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TOWARDS A GALOIS THEORY FOR CROSSED PRODUCTS OF C*-ALGEBRAS

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Introduction.

Let $(\mathscr{M}, G, \alpha)$ be a W*-dynamical system and denote by $(G \times_\alpha \mathscr{M}, \hat{G}, \hat{\alpha})$ its dual system (cf. [7]). Assuming that $\mathscr{M}$ is a factor, A. Connes and M. Takesaki establish in [2, Theorem 4.1] a Galois type correspondence between the set of closed subgroups of G and the set of W*-algebras $\mathscr{N}$ such that $\mathscr{M} \subset \mathscr{N} \subset G \times_\alpha \mathscr{M}$ and $\hat{\alpha}_\tau(\mathscr{N}) = \mathscr{N}$ for all τ in $\hat{G}$. They show that for each such $\mathscr{N}$ there is a unique subgroup H determined by

$$H^\perp = \{\tau \in \hat{G} \mid \hat{\alpha}_\tau(x) = x \ \forall x \in \mathscr{N}\}$$

such that $\mathscr{N} = H \times_\alpha \mathscr{M}$.

When trying to extend this result to C*-algebras there are obvious difficulties. If (A, G, α) is a C*-dynamical system, there is a corresponding dual system $(G \times_\alpha A, \hat{G}, \hat{\alpha})$, but if G is not discrete, it is no longer true that $A \subset G \times_\alpha A$. Rather, $A \subset M(G \times_\alpha A)$ — the multiplier algebra of $G \times_\alpha A$. One aim of this paper is to define a C*-subalgebra $\mathscr{A}_\infty$ of $M(G \times_\alpha A)$ which is large enough to contain (in the natural embedding) every crossed product $H \times_\alpha A$ of A with any closed subgroup H of G; but no larger than we can find each $H \times_\alpha A$ in $\mathscr{A}_\infty$, given H.

In the case where G is discrete and A is simple, we obtain a C*-version of the Connes–Takesaki result concerning Galois correspondence. When G is not discrete, however, no such result seems within reach — in fact it is not even clear how one should formulate a reasonable analogue. If G is discrete, but A is not simple, the Galois correspondence fails.

1. Notation and preliminaries.

Throughout the paper, G will denote a locally compact abelian group and $\hat{G}$ its dual group. If A is a C*-algebra and $\alpha\colon G \to \operatorname{Aut}(A)$ is an automorphic

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representation of G such that each function $t \to \alpha_t(x)$, $x \in A$, is continuous, we call (A, G, α) a C*-dynamical system.

Let $\mathscr{H}$ be a Hilbert space, $A \subset B(\mathscr{H})$. Let $\mathscr{K}$ denote $L^2(G, \mathscr{H})$. Take H to be a closed subgroup of G, and denote by $K(H, A)$ the continuous functions with compact support from H into A. Define an isometric representation of $K(H, A)$ with the $L^1(H, A)$-norm, cf. [6], on $\mathscr{K}$ by

$$(y\xi)(s) = \int_H \alpha_{-s}(y(h))\xi(s-h)\,dh$$

where $y \in K(H, A)$ and $\xi \in \mathscr{K}$. Then $H \times_\alpha A$ is isomorphic to the C*-algebra $\mathscr{A}_H$ generated by $K(H, A)$ on $\mathscr{K}$. When $H = G$, we put $\mathscr{A} = \mathscr{A}_G = G \times_\alpha A$. Recall that A has a natural representation $\imath$ as operators on $\mathscr{K}$, defining

$$(\imath(a)\xi)(s) = \alpha_{-s}(a)\xi(s)\,,$$

and that with $(\lambda_t\xi)(s) = \xi(s-t)$ we have

$$\imath(\alpha_t(a)) = \lambda_t \imath(a)\lambda_{-t}\,,$$

so that (A, G, α) is covariantly represented on $\mathscr{K}$. With this terminology, each y in $K(H, A)$ identifies with $\int_H \imath(y(h))\lambda_h\,dh$.

