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DISINTEGRATION THEORY ON A CONSTANT FIELD OF NON-SEPARABLE HILBERT SPACES

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1. Introduction.

We give simple proofs of some disintegration theorems of Maréchal [8] and Vesterstrøm and Wils [11], and comment on the involved properties of measurability.

Let (Z,Σ,μ) be a finite measure space, i.e. Σ is a Borel structure (σ-algebra) of subsets of the set Z, and μ is a finite positive countably additive function on Σ. A complex function f on Z is called measurable if the counter image of each open set is in Σ.

Let H be a Hilbert space. We call a map $f\colon Z \to H$ scalarly measurable if $z \mapsto (f(z)\mid\eta)$ is measurable for each $\eta \in H$, and strongly measurable if f is scalarly measurable and there exists a separable subspace K of H such that $f(z) \in K$ for almost all z (cf. [6]). Let $\mathscr{L}^2(\mu,\mathrm{H})$ denote the vector space of strongly measurable maps $f\colon Z \to H$ with $\|f\| \in \mathscr{L}^2(\mu)$; let $f(\mu)$ denote the set of strongly measurable functions equal to f almost everywhere, and let $L^2(\mu,H)$ denote the Hilbert space of classes $f(\mu)$, $f \in \mathscr{L}^2(\mu,H)$.

A map $a\colon Z \to \mathscr{L}(H)$ is called scalarly measurable if $z \mapsto (a(z)\xi\mid\eta)$ is measurable for each $\xi,\eta \in H$, and is called a measurable field, if $z \mapsto a(z)\xi$ is strongly measurable for each $\xi \in H$, that is if a is scalarly measurable and to every ξ there exists a separable subspace of H containing almost all values $a(z)\xi$. If H is separable the notions coincide and we use the term measurable. Every bounded measurable field a determines an operator on $\mathscr{L}^2(\mu,H)$ by $(af)(z)=a(z)(f(z))$ and an operator $a(\mu) \in \mathscr{L}(L^2(\mu,H))$ by $a(\mu)(f(\mu))=(af)(\mu)$, $f \in \mathscr{L}^2(\mu,H)$. In this situation a is called a disintegration of $a(\mu)$. Let $\mathscr{D}$ denote the von Neumann algebra on $L^2(\mu,H)$ of operators $\varphi(\mu)$, $\varphi \in \mathscr{L}^\infty(\mu)$. Let O denote the set of bounded maps $a\colon Z \to \mathscr{L}(H)$ such that a and a^* are measurable fields. It is easy to see that O is a sub C*-algebra of $l^\infty(Z,\mathscr{L}(H))$. In [8] and in [11] it is shown that there exists a positive linear map $A \mapsto A\check{}$ of $\mathscr{D}'$ into O with $A\check{}(\mu)=A$. Especially $a \mapsto a(\mu)$ maps O onto $\mathscr{D}'$.

Let $\mathscr{M}=\mathscr{M}(H)$ denote the set of bounded maps $a\colon Z \to \mathscr{L}(H)$ with the

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property that there exists a family $(H_i)_{i\in I}$ of pairwise orthogonal separable subspaces of H, with (Hilbert) sum H, such that for each $i \in I$ H_i is invariant under $a(Z)$, and $z \mapsto a(z)|H_i$ is measurable. Then $\mathscr{M} \subset O$. Our main contribution is a simple proof that $a \mapsto a(\mu)$ maps $\mathscr{M}$ onto $\mathscr{D}'$. We also show that $\mathscr{M}$ is a weakly sequentially closed C*-algebra and we give other characterizations of the maps in $\mathscr{M}$.

We use [1], [3], [4] and [5] freely.

We are indebted to G. A. Elliott for several helpful remarks.

