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ESSENTIAL FUNCTION ALGEBRAS WITH LARGE SILOV BOUNDARY

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1. Introduction.

In this note we construct, starting with a function algebra A on its homomorphism space ΔA, a function algebra B on ΔB which shares a number of properties with A but such that the Silov boundary ∂B of B equals ΔB. The easiest way to do this is to consider $X = \Delta A \times [0,1]$ and to take B the algebra of all continuous functions f on X such that $f \mid \Delta A \times \{0\}$ belongs to A. Then $\Delta B = \partial B = X$ but B is not essential on ΔB. Our construction provides an algebra B which is essential on ΔB and such that ΔB can be considered as a subset of $\mathbf{C}^{n+2}$ if $\Delta A \subset \mathbf{C}^n$.

2. Notation and definitions.

Let X be a compact Hausdorff space. $C(X)$ will be the algebra of continuous complex-valued functions on X with the supremum norm. A function algebra A on X is a closed subalgebra of $C(X)$ containing the constants and separating the points of X. The maximal ideal space of A is denoted by ΔA and its Silov boundary by ∂A. If K is a compact subset of X and $f \in A$ $f \mid K$ denotes the restriction of f to K and $[A \mid K]$ will be the closure in $C(K)$ of the restrictions to K of elements of A. A is called essential on X if the minimal closed subset K of X with the property $f \in C(X)$ and $f \mid K = 0$ implies $f \in A$ equals X.

For a compact subset X in $\mathbf{C}^n$ $P(X)$ will be the closure in $C(X)$ of the algebra of polynomials on X and $R(X)$ is the closure in $C(X)$ of the algebra of rational functions having poles outside of X.

For information on function algebras we refer to [6].

3. The construction.

Let Y be a Swiss cheese (e.g. [5]) such that $R(Y) \neq C(Y)$ and $\Delta R(Y) = \partial R(Y) = Y$. $R(Y)$ is essential on Y. Let A be a function algebra on a metrizable compact space X with $\Delta A = X$. Choose a dense sequence $\{x_n\}$ in X.

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Consider in $X \times \mathbf{C}^2$ the following subsets:

$$\tilde{X} = \{(x,0,0) : x \in X\}$$

$$Y_n = \{(x_n, t, y) : y \in tY, 0 \leqq t \leqq a_n\}$$

where $tY = \{ty : y \in Y\}$, $n \in \mathbf{N}$ and $\{a_n\}$ is a decreasing sequence of positive numbers having 0 as limit. Let Z be the union of the sets $\tilde{X}$ and $\{Y_n\}$ in $X \times \mathbf{C}^2$. Let B be the algebra of all continuous functions f on Z such that

(i) the function on X defined by $x \mapsto f(x,0,0)$ belongs to A (abbreviation $f \mid \tilde{X} \in A$)
(ii) the function on $Y_{n,\alpha} = Y_n \cap \{t = \alpha\}$ defined by $y \mapsto f(x_n, \alpha, y)$ belongs to $R(\alpha Y)$ for all fixed n and $0 < \alpha \leqq a_n$ (abbreviation $f \mid Y_{n,\alpha} \in R(Y_{n,\alpha})$.

Then it is clear that

1. Z is compact in $X \times \mathbf{C}^2$ and connected if X is,
2. B is a function algebra on Z,

and we will show

3. $\Delta B = Z$,
4. $\partial B = Z$,
5. B is essential on Z.

4. Proofs.

PROOF OF ASSERTION 3. Consider the function t on Z and let $\varphi \in \Delta B$. Then $\varphi(t) \in [0, a_1]$, let $\varphi(t) = \alpha$. Let f_α be a continuous function on $[0, a_1]$ of the variable t peaking at α and consider f_α as a function on Z. Then $f_\alpha \in B$. Using this function it is easily seen that

$$\varphi \in \Delta[B \mid Z \cap \{t = \alpha\}] .$$

If $\alpha = 0$, then $\varphi \in \Delta[B \mid \tilde{X}] = \tilde{X}$, since $B \mid \tilde{X} \cong A$.