Recall further that the dual action $\hat{\alpha}$ of $\hat{G}$ on $\mathscr{A}$ is given by $\hat{\alpha}_\sigma = \operatorname{Ad} u_\sigma$, where the unitary group on $\mathscr{K}$ is defined by

$$(u_\sigma\xi)(s) = (s, \sigma)\xi(s)\,.$$

If G is discrete, it was shown in [4, Theorem 4] that $\imath(A)$ is the fixed-point algebra under $\hat{\alpha}$ i.e. $\imath(A)$ equals

$$\mathscr{A}^{\hat{\alpha}} = \{x \in \mathscr{A} \mid \hat{\alpha}_\sigma(x) = x \ \forall \sigma \in \hat{G}\}\,.$$

If G is not discrete, the characterization of $\imath(A)$ is more complicated. It was shown in [4, Theorem 4] that a in $M(\mathscr{A})$ belongs to $\imath(A)$ if and only if

(i) $a\lambda_f \in \mathscr{A}$ and $\lambda_f a \in \mathscr{A}$ for every f in $L^1(G)$;
(ii) The function $t \to \lambda_t a \lambda_{-t}$ is norm-continuous;
(iii) $\hat{\alpha}_\sigma(a) = a$ for every σ in $\hat{G}$.

In section 2, we generalize this characterization from $\imath(A) = \mathscr{A}_{\{0\}}$ to $\mathscr{A}_H$, where H is any closed subgroup of G. In section 3, we specialize to the case where G is discrete, thus $\alpha(A) = \mathscr{A}^{\hat{\alpha}}$, and A is simple. We show that under these conditions there are no $\hat{\alpha}$-invariant algebras between $\mathscr{A}^{\hat{\alpha}}$ and $\mathscr{A}$ other than the algebras $\mathscr{A}_H$.

2. A characterization of $\mathscr{A}_H$ and $M(\mathscr{A}_H)$ in $M(\mathscr{A})$.

For each closed subgroup H of G let $H^{\perp}$ denote the annihilator of H in $\hat{G}$ and identify $\hat{H}$ with $\hat{G}/H^{\perp}$.

Our main result in this section is the following

THEOREM 2.1. *Let (A, G, α) be a C*-dynamical system, $(\mathscr{A}, \hat{G}, \hat{\alpha})$ its dual system. Let H be a closed subgroup of G. Then $\mathscr{A}_H$ is the C*-algebra consisting of those x in $M(\mathscr{A})$ for which*

(M1) *$x\lambda_f \in \mathscr{A}$ and $\lambda_f x \in \mathscr{A}$ for every f in $L^1(G)$;*
(M2) *The function $t \to \lambda_t x \lambda_{-t}$ is norm-continuous on G;*
(H1) *$\hat{\alpha}_\sigma(x) = x$ for every σ in $H^{\perp}$;*
(H2) *The function $h \to \lambda_h x$ is norm-continuous on H.*

2.2. Define $\mathscr{A}_\infty$ to be the C*-subalgebra of $M(\mathscr{A})$ consisting of those x in $M(\mathscr{A})$ for which

(M1) $x\lambda_f \in \mathscr{A}$ and $\lambda_f x \in \mathscr{A}$ for every f in $L^1(G)$;
(M2) The function $t \to \lambda_t x \lambda_{-t}$ is norm-continuous on G;
(M3) The function $\gamma \to \hat{\alpha}_\gamma(x)$ is norm-continuous on $\hat{G}$.

PROPOSITION 2.3. *$\mathscr{A}_H \subseteq \mathscr{A}_\infty$ and furthermore every x in $\mathscr{A}_H$ satisfies*

(H1) *$\hat{\alpha}_\gamma(x) = x$ for every γ in $H^{\perp}$;*
(H2) *The function $h \to \lambda_h x$ is norm-continuous on H.*

PROOF. Let $y \in K(H, A)$, $f \in K(G)$. Then $y\lambda_f = y * f$, where

$$(y * f)(t) = \int_H y(h) f(t-h)\, dh ,$$

so $y\lambda_f \in \mathscr{A}$. That $\lambda_f y \in \mathscr{A}$ is proved analogously. This argument also shows that $y \in M(\mathscr{A})$, since choosing x in $\mathscr{A}$ and a net $\{f_i\}$ in $K(G)$ such that $\lambda_{f_i} x \to x$ we have

$$yx = \lim y\lambda_{f_i} x = \lim (y * f_i) x \in \mathscr{A} .$$

Now $(\lambda_t y \lambda_{-t})(s) = \alpha_t(y(s)) \in K(H, A)$ for every t in G, so (M2) is easily verified, and $(\hat{\alpha}_\gamma(y))(t) = (t, \gamma) y(t)$ immediately yields (H1). Finally, $\lambda_h y$ belongs to $K(H, A)$, because

$$\lambda_h y(k) = \alpha_h(y(k-h)) ,$$

which proves (H2). Since $K(H, A)$ is dense in $\mathscr{A}_H$, the proof is complete.