2. Preliminaries.

Let f be a strongly measurable map of Z into H; a point $\xi \in H$ is called an essential value of f, if for each $\varepsilon > 0$ the counter image of $\{\eta \in H \mid \|\xi - \eta\| < \varepsilon\}$ has positive measure. The set of essential values of f is a closed separable subset of H, equivalent functions have the same essential values, and f is equivalent to a strongly measurable function taking essential values only.

If $(f_n)_{n\in \mathsf{N}}$ is a convergent sequence of functions in $\mathscr{L}^2(\mu, H)$, tending to f in $\mathscr{L}^2(\mu, H)$, then the set of essential values of f is contained in the closure of the union of the sets of essential values of the functions f_n. In fact there exists a subsequence $(n_i)_{i\in \mathsf{N}}$ of N and a set $Y \in \Sigma$ with complement of measure zero, such that $f_{n_i}(y) \to f(y)$ and $f_n(y)$ is an essential value of f_n for each n when $y \in Y$, and then any essential value of f is contained in

$$\overline{f(Y)} \subseteqq \overline{\bigcup_{n\in \mathsf{N}} f_n(Y)}\,.$$

Let K be a closed subspace of H; then $\mathscr{L}^2(\mu, K)$ is a subspace of $\mathscr{L}^2(\mu, H)$, and there is a natural isometry of $L^2(\mu, K)$ into $L^2(\mu, H)$. If a function $g \in \mathscr{L}^2(\mu, H)$ is orthogonal to $\varphi(\mu)\xi(\mu)$ for each $\varphi \in \mathscr{L}^\infty(\mu)$ and each $\xi \in K$ (identified with the corresponding constant function $z \mapsto \xi$ on Z), then any essential value of g is orthogonal to K and $g(\mu) \in L^2(\mu, K^\perp)$. In fact, if there exists $Y \in \Sigma$ with positive measure and $\xi \in K$, such that $(g(y) \mid \xi) \neq 0$ for all $y \in Y$, then

$$0 \neq \int_Y |(g(y) \mid \xi)|\, d\mu(y) = (g(\mu) \mid \varphi(\mu)\xi(\mu))\,,$$

when $\varphi(y) = \overline{\operatorname{sign}\,(g(y) \mid \xi)}$, $y \in Y$, and $\varphi(z) = 0$, $z \notin Y$.

From this follows that $L^2(\mu, K)^\perp = L^2(\mu, K^\perp)$, and that $L^2(\mu, K)$ is spanned by $\{\varphi(\mu)\xi(\mu) \mid \varphi \in \mathscr{L}^\infty(\mu),\ \xi \in K\}$. If $(K_i)_{i\in I}$ is a family of pairwise orthogonal subspaces of H with sum K, then $(L^2(\mu, K_i))_{i\in I}$ is a family of pairwise orthogonal subspaces of $L^2(\mu, H)$ with sum $L^2(\mu, K)$. If E is the (orthogonal) projection on K, then $E(\mu)$ is the projection on $L^2(\mu, K)$.

3. Disintegration.

LEMMA 1. *Let $\mathscr{A}$ be a separable C*-algebra contained in $\mathscr{D}'$. Let K be a separable subspace of H. There exists a separable subspace $\tilde{K}$ of H, containing K, such that $L^2(\mu, \tilde{K})$ is invariant under $\mathscr{A}$.*

