$$N^\pm(t; P) = c_P^\pm t^{n/l} + O(t^{(n-\frac{1}{2}+\varepsilon)/l}) \quad \text{for } t \to \infty , \tag{5.19}$$

here

$$c_P^\pm = \frac{1}{n(2\pi)^n} \int_\Sigma \int_{S_x} \sum |\lambda_j^\pm(p^0(x,\xi))|^{-n/l}\, d\omega\, dx , \tag{5.20}$$

where the sum is over the positive, respectively negative, eigenvalues of $p^0(x, \xi)$.

PROOF. The result follows from a direct application of the method of proof of [10′, Proposition 8.9] (one applies Theorem 5.4 above to $P_{\pm a} = (P^2)^{\frac{1}{2}} \pm aP$, for some $a \in]0, 1[$).

This result is new also for differential operators.

6. Further developments.

One of the reasons for working with Sobolev estimates in the above theory is that this is very well suited for a treatment of pseudo-differential *boundary value problems* in the framework developed in [10], [11]. On the other hand it is possible that some estimates for the Dirichlet problem (for pseudo-differential operators of even order on an open subset Ω of Σ with smooth boundary Γ, satisfying the transmission property with respect to Γ) can be obtained more directly on the basis of the above estimates for $(P-\lambda I)^{-1}$ on Σ, by a generalization of differential operator methods; other boundary problems are easily included, cf. [11]. We intend to take up this subject elsewhere, and conclude the present work with a generalization of Theorem 5.4 to Douglis–Nirenberg elliptic systems.

We first show the following extension of a theorem of Ky Fan [9]:

PROPOSITION 6.1. *Let A and B be compact operators in a Hilbert space H, and let $s_j(A)$, respectively $s_j(B)$, be the sequences of s-numbers of A, respectively B ($s_j(A)=\lambda_j(A^*A)^{\frac{1}{2}}$ for $j\in\mathbf{N}$, etc.), counted with multiplicity and arranged non-increasingly. Let there be given positive constant a, b and c, and $\beta>\alpha>0$, $\gamma>\alpha>0$ so that*

$$|s_j(A)-aj^{-\alpha}| \leqq bj^{-\beta} \tag{6.1}$$

$$|s_j(B)| \leqq cj^{-\gamma} \tag{6.2}$$

for all j. Then there exists $c'>0$ so that

$$|s_j(A+B)-aj^{-\alpha}| \leqq c'j^{-\beta'} \quad \textit{for all } j\,, \tag{6.3}$$

where

$$\beta' = \min\{\beta,\gamma(1+\alpha)/(1+\gamma)\}\,. \tag{6.4}$$

PROOF. We use that, as shown in [9],

$$s_{j+k-1}(A+B) \leqq s_j(A)+s_k(B) \tag{6.5}$$

for all j,k. Let $d\in\,]0,1[$, to be chosen later. For each $m\in\mathbf{N}$, let $k=[m^d]+1$ and let $j=m-[m^d]$. Then (6.1)–(6.2) imply (using that $(1+x)^k\leqq 1+c_1x$ for small x)

$$\begin{aligned} s_m(A+B) &\leqq a(m-[m^d])^{-\alpha}+b(m-[m^d])^{-\beta}+c([m^d]+1)^{-\gamma} \\ &\leqq am^{-\alpha}\left(1-\frac{[m^d]}{m}\right)^{-\alpha}+bm^{-\beta}\left(1-\frac{[m^d]}{m}\right)^{-\beta}+cm^{-d\gamma} \\ &\leqq am^{-\alpha}+bm^{-\beta}+c_2m^{-\alpha+d-1}+c_3m^{-\beta+d-1}+cm^{-d\gamma} \\ &\leqq am^{-\alpha}+c_4m^{-\beta'}, \end{aligned}$$

where $\beta'=\min\{\beta,\alpha-d+1,\beta-d+1,d\gamma\}$. Taking $d=(1+\alpha)/(1+\gamma)$, we have (6.4). This shows that

$$s_j(A+B)-aj^{-\alpha} \leqq c_4j^{-\beta'};$$

the other estimate is shown similarly on the basis of the formula

$$s_j(A+B) \geqq s_{j+k-1}(A)-s_k(B).$$

The next step is the observation

LEMMA 6.2. *Let A be a selfadjoint positive operator in H with compact inverse. Let $a>0$ and let $\beta>\alpha>0$. There exists $c_1>0$ such that*

(6.6) $$|s_j(A^{-1})-aj^{-\alpha}| \leqq c_1j^{-\beta} \quad \textit{for all } j \in \mathbf{N},$$

if and only if there exists $c_2>0$ so that

(6.7) $$|N(t;A)-a^{1/\alpha}t^{1/\alpha}| \leqq c_2t^{(1+\alpha-\beta)/\alpha} \quad \textit{for all } t>0.$$

PROOF. Note first that (6.6) implies $s_j(A^{-1})\sim aj^{-\alpha}$; hence since $s_j(A^{-1}) =\lambda_j(A)^{-1}$, (6.6) holds if and only if

(6.8) $$|\lambda_j(A)-a^{-1}j^{\alpha}| \leqq c_4j^{2\alpha-\beta} \quad \text{for all } j \in \mathbf{N},$$

for some $c_4>0$. Next, note that the functions $j\mapsto\lambda_j(A)$ and $t\mapsto N(t;A)$ are essentially inverse functions of one another. Consider e.g. the inequality

(6.9) $$\lambda_j(A) \leqq a^{-1}j^{\alpha}+c_4j^{2\alpha-\beta} \quad [\equiv\varphi(j)].$$

