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A MAXIMAL SUBALGEBRA OF $R(X)$

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Introduction.

Let X be a compact set in the complex plane, consisting of an annular domain U together with the boundary of U consisting of two disjoint simple closed curves Γ_1 and Γ_2, Γ_1 lying in the bounded component of $\mathbf{C} \setminus \Gamma_2$.

Let V denote the bounded component of $\mathbf{C} \setminus \Gamma_1$. Let A be the function algebra on X consisting of all elements of $R(X)$ which have a continuous extension to $X \cup V$, which is analytic on V. $R(X)$ denotes the function algebra on X consisting of uniform limits on X of restrictions to X of rational functions whose zero sets miss X. We will prove: A is maximal in $R(X)$.

If Γ_1 is not too wild A equals the algebra $P(X)$ of uniform limits of polynomials on X by Mergelyan's theorem since every element of A is actually analytic on $U \cup \Gamma_1 \cup V$, i.e. on the bounded component of $\mathbf{C} \setminus \Gamma_2$. This happens if Γ_1 is rectifiable for instance. However it may happen that $A \neq P(X)$.

For instance, assume that Γ_1 contains an arc α of positive two-dimensional Lebesgue measure, then it is well-known that there are functions which are continuous on the extended complex plane and analytic off α but are not approximable by polynomials on X.

In [2] and [3] it is shown that for well-behaving Γ_1, $P(X)$ is a maximal subalgebra of $R(X)$, i.e. if B is a function algebra on X such that $P(X) \subset B \subset R(X)$ either $B = P(X)$ or else $B = R(X)$. The purpose of this paper is to prove the analogous theorem for general curves Γ_1. The proof given here uses only basic facts from the theory of function algebras, together with Rossi's local maximum modulus principle [5] and a version of Wermer's famous maximality theorem [7]. The first part of the proof is similar to the first part of the proof of lemma 3.2 contained in [1] but is given here for reasons of completeness. (See also the proof of theorem 10.7 in [8].) For the basic facts on function algebras we refer to [4] or [6].

THEOREM. *Let X and A be as above. Then A is a maximal subalgebra of $R(X)$.*

PROOF. Let B be a function algebra on X such that $A \subset B \subset R(X)$ and let

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$\pi\colon \Delta B \to \Delta A$ be the projection map defined by $\pi\varphi = \varphi \mid A$ (Δ denoting maximal ideal space). By Arens' theorem (see e.g. [4]) $\Delta A = X \cup V$. Suppose $y \in \Delta B$ such that $\pi y = x \in \Gamma_2$. Let m be a representing measure for y on X then

$$\hat{f}(y) = \int_X f\,dm \qquad \text{for all } f \in B .$$

(here $\hat{f}$ denotes the Gelfand transform of f).

For $f \in P(X) \subset B$

$$f(x) = \hat{f}(y) = \int_X f\,dm ,$$

so since x is a peak point for $P(X)$ m has to be point mass at x, hence $y = x$.

Let $y_1, y_2 \in \Delta B$ such that $x = \pi(y_1) = \pi(y_2) \in U$.

Let Γ be a simple closed curve in X, with x in the bounded component of $\mathbf{C} \setminus \Gamma$ and such that $\overline{\Gamma \setminus \Gamma_2}$, the closure of $\Gamma \setminus \Gamma_2$, is polynomially convex and such that $\hat{\Gamma}$, the polynomially convex hull of Γ, is contained in X. Let $Y = \pi^{-1}(\hat{\Gamma})$. Then by Rossi's local maximum modulus principle $\partial B_Y \subset \pi^{-1}(\Gamma)$ (∂ denoting Silov boundary and B_Y the function algebra on Y generated by the restrictions to Y of elements of B). Let μ_i be a representing measure for y_i on $\pi^{-1}(\Gamma)$ for the algebra B_Y $(i = 1, 2)$ and let $\tilde{\mu}_i = \mu_i \circ \pi^{-1}$ on Γ. Then

$$\int_\Gamma f\,d\tilde{\mu}_i = \int_{\pi^{-1}(\Gamma)} \hat{f}\,d\mu_i = \hat{f}(y_i) = f(x) \qquad (i = 1, 2)$$

for all $f \in P(X)$. Since $P(\Gamma)$ is a Dirichlet algebra on Γ, $\tilde{\mu}_1 = \tilde{\mu}_2$, so $\mu_1 = \mu_2$ on $\Gamma_2 \cap \pi^{-1}(\Gamma) = \Gamma_2 \cap \Gamma$. Now suppose $y_1 \neq y_2$; let $g \in B$ such that $\hat{g}(y_1) = 1$, $\hat{g}(y_2) = 0$, so $\alpha = \hat{g}(\mu_1 - \mu_2)$ represents $y_1 \in \Delta B_Y$. Hence

$$\int_\Gamma f\,d\tilde{\alpha} = f(x) \qquad \text{for all polynomials } f .$$

Since $\tilde{\alpha} = 0$ on $\Gamma \cap \Gamma_2$ and $\overline{\Gamma \setminus \Gamma_2}$ is polynomially convex this is absurd. So $y_1 = y_2$. Hence π is one-to-one over $U \cup \Gamma_2$.

Now

$$\Delta B_{\Gamma_1} = \{y \in \Delta B : |\hat{g}(y)| \leqq \|g\|_{\Gamma_1} \text{ for all } g \in B\} .$$

Since $B_{\Gamma_1} \supset P(\Gamma_1)$ it follows from Wermer's maximality theorem that $B_{\Gamma_1} = P(\Gamma_1)$ or else $B_{\Gamma_1} = C(\Gamma_1)$. In the first case every element of B extends analytically to V, hence $B = A$. In the second case $\Delta B_{\Gamma_1} = \Gamma_1$. If there were $y \in \Delta B$ such that $\pi y = x \in V$ choose $g \in B$ such that $|\hat{g}(y)| > \|g\|_{\Gamma_1}$. The component M of $\{z \in \Delta B : |\hat{g}(z)| \geqq |\hat{g}(y)|\}$ containing y then satisfies $M \cap \partial B \subset M \cap (\Gamma_1 \cup \Gamma_2) = \emptyset$, in contradiction with Rossi's local maximum modulus principle. So $\Delta B = X$, hence the function $z - \alpha$ is invertible in B for all $\alpha \in V$. So $B = R(X)$.

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