PROOF. We may assume $1 \in \mathscr{A}$. Let $(A_i)_{i \in \mathrm{N}}$ be a dense subset of $\mathscr{A}$, and let $(\xi_j)_{j \in \mathrm{N}}$ be a dense subset of K. For each $(i, j) \in \mathrm{N} \times \mathrm{N}$, let $H_{i,j}$ be the subspace of H spanned by the set of essential values of any function in $\mathscr{L}^2(\mu, H)$ with class $A_i \xi_j(\mu)$. Let K_1 be the separable subspace of H spanned by $\bigcup_{(i,j) \in \mathrm{N} \times \mathrm{N}} H_{i,j}$. For any $A \in \mathscr{A}$ and $\xi \in K$, $A\xi(\mu)$ is limit in $L^2(\mu, H)$ of a sequence of vectors of form $A_i \xi_j(\mu)$; therefore K_1 contains the essential values of $A\xi(\mu)$. In the same way we let K_2 be the subspace of H spanned by the essential values of all $A\xi(\mu)$, $A \in \mathscr{A}$, $\xi \in K_1$, and so on. Then $\tilde{K} = \overline{\bigcup_{n \in \mathrm{N}} K_n}$ contains the essential values of all $A\xi(\mu)$, $A \in \mathscr{A}$, $\xi \in \tilde{K}$. All essential values of $A\varphi(\mu)\xi(\mu) = \varphi(\mu)A\xi(\mu)$, $\varphi \in \mathscr{L}^\infty(\mu)$, is contained in $\tilde{K}$, so $A\varphi(\mu)\xi(\mu) \in L^2(\mu, \tilde{K})$, and $AL^2(\mu, \tilde{K}) \subseteqq L^2(\mu, \tilde{K})$.

LEMMA 2. *There exists a family $(H_i)_{i \in I}$ of pairwise orthogonal separable subspaces of H, with sum H, such that $L^2(\mu, H_i)$ is invariant under $\mathscr{A}$ for each $i \in I$.*

PROOF. Let $(H_i)_{i \in I}$ be a maximal family of pairwise orthogonal separable subspaces of H with each $L^2(\mu, H_i)$ invariant under $\mathscr{A}$. Then $L = \bigcap_{i \in I} H_i^\perp$ is a subspace of H with $L^2(\mu, L)$ invariant under $\mathscr{A}$. Let K be any separable subspace of L, and let $\tilde{K}$ be the smallest subspace of H containing K with $L^2(\mu, K)$ invariant under $\mathscr{A}$. $\tilde{K}$ is a separable subspace of L, so by maximality $\tilde{K} = \{0\}$; therefore $L = \{0\}$.

THEOREM. *Let (Z, Σ, μ) be a finite measure space, and H a Hilbert space. Let $\mathscr{D}$ denote the algebra of multiplication operators on $L^2(\mu, H)$, and let $\mathscr{A}$ be a separable sub C*-algebra of $\mathscr{D}'$. There exist a family $(H_i)_{i \in I}$ of pairwise orthogonal separable subspaces of H, with sum H, and a field π on Z of representations $\pi(z)$ of $\mathscr{A}$ on H, with the properties*

(a) $\forall i \in I\ \forall A \in \mathscr{A}\ \forall z \in Z\colon \pi(z)(A)H_i \subseteqq H_i$, *and*
(b) *for each $A \in \mathscr{A}$, $z \mapsto \pi(z)(A)$ is a measurable field $\check{A}$ on Z with $\check{A}(\mu) = A$.*

PROOF. Choose $(H_i)_{i \in I}$ as in Lemma 2. Define

$$\mathscr{A}_i = \{A \mid L^2(\mu, H_i) \mid A \in \mathscr{A}\}.$$

For each $i \in I$ there exists a field π_i of representations $\pi_i(z)$ of $\mathscr{A}_i$ on H_i such that $z \mapsto \pi_i(z)(A \mid L^2(\mu, H_i))$ is a measurable field $\check{A_i}$ on Z with $\check{A_i}(\mu) = A \mid L^2(\mu, H_i)$ for each $A \in \mathscr{A}$ (see [4, 8.3.1. Lemme]); then

$$\|\pi_i(z)(A \mid L^2(\mu, H_i))\| \leqq \|A\| .$$

Define $\pi(z)(A)$ as the operator on H satisfying

$$\pi(z)(A) \mid H_i = \pi_i(z)(A \mid L^2(\mu, H_i)) \quad \text{for each } i \in I .$$

Using the notation $\mathscr{M}$ introduced in Section 1, we have the immediate consequence:

COROLLARY. *Each $A \in \mathscr{D}'$ has a disintegration in $\mathscr{M}$.*

4. The notion of measurability.

Let B be a von Neumann algebra on the Hilbert space H. It is well known that the following conditions on B are equivalent

a) B has separable predual B_*,
b) B is of countable type and countably generated,
c) the center of B is of countable type and B is countably generated,
d) B has a faithful normal representation on a separable Hilbert space,

cf. [3, Chapitre I, § 7, ex. 3 and § 3, ex. 4].