2.4. Take B_0 to be the set of x in $B(\mathscr{H})$ for which there is some $K \geqq 0$ such that

$$\int_{H^\perp} \langle \hat{\alpha}_\gamma(x^*x)\xi \mid \xi \rangle \, d\gamma \leqq K\|\xi\|^2$$

for all ξ in $\mathscr{H}$. Define $I_H\colon B_0{}^*B_0 \to B(\mathscr{H})$ by

$$\langle I_H(y)\xi \mid \eta \rangle = \int_{H^\perp} \langle \hat{\alpha}_\gamma(y)\xi \mid \eta \rangle \, d\gamma$$

where ξ and η belong to $\mathscr{H}$.

LEMMA 2.5. *Take f_1, f_2, g_1, g_2 to be in $K(G)$, the set of continuous functions with compact support from G into $\mathbf{C}$. Let $f=f_1 * f_2$, $g=g_1 * g_2$. If $b \in B(\mathscr{H})$ then $\lambda_f b \lambda_g \in B_0^* B_0$ and*

$$\|I_H(\lambda_f b \lambda_g)\| \leqq \|b\| \int_G \int_H |f(t)g(h-t)| \, dh \, dt \, .$$

If $\hat{\alpha}_\gamma(b)=b$ for all γ in $H^\perp$ we get

$$I_H(\lambda_f b \lambda_g) = \int_G \int_H f(t+h)g(-t)\lambda_{t+h} b \lambda_{-t} \, dh \, dt \, .$$

PROOF. Let $\xi \in \mathscr{H}$, $b \geqq 0$. Put $g^*(t)=\overline{g(-t)}$. Then

$$\int_{H^\perp} \langle \hat{\alpha}_\gamma(\lambda_{g^*} b \lambda_g)\xi \mid \xi \rangle \, d\gamma$$

$$\leqq \|b\| \int_{H^\perp} \int_G \int_G g^*(t)g(s)(s+t,\gamma)\langle \lambda_{s+t}\xi \mid \xi \rangle \, ds \, dt \, d\gamma$$

$$= \|b\| \int_{H^\perp} \int_G \int_G g^*(t)g(s)(s+t,\gamma)\langle \lambda_{s+t}\xi \mid \rangle \, ds \, dt \, d\gamma$$

$$= \|b\| \int_{H^\perp} \int_G (g^* * g)(u)(u,\gamma)\langle \lambda_u \xi \mid \xi \rangle \, du \, d\gamma$$

$$= \|b\| \int_H (g^* * g)(h)\langle \lambda_h \xi \mid \xi \rangle \, dh$$

$$\leqq \|b\| \, \|\xi\|^2 \int_H |(g^* * g)(h)| \, dh$$

$$\leqq \|b\| \, \|\xi\|^2 \int_G \int_H |g^*(t)g(h-t)| \, dh \, dt$$

We have here used the Poisson formula ([3, 31.46(e)(xi)])

$$\int_{H^\perp}\int_G (u,\gamma)h(u)\,du\,d\gamma = \int_{H^\perp}\hat{h}(\gamma)\,d\gamma = \int_H h(u)\,du$$

for each h in $L^1(G)$ with $\hat{h}$ in $L^1(\hat{G})$.

This shows that $b^{\frac{1}{2}}\lambda_g \in B_0$, thus $b^{\frac{1}{2}}\lambda_{f^*} \in B_0$ and so

$$\lambda_f b\lambda_g = (b^{\frac{1}{2}}\lambda_{f^*})^*(b^{\frac{1}{2}}\lambda_g) \in B_0^* B_0 .$$

The estimate of $\|I_H(\lambda_f b\lambda_g)\|$ follows by direct computation.