Set $t=\varphi(j)$ (defined for $j\in\mathbf{R}_+$), then (6.9) implies

(6.10) $$N(t;A) \geqq \varphi^{-1}(t) \quad \text{for sufficiently large } t.$$

Now $a^{-1}j^{\alpha}+c_4j^{2\alpha-\beta}=t$ implies

$$(at)^{1/\alpha} = j(1+c_4'j^{\alpha-\beta})^{1/\alpha}$$

and hence, since $t \leqq c_5 j^\alpha$,

$$\begin{aligned}\varphi^{-1}(t) = j &= (at)^{1/\alpha}(1+c_4' j^{\alpha-\beta})^{-1/\alpha} \\ &\geqq (at)^{1/\alpha}(1-c_6 t^{(\alpha-\beta)/\alpha}) \\ &= a^{1/\alpha} t^{1/\alpha} - c_7 t^{(1+\alpha-\beta)/\alpha} .\end{aligned}$$

This shows part of the implication (6.8) $\Leftrightarrow$ (6.7); the remaining implications are shown similarly.

THEOREM 6.3. *Let $\{E_s\}_{s=1,\ldots,q}$ be a family of hermitian vector bundles over Σ of dimensions $r_s > 0$, let $\{m_s\}_{s=1,\ldots,q}$ be a sequence of positive numbers with*

$$m_1 > m_2 > \ldots > m_q > 0 , \tag{6.11}$$

and let $P = (P_{st})_{s,t \leqq q}$ be a selfadjoint system of pseudo-differential operators P_{st} from E_t to E_s of orders $m_t + m_s$, P being strongly Douglis–Nirenberg elliptic (i.e. the symbol matrix $(p^0_{st}(x,\xi))_{s,t \leqq q}$ is positive definite for all $(x,\xi) \in T^*(\Sigma) \setminus 0$). Assume that P is positive and denote $P^{-1} = \tilde{P}$ $(= (\tilde{P}_{st})_{s,t \leqq q})$. Then the eigenvalues of P satisfy, for any $\varepsilon > 0$,

$$N(t;P) = c_{P'} t^{n/l} + O(t^\sigma) \quad \textit{for } t \to \infty , \tag{6.12}$$

where

$$c_{P'} = \frac{1}{n(2\pi)^n} \int_\Sigma \int_{S_x} \operatorname{tr}\,[\tilde{p}^0_{qq}(x,\xi)^{n/l}]\, d\omega\, dx , \tag{6.13}$$

with

$$l = 2m_q,\ l' = m_q + m_{q-1},\ \textit{and}\ \sigma = \max\left\{\frac{n-\frac{1}{2}+\varepsilon}{l}, \frac{n(n+l)}{l(n+l')}\right\} .$$

PROOF. We first note that

$$\tilde{P} = \begin{bmatrix} 0 \ldots 0 & 0 \\ \vdots \quad\ \vdots & \vdots \\ 0 \ldots 0 & 0 \\ 0 \ldots 0 & \tilde{P}_{qq} \end{bmatrix} + T , \tag{6.14}$$

where T is of order $\leqq -m_q - m_{q-1} = -l'$. Then the s-numbers of T satisfy

$$|s_j(T)| \leqq c_1 j^{-l'/n} ,$$

by a theorem of Agmon [1]. $\tilde{P}_{qq}$ is the inverse of an elliptic positive selfadjoint pseudo-differential operator P' of order l, whose eigenvalues are estimated by Theorem 5.4, with $c_{P'}$ satisfying (6.13). By Lemma 6.2, this gives

$$|s_j(\tilde{P}_{qq}) - c_{P'}^{l/n} j^{-l/n}| \leqq c_1 j^{-(l+\frac{1}{2}-\varepsilon)/n} .$$

Applying Proposition 6.1 to (6.14) with $\alpha = l/n$, $\beta = (l+\frac{1}{2}-\varepsilon)/n$, $\gamma = l'/n$, we then find that

$$|s_j(\tilde{P}) - c_{P'}^{l/n} j^{-l/n}| \leqq c_2 j^{-\beta'} ,$$

where

$$\beta' = \min\left\{\frac{l+\frac{1}{2}-\varepsilon}{n}, \frac{l'(n+l)}{n(n+l')}\right\} .$$

Then

$$(1+\alpha-\beta')/\alpha = \max\left\{\frac{n-\frac{1}{2}+\varepsilon}{l}, \frac{n(n+l)}{l(n+l')}\right\} ,$$

so that (6.12) holds by Lemma 6.2.

(When [14] can be applied to $(\tilde{P}_{qq})^{-1}$, e.g. when $\dim E_q = 1$, the above argument gives

$$\sigma = \max\left\{\frac{n-1}{l}, \frac{n(n+l)}{l(n+l')}\right\}$$

in (6.12).)

For example, if

$$P = \begin{pmatrix} \Delta^2 & A \\ A^* & -\Delta \end{pmatrix},$$

where A is a suitable 3rd order operator, and $n=3$, then $n/l = 3/2$, and $\sigma = \max\{(3-\frac{1}{2}+\varepsilon)/2, (3\cdot 5)/(2\cdot 6)\} = \frac{5}{4}+\varepsilon'$. ($\varepsilon' = 0$ if $-\Delta$ acts on scalar functions.)

The principal estimate ((6.12) with the O-term replaced by $o(t^{n/l})$) was shown by Kozevnikov [16]. (See also the simple proof by the author in C.I.M.E. III, 1973.)

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