PROPOSITION 1. *Let B be a von Neumann algebra on the Hilbert space H. The following conditions on B are equivalent.*

a) *Every non-zero central projection in B has a non-zero central subprojection E with BE countably generated.*

b) *For each central projection E in B of countable type with respect to the center of B, BE is countably generated.*

c) *B is isomorphic to a direct product of algebras with faithful normal representations on separable Hilbert spaces.*

d) *B is isomorphic to a sub von Neumann algebra of a direct product of algebras $\mathscr{L}(H_i)$, with each Hilbert space H_i separable.*

e) *B has a separating family of normal representations on separable Hilbert spaces.*

f) *For each normal state φ of B, the Hilbert space of the representation π_φ associated to φ by the G.N.S. construction is separable.*

g) *For each $\xi \in H$, $B\xi$ is separable.*

h) *For each projection $E' \in B'$ of countable type in B', $E'H$ is separable.*

i) *There exists a family $(H_i)_{i \in I}$ of pairwise orthogonal separable subspaces of H, with sum H, each invariant under B.*

PROOF. We first prove that a), b), c), d) and e) are equivalent.

The implications a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) $\Rightarrow$ e) are straightforward.

Assume e); then the supports of the normal representations in the separating family form a family $(E_i)_{i \in I}$ of central projections in B, with $\sup_{i \in I} E_i = 1$, such that each BE_i is countably generated. This implies a).

Next we prove b) $\Rightarrow$ g) $\Rightarrow$ h) $\Rightarrow$ i) $\Rightarrow$ e) and g) $\Rightarrow$ f) $\Rightarrow$ e).

Assume b); when $\xi \in H$, the smallest projection E in the center of B with $E\xi = \xi$ is of countable type in the center; thus BE has a strongly dense countable subset, and $B\xi$ is separable, that is g) holds.

Any projection E' in B' of countable type is sum of a sequence of projections on pairwise orthogonal spaces of form $\overline{B\xi_n}$, $\xi_n \in H$, so g) implies h).

By [3, Chapitre III, § 1, Lemme 7], there exists a family of pairwise orthogonal projections of countable type in B', with sum 1; therefore h) implies i). It is trivial that i) implies e).

To prove that g) implies f) we may assume that each normal state of B is a vector state ω_ξ, because g) is equivalent to the space free condition a); then the space of the corresponding representation is isometric with the separable space $\overline{B\xi}$. It is trivial that f) implies e).

DEFINITION. We call a von Neumann algebra locally countably generated (l.c.g.), if it satisfies the equivalent conditions in Proposition 1.

Any sub von Neumann algebra of a product of l.c.g. algebras is l.c.g., by Condition d) of Proposition 1.

PROPOSITION 2. *Let $(B_n)_{n \in \mathbf{N}}$ be a sequence of* l.c.g. *algebras on a Hilbert space H. The von Neumann algebra spanned by $\bigcup_{n \in \mathbf{N}} B_n$ is* l.c.g.

PROOF. This can be shown on the basis of Proposition 1 i), by a combinatorial argument combined with Zorn's lemma. We are indebted to the referee for the following simple proof, also suggested by G. A. Elliott.

Note that $B_n K$ is separable for each $n \in \mathbf{N}$ and each separable subspace K of H. For each $\xi \in H$, $\overline{B\xi}$ is the closed linear span of

$$\bigcup_{n=1}^{\infty} \bigcup_{i_1=1}^{n} \bigcup_{i_2=1}^{n} \cdots \bigcup_{i_n=1}^{n} B_{i_1} B_{i_2} \cdots B_{i_n} \xi ,$$

so $B\xi$ is separable, and Proposition 1 g) applies.