Assume that $\hat{\alpha}_\gamma(b)=b$ for every γ in $H^\perp$. Then

$$\begin{aligned} I_H(\lambda_f b\lambda_g) &= \int_{H^\perp}\hat{\alpha}_\gamma(\lambda_f b\lambda_g)\,d\gamma \\ &= \int_{H^\perp}\int_G\int_G \hat{\alpha}_\gamma(\lambda_t b\lambda_s)f(t)g(s)\,dt\,ds\,d\gamma \\ &= \int_{H^\perp}\int_G\int_G (t+s,\gamma)\lambda_t b\lambda_s f(t)g(s)\,dt\,ds\,d\gamma \\ &= \int_{H^\perp}\int_G\int_G (u,\gamma)\lambda_t b\lambda_{u-t} f(t)g(u-t)\,dt\,du\,d\gamma \\ &= \int_G\int_H \lambda_t b\lambda_{h-t} f(t)g(h-t)\,dh\,dt , \end{aligned}$$

using the Poisson formula again.

LEMMA 2.6. *With b in $\mathscr{A}$, f and g as in lemma 2.5, we get*

$$I_H(\lambda_f b\lambda_g) \in \mathscr{A}_H .$$

PROOF. (i) Assume that $b \in \iota(A)$, that is, $b=\iota(a)$, $a \in A$ and $(\iota(a)\xi)(s) =\alpha_{-s}(a)\xi(s)$. The formula above yields

$$I_H(\lambda_f b\lambda_g) = \int_H\left(\int_G \iota(\alpha_t(a))f(t)g(h-t)\,dt\right)\lambda_h\,dh .$$

Now recall that by definition $\mathscr{A}_H$ is the closed span of operators of the form $\int_H \iota(y(h))\lambda_h\,dh$, where $y \in K(H,A)$. But this is exactly the form of $I_H(\lambda_f b\lambda_g)$ above, as

$$y(h) = \int_G \alpha_t(a)f(t)g(h-t)\,dt \in K(H,A) .$$

(ii) Assume that $b=a\otimes\varphi$, $\varphi\in K(G)$, $a\in A$, that is, $b(s)=a\varphi(s)$ for every s in G. Then the result follows from (i) and the computation

$$I_H(\lambda_f b\lambda_g) = \int_H \left(\int_G \iota(\alpha_t(a))f(t)g(h-t)\,dt\right)\lambda_\varphi)\lambda_h\,dh\,.$$

(iii) Assume b to be arbitrary in $\mathscr{A}$. Given $\varepsilon>0$ there are finite sets (φ_k) in $K(G)$ and (a_k) in A such that

$$\|b-\sum a_k\otimes\varphi_k\| < \frac{1}{M}\varepsilon\,,$$

where $M=\int_G\int_H|f(t)g(h-t)|\,dh\,dt$. Thus $I_H(\lambda_f(\sum a_k\otimes\varphi_k)\lambda_g)\in\mathscr{A}_H$ and

$$\|I_H(\lambda_f(b-\sum a_k\otimes\varphi_k)\lambda_g)\| \leqq M\|b-\sum a_k\otimes\varphi_k\| < \varepsilon\,.$$

From this the conclusion follows, since $\mathscr{A}_H$ is norm-closed.

Proof of main theorem. That every x in $\mathscr{A}_H$ satisfies M1–3 and H1–2 follows from proposition 2.3. Take x in $M(\mathscr{A})$ and assume that it satisfies M1–2 and H1–2. Let $f\in K(G)$, $\varepsilon>0$. Then $\lambda_f x\in\mathscr{A}$, so by lemma 2.6 we get

$$b = I_H(\lambda_{f*f*f}x\lambda_{f*f*f})\in\mathscr{A}_H\,.$$

Choose a neighbourhood U of e in G such that

$$\|\lambda_t\lambda_h x\lambda_{-t}-x\| < \varepsilon,\qquad \forall t\in U,\ \forall h\in U\cap H\,.$$

Choose V to be a neighbourhood of e such that $V=-V$ and $V+V\subset U$, and let f be chosen so $g=f*f*f$ has support in V and furthermore

$$\int_G\int_H g(t+h)g(t)\,dh\,dt = 1\,.$$

Then

$$\begin{aligned}\|b-x\| &= \|I_H(\lambda_g x\lambda_g)-x\|\\ &= \left\|\int_G\int_H g(t+h)g(t)(\lambda_{t+h}x\lambda_{-t}-x)\,dh\,dt\right\|\,.\end{aligned}$$