If $\alpha \neq 0$ then $Z \cap \{t = \alpha\}$ consists of a finite union of disjoint sets $Y_{n,\alpha}$:

$$Z \cap \{t = \alpha\} = Y_{1,\alpha} \cup \ldots \cup Y_{n(\alpha),\alpha} ,$$

with $n(\alpha)$ the greatest integer such that $a_{n(\alpha)} \geqq \alpha$. Let μ be a Jensen representing measure for $\varphi \in \Delta[B \mid Z \cap \{t = \alpha\}]$ on $Z \cap \{t = \alpha\}$. If $\mu(Y_{n,\alpha}) > 0$ for some n then consider a function $g_{n,\alpha} \in B$ such that $g_{n,\alpha} = 1$ on $Y_{n,\alpha}$, $g_{n,\alpha} = 0$ on $Z \setminus Y_n \cap \{t \geqq \alpha/2\}$ and $0 \leqq g_{n,\alpha} < 1$ on $Z \setminus Y_{n,\alpha}$. Now

$$\log|\varphi(1 - g_{n,\alpha})| \leqq \int_{Z \cap \{t = \alpha\}} \log|1 - g_{n,\alpha}| \, d\mu = -\infty$$

since $\mu(Y_{n,\alpha}) > 0$. So because μ represents φ we have $\mu(Y_{n,\alpha}) = 1 = \|\mu\|$. Hence

$$\varphi \in \Delta[B \mid Y_{n,\alpha}] = Y_{n,\alpha}$$

since $B \mid Y_{n,\alpha} \cong R(Y_{n,\alpha})$. So if $\varphi \in \Delta B$ then $\varphi \in Z$. Hence $\Delta B = Z$.

PROOF OF ASSERTION 4. Consider a peak point $y_0 \in Y$ for the algebra $R(Y)$ with peak function h (these points are dense in Y). Now $z = (x_n, \alpha, \alpha y_0)$ where $0 < \alpha \leqq a_n$ is in Z and the function $h(y/t)g_{n,\alpha}$ peaks at z and belongs to B. Moreover points z of the above type are dense in Z. Hence $\partial B = Z$.

PROOF OF ASSERTION 5. Let K be a proper compact subset of Z then $Z \setminus K$ contains an open subset of some $Y_{n,\alpha}$, $\alpha > 0$. Since $R(Y_{n,\alpha})$ is essential on $Y_{n,\alpha}$ not every element of $C(Z)$ vanishing on K belongs to B, hence B is essential on Z.

5. Remarks.

In [4] Glicksberg posed the following question: if $B_1 \subset B_2 \subset C(\Delta B_1)$ and $\partial B_1 = \partial B_2$, must $\Delta B_1 = \Delta B_2$?

The answer is negative and a well-known example is the following: let A_1 and A_2 be function algebras on X such that $A_1 \subset A_2$ and $X = \Delta A_1 \neq \Delta A_2$. Then the algebras B_1 and B_2 on $X \times [0,1]$ of all continuous extensions to $X \times [0,1]$ of elements of A_1, respectively A_2, on $X \times \{0\}$ provide a counter example.

Using our construction it is possible to obtain B_1 and B_2 essential algebras on ΔB_1: start with A_1 (on a metrizable space X) and construct B_1 as in section 3. Let B_2 be all continuous functions f on Z such that $f \mid Y_{n,\alpha} \in R(Y_{n,\alpha})$, $0 < \alpha \leqq a_n$, $n \in \mathbb{N}$, and $f \mid \tilde{X} \in A_2$ (notation of section 3). Then

$$\Delta B_2 = Z \cup \{(z,0,0) : z \in \Delta A_2 \setminus X\} \neq \Delta B_1 = Z .$$

But $\partial B_1 = \partial B_2 = Z$.

In [1] and [2] Csordas and Reiter asked for a solution of the following problem: is there a non-separating essential function algebra B on a (connected) space X for which $\Delta B = \partial B = X$? (a function algebra A on X is called separating if for every proper compact subset Y of X and every point x in $X \setminus Y$ there exists $f \in A$ such that $f(x) \notin f(Y)$). The answer is positive and given by Eifler [3].

Starting with a non-separating function algebra A on a connected metrizable space X with $\Delta A = X$ and using the construction in section 3 we find a function algebra B satisfying all statements in the question of Csordas and Reiter. For A one can take $P(X)$ where X is the unit polydisc in $\mathbb{C}^2$, B then becomes a function algebra on a compact set in $\mathbb{C}^4$. Eifler's example shows some similarity with ours: the crucial ingredient is again the algebra $R(Y)$, Y a Swiss cheese. His algebra B is a subalgebra of $R(Y)$ which is a function algebra on a quotient space of Y.

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