PROPOSITION 3. *Let (Z, Σ) be a measurable space, i.e. Σ is a Borel structure on the set Z. Let H be a Hilbert space, and let a be a bounded mapping of Z into*

$\mathscr{L}(H)$. Let $\mathfrak{A}$ denote the von Neumann algebra generated by $a(Z)$. Let B be a von Neumann algebra on H containing $\mathfrak{A}$.

The following conditions are equivalent.

a) *There exists a family $(H_i)_{i \in I}$ of pairwise orthogonal separable subspaces of H, with sum H, such that for each $i \in I$ the space H_i is invariant under $a(Z)$ and $z \mapsto a(z) \mid H_i$ is measurable.*

b) *For each $\xi \in H$, $a(Z)\xi$ and $a(Z)^*\xi$ are separable, and a is scalarly measurable.*

c) $\mathfrak{A}$ *is* l.c.g., *and a is scalarly measurable.*

d) $\mathfrak{A}$ *is* l.c.g., *and for each normal state φ on B, $\varphi \circ a$ is measurable.*

e) $\mathfrak{A}$ *is* l.c.g., *and for each normal representation π of $\mathfrak{A}$ on a separable Hilbert space, $\pi \circ a$ is measurable.*

f) *There exists a separating family $(\pi_i)_{i \in I}$ of normal representations of $\mathfrak{A}$ on separable Hilbert spaces, with $\pi_i \circ a$ measurable for each $i \in I$.*

PROOF. It is easy to prove d) $\Rightarrow$ c) $\Rightarrow$ a) $\Rightarrow$ b), and d) $\Rightarrow$ e) $\Rightarrow$ f). We prove b) $\Rightarrow$ f) $\Rightarrow$ d) below.

Assume b). Let V denote $a(Z) \cup a(Z)^*$. Given $\xi \in H$, define $K_0 = \xi$ and recursively $K_n = V K_{n-1}$, $n \in \mathbf{N}$; the closed span $\bar{\xi}$ of $\bigcup_{n \in \mathbf{N}} K_n$ is separable and invariant under $\mathfrak{A}$. The representations $A \mapsto A \mid \bar{\xi}$ of $\mathfrak{A}$ for $\xi \in H$ separates $\mathfrak{A}$. Thus b) implies f).

Assume f). By Condition e) of Proposition 1, $\mathfrak{A}$ is l.c.g. The set of normal functionals φ on $\mathfrak{A}$, for which $\varphi \circ a$ is measurable, is a closed subspace of $\mathfrak{A}_*$ separating the points of $\mathfrak{A}$, because it contains all functionals of form $\psi \circ \pi_i$, $\psi \in \pi_i(\mathfrak{A})_*$; by the bipolar theorem it contains all normal states of $\mathfrak{A}$. This proves d).

The set of bounded maps of Z into $\mathscr{L}(H)$ satisfying the equivalent conditions of Proposition 3 we denote $\mathscr{M} = \mathscr{M}(H)$, cf. Section 1.

PROPOSITION 4. *In its natural representation on $l^2(Z,H)$, $\mathscr{M}$ is a weakly sequentially closed* C*-*algebra.*

PROOF. This follows from the fact that by Proposition 2 for any sequence $(a_n)_{n \in \mathbf{N}}$ of maps in $\mathscr{M}$ the union $\bigcup_{n \in \mathbf{N}} a_n(Z)$ is contained in some l.c.g. algebra.