This integrand is non-zero only if $t+h\in V$ and $t\in V$, that is, $t\in V$ and $h\in V-V\subset U$, so by the choice of U,

$$\|b-x\| \leqq \varepsilon\int_G\int_H g(t+h)g(t)\,dh\,dt = \varepsilon\,.$$

Remark 2.7. The characterization of $\mathscr{A}_H$ may also be formulated in the following way:

$\mathscr{A}_H$ is the C*-subalgebra of $\{x \in \mathscr{A}_\infty \mid \hat{\alpha}_\sigma(x)=x, \forall \sigma \in H^\perp\}$ consisting of those elements x such that $h \to \lambda_h x$ is norm-continuous on H. In particular, $\iota(A)=(\mathscr{A}_\infty)^{\hat{\alpha}}$.

This formulation amounts to saying that $\mathscr{A}_H$ is characterized by satisfying conditions M1–3 and H1–2. However, as we have seen above, (M3) is free, since it follows from the other four conditions. No such interdependence remains between the remaining four conditions, as the following examples will show.

EXAMPLE 2.8. Suppose that (A, G, α) is a C*-dynamical system where α is trivial. Then $\mathscr{A}=C_0(\hat{G}, A)$ $(=C^*(G)\otimes A)$ and by [1, Corollary 3.4] $M(\mathscr{A}) = C^b(\hat{G}, M(A))$, where $M(A)$ is endowed with the strict topology. Note that for x in $C^b(\hat{G}, M(A))$

$$(\hat{\alpha}_\tau(x))(\sigma) \;=\; x(\sigma-\tau)$$

for all τ, σ in $\hat{G}$. Condition (M1) asserts that $x(\sigma) \in A$ for all σ in $\hat{G}$, (M2) is empty, (H1) requires x to be periodic, thus a function on $\hat{H}$, whereas (H2) forces $x(\sigma) \to 0$ as $\sigma \to \infty$ in $\hat{H}$, because $(\lambda_t x)(\sigma)=(t,\sigma)x(\sigma)$. Consequently

$$\mathscr{A}_H \;=\; C_0(\hat{H}, A)\,.$$

Note that $\mathscr{A}_\infty = C^b_{\text{unif.}}(\hat{G}, A)$, as (M3) requires x to be uniformly continuous.

If $H=\{e\}$, (M1) and (H1) are necessary to show $\mathscr{A}_{\{e\}}=A$.

If $H=G$, (M1) and (H2) are necessary to show $\mathscr{A}_G=C_0(\hat{G}, A)$.

So in this example, all conditions but (M2) are in play.

EXAMPLE 2.9. Consider the C*-dynamical system $(C_0(G), G, \alpha)$ where $(\alpha_t(x))(s)=x(s-t)$ for each x in $C_0(G)$. Then $\mathscr{A}=C(L^2(G))$ (see e.g. [6, Proposition 3.3]), whence $M(\mathscr{A})=B(L^2(G))$. To simplify matters assume that G is compact. Then each operator λ_f, $f \in L^1(G)$, is compact, being unitarily equivalent with the multiplication operator $m_{\hat{f}}$ on $L^2(\hat{G})$. Thus condition (M1) is empty.

If $H=\{e\}$, condition (H1) implies that $x \in L^\infty(G)$ and (M2) that $x \in C^b(G)$. (H2) is empty. Thus (H1) and (M2) characterize A.

2.10. We shall now see that theorem 2.1 can be used to characterize the multiplieralgebra $M(\mathscr{A}_H)$. First note that since $\mathscr{A}_H \subset M(\mathscr{A})$ and $\mathscr{A}_H$ contains an approximate identity for $\mathscr{A}$ we have $M(\mathscr{A}_H) \subset M(\mathscr{A})$, cf. [1, Proposition 2.6]. Using the terminology of theorem 2.1 we have

THEOREM 2.11. *$M(\mathscr{A}_H)$ is the C*-algebra consisting of those x in $M(\mathscr{A})$ for which*

(H1) *$\hat{\alpha}_\gamma(x)=x$ for every γ in $H^\perp$;*

(H3) *The function $t \to \lambda_t(xy+zx)\lambda_{-t}$ is norm-continuous on G for all y and z in $\mathscr{A}_H$.*

PROOF. It is obvious from 2.1 that the elements of $M(\mathscr{A}_H)$ satisfy (H1) and (H3). So suppose $x \in M(\mathscr{A})$ which satisfies (H1) and (H3). Let $y \in \mathscr{A}_H$. We want to prove that yx satisfies (M1), (M2), (H1) and (H2).

