MATH. SCAND. 43 (1978), 308-310

SCATTERED C*-ALGEBRAS II

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This note is a continuation of [3], in which scattered C*-algebras were introduced as those C*-algebras, for which each positive functional is a countable sum of pure functionals. This corresponds in the commutative case to an algebra $C_0(X)$, where X is a locally compact, scattered space.

In [3] it was proved, among other things, that if a C*-algebra A has a decomposition series $(I_p)_{0 \leqq p \leqq \alpha}$, such that each algebra I_{p+1}/I_p is isomorphic with an elementary C*-algebra, then A is scattered. In the following we shall prove the converse of this result.

LEMMA 1. *Let A be a C*-algebra with continuous trace. If A is scattered, then $\hat{A}$ is a scattered space.*

PROOF. We will prove, that each bounded Radon measure on $\hat{A}$ has an atom. Then result then follows from [4, 19.7.6].

Let μ be such a measure. From [1, 4.5] we see, that there exists a compact set $E \subseteqq \hat{A}$ with $\mu(E) > 0$ and an element $e \in A$, such that $\pi(e)$ is a one-dimensional projection for all $\pi \in E$. Moreover, the function $\pi \to \operatorname{tr} \pi(xe)$ is continuous on E for each $x \in A$, where tr denotes the ordinary trace. Define a positive functional φ on A by

$$\varphi(x) = \int_E \operatorname{tr}(\pi(xe))\, d\mu(\pi), \qquad x \in A .$$

Since $\hat{A}$ is scattered, we have $\varphi = \sum \lambda_i \psi_i$, where each ψ_i is a pure state. Therefore, since $\varphi(e) = \mu(E) > 0$, there is a pure state ψ and a number $\lambda > 0$, such that $\lambda\psi \leqq \varphi$ and $\psi(e) = \alpha > 0$.

Now, let π_0 denote the irreducible representation corresponding to ψ, and suppose, that $\mu\{\pi_0\} = 0$. For each $\varepsilon > 0$ we can find an open set $D \subset \hat{A}$ with $\mu(D) < \varepsilon$ and $\pi_0 \in D$, and a continuous function $f\colon \hat{A} \to [0,1]$, such that $f(\pi_0) = 1$ and $f(\pi) = 0$ for $\pi \notin D$. Let $f \cdot e$ denote the unique element in A, for which $\pi(f \cdot e) = f(\pi)\pi(e)$ for all $\pi \in \hat{A}$. (Dauns–Hofmann's theorem, see f.ex. [2] about this). We then have

Received May 25, 1977; in revised form December 5, 1977.

$$\varphi(f\cdot e) = \int_E f(\pi)\,\mathrm{tr}\,\pi(e)\,d\mu(\pi) < \varepsilon$$

and for some vector ξ in the Hilbert space for π_0

$$\psi(f\cdot e) = (\pi_0(f\cdot e)\xi \mid \xi) = (\pi_0(e)\xi \mid \xi) = \psi(e) = \alpha .$$

Therefore, since $\lambda\psi \leqq \varphi$, we have $\lambda\alpha < \varepsilon$, which is impossible for subsets D with sufficiently small measure. Consequently $\mu\{\pi_0\} > 0$, and the lemma is proved.

THEOREM 2. *A C*-algebra A is scattered if and only if A has a decomposition series $(I_p)_{0 \leqq p \leqq \alpha}$, such that each algebra I_{p+1}/I_p is elementary.*

PROOF. One half of the theorem follows from [3, proposition 2.6]. Next, suppose, that A is scattered. Since each quotient algebra of a scattered C*-algebra is scattered [3, proposition 2.4] the conclusion follows by transfinite induction, if it is proved, that A has an elementary ideal. But A is of type I [3, theorem 2.3] and therefore has an ideal J with continuous trace, and this ideal is scattered [3, proposition 2.4]. By lemma 1 there is an isolated point in $\hat{J}$, which gives the desired conclusion (as in [3, lemma 3.3]).

From this theorem and the proof we get

COROLLARY 3. *A C*-algebra A is scattered, if and only if A is of type* I *and $\hat{A}$ is scattered.*

If A is separable, then the above condition A is of type I can be omitted. Because, if $\hat{A}$ is scattered, then Prim (A) is scattered, so A has a decomposition series as in theorem 2 by [1, 4.7.3.].

Finally it should be remarked, that C*-algebras as the ones discussed here previously have been considered by Wojtaszyk [5] in the separable case. The C*-algebras investigated in [5] are those with separable dual space, which by theorem 2 and [3, theorem 3.1] are exactly the separable, scattered C*-algebras.

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