EXAMPLE 1. A scalarly measurable field a, such that $a(Z)\xi$ is separable for each $\xi \in H$, but $a(Z)^*\xi$ is not:

Let Z be $[0,1]$, Σ the topological Borel structure on Z, and $H = l^2(Z)$. Let ξ_t denote the characteristic function of t, $t \in Z$. Define a field $a\colon Z \to \mathscr{L}(H)$ by

$$a(t)\xi = 2^{-\frac{1}{2}}(\xi \mid \xi_0+\xi_t)\xi_0, \qquad \xi \in H,\ t \in Z;$$

then $a(t)^*\xi = 2^{-\frac{1}{2}}(\xi \mid \xi_0)(\xi_0+\xi_t)$, so $t \mapsto a(t)^*$ is not a measurable field.

EXAMPLE 2. A field a of one-dimensional projections, such that $a(Z)\xi$ is strongly measurable for each $\xi \in H$, but $a(Z)\xi$ is not separable when $\xi \neq 0$: Let

$$Z = \{(s,t) \in [0,1]^2 \mid |s-t| \geqq 3^{-1}\}.$$

Further let Σ be the topological Borel structure, μ the restriction to Z of two-dimensional Lebesgue measure, and $H = l^2([0,1])$. Define ξ_t, $t \in [0,1]$, as above, and define

$$a(s,t)\xi = \tfrac{1}{2}(\xi \mid \xi_s+\xi_t)(\xi_s+\xi_t), \qquad (s,t) \in Z.$$

5. Constant fields of von Neumann algebras.

Let a finite measure space (Z,Σ,μ) and a Hilbert space H be given.

Let $\mathscr{D}_1$ denote the maximal abelian von Neumann algebra of multiplication operators on $L^2(\mu)$. Let B be a von Neumann algebra on H, let $\mathscr{M}(B)$ denote $\{a \in \mathscr{M} \mid a(Z) \subseteq B\}$, and let $B(\mu)$ denote the von Neumann algebra on $L^2(\mu,H)$ spanned by all the operators $a(\mu)$, $a \in \mathscr{M}(B)$.

It is well known that $L^2(\mu,H)$ is naturally isomorphic to the Hilbert space tensor product $L^2(\mu)\otimes H$; under this isomorphism $\mathscr{D}$ corresponds to the von Neumann algebra tensor product $\mathscr{D}_1 \otimes \mathbf{C}_H$, $\mathscr{D}'$ corresponds to $\mathscr{D}_1 \otimes \mathscr{L}(H)$, $\{b(\mu) \mid b \in B\}$ corresponds to $\mathbf{C}_{L^2(\mu)} \otimes B$, and $B(\mu)$ corresponds to a von Neumann algebra containing $\mathscr{D}_1 \otimes B$; since $B(\mu)$ and $B'(\mu)$ commutes, and $(\mathscr{D}_1 \otimes B')' = \mathscr{D}_1 \otimes B$ (by [10], cf. [11, Corollary 4.2]), $B(\mu)$ corresponds to $\mathscr{D}_1 \otimes B$, and $B(\mu)' = B'(\mu)$.

If $\mathfrak{A}$ is a von Neumann algebra on H, the von Neumann algebra spanned by the tensor products $\mathbf{C}_{L^2(\mu)} \otimes \mathfrak{A}'$ and $\mathbf{C}_{L^2(\mu)} \otimes B'$ is the tensor product of $\mathbf{C}_{L^2(\mu)}$ with the von Neumann algebra spanned by $\mathfrak{A}'$ and B', so $(\mathscr{D}_1 \otimes \mathfrak{A}') \cup (\mathscr{D}_1 \otimes B')$ spans $\mathscr{D}_1 \otimes (\mathfrak{A}' \cup B')''$, and

$$(\mathscr{D}_1 \otimes \mathfrak{A}) \cap (\mathscr{D}_1 \otimes B) = \mathscr{D}_1 \otimes (\mathfrak{A} \cap B),$$

and $\mathfrak{A}(\mu) \cap B(\mu) = (\mathfrak{A} \cap B)(\mu)$.