Take f in $L^1(G)$; then $\lambda_f yx \in \mathscr{A}x \subset \mathscr{A}$ since y satisfies (M1). In order to prove that $yx\lambda_f \in \mathscr{A}$ it suffices to prove (using (H3)) that

$$b = \int_G g(t)\lambda_t yx\lambda_{-t}\lambda_f \, dt \in \mathscr{A}$$

for all f, g in $K(G)$. Now

$$\begin{aligned} b &= \int_G \int_G f(s)g(t)\lambda_t yx\lambda_{s-t} \, ds\, dt \\ &= \int_G \int_G f(s+t)g(t)\lambda_t yx\lambda_s \, dt\, ds \\ &= \int_G \lambda_{h_s} yx\lambda_s \, ds \in \mathscr{A} \end{aligned}$$

since $\lambda_{h_s} y \in \mathscr{A}$, where $h_s(t)=f(s+t)g(t)$.

This proves that yx satisfies (M1). Since yx obviously satisfies (M2), (H1) and (H2), we have that $yx \in \mathscr{A}_H$.

3. A Galois correspondence for discrete crossed products.

In this section we have the following main result:

THEOREM 3.1. *Let (A, G, α) be a C*-dynamical system with G discrete and A simple. Denote by $(\mathscr{A}, \hat{G}, \hat{\alpha})$ the dual system. Take $\mathscr{B}$ to be an $\hat{\alpha}$-invariant C*-subalgebra of $\mathscr{A}$ which contains $\iota(A)$. Then $\mathscr{B}$ is isomorphic to $\mathscr{A}_H = H \underset{\alpha}{\times} A$ where H is the subgroup of G that annihilates*

$$\Gamma = \{\gamma \in \hat{G} \mid \hat{\alpha}_\gamma(b)=b \ \forall\, b \in \mathscr{B}\}.$$

We prove this in two steps. The first is a direct corollary of theorem 2.1:

LEMMA 3.2. *Let (A, G, α) be a C*-dynamical system with G discrete. For each subgroup H of G we have*

$$\mathscr{A}_H = \{x \in \mathscr{A} \mid \hat{\alpha}_\gamma(x)=x \ \forall\, \gamma \in H^\perp\}.$$

LEMMA 3.3. *Let* (B, K, β) *be a* C*-*dynamical system with* K *compact and the fixed-point algebra* B^β *simple. Assume that for each* γ *in* $\mathrm{Sp}(\beta)$ *there is a unitary eigenoperator* u_γ *in* $M(B)$ (*i.e.* $\beta_t(u_\gamma) = (t, \gamma) u_\gamma$). *If* D *is a* β-*invariant* C*-*subalgebra of* B *containing* B^β *put*

$$H = \{t \in K \mid \beta_t(x) = x, \forall x \in D\},$$

$$B^H = \{x \in B \mid \beta_t(x) = x, \forall t \in H\}.$$

Then $D = B^H$.

PROOF. If $\gamma \in \mathrm{Sp}(\beta \mid D)$ the eigenspace $D^\beta\{\gamma\}$ is non-zero. But then $u_\gamma^* D^\beta\{\gamma\}$ is a closed, non-zero ideal of B^β, since

$$B^\beta u_\gamma^* D^\beta\{\gamma\} = u_\gamma^*(u_\gamma B^\beta u_\gamma^*) D^\beta\{\gamma\} \subseteq u_\gamma^* B^\beta D^\beta\{\gamma\} \subseteq u_\gamma^* D^\beta\{\gamma\}.$$

As B^β is simple this implies that $u_\gamma^* D^\beta\{\gamma\} = B^\beta$, thus $D^\beta\{\gamma\} = u_\gamma B^\beta$. In particular, $\gamma \in \mathrm{Sp}(\beta \mid D)$ if and only if $u_\gamma \in M(D)$. It follows that $\mathrm{Sp}(\beta \mid D)$ is a subgroup Γ of $\hat{K}$ and that

$$D = \bigoplus_{\gamma \in \Gamma} D^\beta\{\gamma\} = \bigoplus_{\gamma \in \Gamma} u_\gamma B^\beta.$$

From this it is immediate that $H = \Gamma^\perp$ and that

$$D = \bigoplus_{\gamma \in \Gamma} u_\gamma B \subset B^H.$$

However, if $0 \neq x \in u_\gamma B^\beta \cap B^H$, then $\gamma \in H^\perp = \Gamma$, whence $x \in D$. Since $B = \bigoplus u_\gamma B^\beta$ we conclude that $B^H \subset D$ and the proof is complete.