LEMMA 3. *Let $\mathfrak{A}$ and B be von Neumann algebras on Hilbert spaces H and K respectively, Φ a normal homomorphism of $\mathfrak{A}$ into B, and $\Phi(\mu)$ the normal homomorphism of $\mathfrak{A}(\mu)$ into $B(\mu)$ corresponding to the homomorphism $1 \otimes \Phi$ of $\mathscr{D}_1 \otimes \mathfrak{A}$ into $\mathscr{D}_1 \otimes B$. Let a be a map in $\mathscr{M}(\mathfrak{A})$. Then $\Phi \circ a \in \mathscr{M}(B)$, and $(\Phi \circ a)(\mu) = \Phi(\mu)(a(\mu))$.*

PROOF. The statement about measurability follows from Proposition 3. If Φ has the form $a \mapsto a \mid E'H$, $a \in \mathfrak{A}$, where E' is a projection in $\mathfrak{A}'$, then $\Phi(\mu)$ is the map $A \mapsto A \mid E'(\mu)L^2(\mu, H)$, $A \in \mathfrak{A}(\mu)$, and the lemma follows easily. The lemma now follows from the known structure of normal homomorphisms, cf. [3, Chapitre I, § 4, Théorème 3].

LEMMA 4. *Let $(H_i)_{i \in I}$ be a family of Hilbert spaces, with Hilbert sum H, and for each $i \in I$ let $a_i \in \mathscr{M}(H_i)$. Assume $\sup_{i \in I, z \in Z} \|a_i(z)\| < \infty$. Then $z \mapsto a(z) = \bigoplus_{i \in I} a_i(z)$ defines a map $a \in \mathscr{M}(H)$, with $a(\mu) = \bigoplus_{i \in I} a_i(\mu)$.*

PROOF. It is clear that $a \in \mathscr{M}(H)$. It is enough to check the equality on each $L^2(\mu, H_i)$, cf. Section 2, and there it is trivial.

PROPOSITION 5. (cf. [11 Theorem 4.1, 2)]). *Let B be a von Neumann algebra on H. Any $A \in B(\mu)$ has a disintegration in $\mathscr{M}(B)$.*

PROOF. If B has a faithful normal representation on a separable Hilbert space, the proposition follows from Lemma 3 and [3, Chapitre II, § 3, Théorème 1]. If B is l.c.g. it follows from the above, Proposition 1, and Lemma 4. We now consider the general case.

Let $A \in B(\mu)$ be given. Let $a \in \mathscr{M}$ be a disintegration of A, and let $\mathfrak{A}$ be the von Neumann algebra spanned by $a(Z)$. Then $\mathfrak{A} \cap B$ is l.c.g., and $A \in \mathfrak{A}(\mu) \cap B(\mu) = (\mathfrak{A} \cap B)(\mu)$, so A has a disintegration in $\mathscr{M}(\mathfrak{A} \cap B) \subseteq \mathscr{M}(B)$.

REMARK. It is easy to generalize the contents of this paper to the framework of Radon measure spaces (see [2], cf. also [9]). By use of Proposition 3 and the structure of Radon measure spaces we see that the relevant definition of $\mathscr{M}$ is that $\mathscr{M}$ is the set of bounded scalarly measurable maps $a: Z \to \mathscr{L}(H)$ such that the von Neumann algebra spanned by $a(K)$ is l.c.g. for each compact subset K of Z.

6. Non-constant fields of Hilbert spaces.

Let a finite measure space (Z, Σ, μ) and a field $H = (H(z))_{z \in Z}$ of Hilbert spaces be given.

By a measurable structure on H we shall here understand a family $\mathscr{F}$ of vector fields $\eta \in \prod_{z \in Z} H(z)$ with the property:

There exists a family $(\xi_i)_{i \in I}$ of fields in $\mathscr{F}$, such that $(\xi_i(z))_{i \in I}$ spans $H(z)$ for each $z \in Z$, such that for each $i \in I$ $(\xi_i \mid \xi_j) = 0$ everywhere for all but countably many $j \in I$, and such that $\mathscr{F}$ consists of the fields ξ equal a.e. to a field η with

$(\eta \mid \xi_i)$ measurable for each $i \in I$ and $(\eta \mid \xi_i)=0$ everywhere for all but countably many $i \in I$.

We call $(\xi_i)_{i \in I}$ a fundamental family of measurable vector fields.

Our conditions are analogous to the conditions used in [8], compare [7] for a much more general, and less elementary, theory.

Constant fields are measurable.

Now assume given a measurable structure $\mathscr{F}$ with fundamental family $(\xi_i)_{i \in I}$.

It is easy to see that $\mathscr{F}$ is a vector space, invariant under multiplication with measurable functions.

It is easy to show, by Zorn's lemma, that there exists a family $(I(l))_{l \in L}$ of pairwise disjoint countable subsets of I, with union I, such that $(\xi_i \mid \xi_j)=0$ whenever $i \in I(l), j \in I(m), l \neq m$. For shortness, we call such a family a splitting of I.

Since we can orthonormalize $(\xi_i)_{i \in I(l)}$ for each $l \in L$, we see that $\mathscr{F}$ has a fundamental family which at each point z of Z is an orthogonal system containing a basis of $H(z)$. It follows that $(\xi \mid \eta)$ is measurable for all $\xi, \eta \in \mathscr{F}$.

If $\eta \in \mathscr{F}$ and $(\eta \mid \xi_i) \neq 0$, say that η and i are associated. Call i inessential for η if $(\eta \mid \xi_i)=0$ a.e., and essential otherwise. Then the set of essential indices of η is countable, equivalent vector fields in $\mathscr{F}$ have the same essential indices, and η is equivalent to a field in $\mathscr{F}$ associated with essential indices only.

When $\eta \in \mathscr{F}$ let $\eta(\mu)$ denote the equivalence class of η, and let $H(\mu)$ denote the set $\{\eta(\mu) \mid \eta \in \mathscr{F}, \|\eta\| \in \mathscr{L}^2(\mu)\}$. $H(\mu)$ is a Hilbert space.

Let J be a subset of I, with $(\xi_j \mid \xi_k)=0$ when $j \in J$, $k \in I \setminus J$. Let $H_J(\mu)$ denote the set of $\eta(\mu)$ in $H(\mu)$ with all essential indices in J; then $H_J(\mu)$ is the closed subspace of $H(\mu)$ spanned by

$$\{\varphi(\mu)\xi_j(\mu) \mid \varphi \in \mathscr{L}^\infty(\mu),\ j \in J\},$$

and $H_J(\mu)^\perp = H_{I \setminus J}(\mu)$. If $(I(l))_{l \in L}$ is a splitting of I, $H(\mu)$ is the sum of the spaces $H_{I(l)}(\mu)$, $l \in L$.

Call a field a of operators $a(z) \in \mathscr{L}(H(z))$ measurable if $a\eta \in \mathscr{F}$ when $\eta \in \mathscr{F}$. If a is also bounded, let $a(\mu)$ denote the corresponding operator on $H(\mu)$. Let $\mathscr{D}$ denote the algebra of operators $\varphi(\mu)$ of multiplication with functions $\varphi \in \mathscr{L}^\infty(\mu)$ on $H(\mu)$.

The proof of Theorem 1 can now be carried over in this framework.

Let $\mathscr{A}$ be a separable sub C*-algebra of $\mathscr{D}'$. There exists a splitting $(I(l))_{l \in L}$ of I, and a field π on Z of representations $\pi(z)$ of $\mathscr{A}$ on $H(z)$, such that $H_{I(l)}$ is $\mathscr{A}$-invariant for each l, and $z \mapsto \pi(z)(A)$ is a measurable field $A^\vee$ with $A^\vee(\mu)=A$.

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