PROOF OF THEOREM 3.1. Using that $\iota(A) = \mathscr{A}^{\hat{\alpha}}$ when G is discrete we see that $(\mathscr{A}, \hat{G}, \hat{\alpha})$ becomes a dynamical system of the kind described in lemma 3.3. The unitary eigenoperators are the λ_t. We conclude that

$$\mathscr{B} = \bigoplus_{t \in H^\perp} \lambda_t \mathscr{A}^{\hat{\alpha}} = \bigoplus_{t \in H^\perp} \lambda_t \iota(A) = \mathscr{A}^{\hat{\alpha} \mid H^\perp},$$

and so lemma 3.2 yields the desired result.

That A has to be simple in order for theorem 3.1 to hold follows from the ensuing

EXAMPLE 3.4. Let A be a C*-algebra, I a non-trivial ideal of A. Let $G = \{e, \varepsilon\}$ be the two-element group, and α the trivial action of G on A. Then

$$\mathscr{A} = \{f \colon G \to A\}$$

equipped with the multiplication $(fg)(t)=\sum f(s)g(t-s)$ and involution $f^*(t)=f(t)^*$.

We have $\hat{G}=\{e,\sigma\}$ where e is the identity, $\sigma(\varepsilon)=-1$. The dual action of $\hat{G}$ on $\mathscr{A}$ is given by

$$\hat{\alpha}_e(f) = f, \quad \hat{\alpha}_\sigma(f)(t) = \sigma(t)f(t) .$$

Take $\mathscr{B}=\{f\in\mathscr{A} \mid f(e)\in A,\ f(\varepsilon)\in I\}$. Then

$$\iota(A) = \{f\in\mathscr{A} \mid f(\varepsilon)=0\} \subset \mathscr{B} \quad \text{and} \quad \mathscr{B} \text{ is } \hat{\alpha}\text{-invariant},$$

but $\iota(A)\neq\mathscr{B}\neq\mathscr{A}$.

REMARK 3.5. Note that we do not get a Galois correspondence for the $\hat{\alpha}$-invariant subalgebras of $M(\mathscr{A})$ even if A is simple and G is discrete. This is seen by taking A simple without unit and $G=\{e\}$. Then $\mathscr{A}=A$ and we will in general have many C*-subalgebras between A and $M(A)$.

REMARK 3.6. As pointed out to us by G. A. Elliott theorem 3.1 remains true if A is only assumed to be G-simple (no non-trivial G-invariant closed ideals, see [5]). If namely the unitary family $\{u_\gamma\}$ in Lemma 3.3 is commutative, each of the ideals $u_\gamma^* D^\beta\{\gamma\}$ will be invariant under all automorphisms of B^β of the form $\mathrm{Ad}\, u_\delta$. This being excluded (when $B^\beta=A$ is G-simple) we may again conclude that $D^\beta\{\gamma\}=u_\gamma B^\beta$.

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REFERENCES

1. C. A. Akemann, G. K. Pedersen, and J. Tomiyama, *Multipliers of C*-algebras*, J. Functional Analysis 13 (1973), 277–301.
2. A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tôhoku Math. J. 29 (1977), 473–575.
3. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, II (Grundlehren Math. Wissensch. 152), Springer-Verlag, Berlin - Heidelberg - New York, 1970.
4. M. B. Landstad, *Duality theory of covariant systems*, Trans. Amer. Math. Soc. to appear.
5. D. Olesen and G. K. Pedersen, *Applications of the Connes spectrum to C*-dynamical systems*, J. Functional Analysis 30 (1978), 179–197.

6. H. Takai, *On a duality for crossed products of* C*-*algebras*, J. Functional Analysis 19 (1975), 25–39.
7. M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type* III, Acta Math. 131 (1973), 249–310.

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